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b Generalized Closed Sets In Grill Topological Spaces

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Abstract: *The purpose of this paper is to introduce and study a new class of b generalized closed sets defined in terms of a grill G on X. The characterization of such sets along with certain other properties of them are obtained.*

Keywords : *b – closed, b*g closed, $G_{(b*g)*}$ closed,*
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I. INTRODUCTION

It is found from literature that during recent years many topologists are interested in the study of generalized types of closed sets. For instance, a certain form of generalized closed sets was initiated by Levine [7], Following the trend, we have introduced and investigated a kind of generalized closed sets, the definition being formulated in terms of grills. The concept of grill was first introduced by choquet [2] in the year 1947. From subsequent investigations it is revealed that grills can be used as an extremely useful device for investigation of a number of topological problems.

II. PRELIMINARIES

Definition 2.1: A non-empty collection G of non-empty subsets of a topological space X is called a Grill if
(i) $A \in B$ and $A \subseteq B \subseteq X \implies B \in G$ and
(ii) $A, B \subseteq X$ and $A \cup B \in G \implies A \in G$ or $B \in G$.

Let G be a grill on a topological space (X,τ) In an operator $\emptyset$:P(X)→ P(X) was defined by $\emptyset(A) = \{x \in X / U \cap A \in G, \forall\, U \in \tau(x)\}$, $\tau(x)$ denotes the neighborhood of x. Also the map Ψ:P(X)→ P(X) given by $\Psi(A) = A \cup \emptyset(A)$ for all $A \in P(X)$. Corresponding to a grill G, on a topological space(X,τ) there exist a unique topology τ_G on X given by $\tau_G = \{U \subseteq X / \Psi(X - U) = X - U\}$ where for any $A \subseteq X$, $\Psi(A) = A \cup \phi(A) = \tau_G - cl(A)$. Thus a subset A of X is τ_G – closed (resp. τ_G – dense in itself) if $\psi(A) = A$ or equivalently if $\emptyset(A) \subseteq A$ (resp $A \subseteq \emptyset(A)$).

In the next section, we introduce and analyze a new class of generalized closed sets, namely $G_{(b*g)*}$ closed sets in terms of a given grill G. The definition having a close bearing to the above operator $\emptyset$.

Throughout the paper, by a space X we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, we shall adopt the usual notations int(A) and cl(A) respectively for the interior and closure of A in (X, τ). Again τ_G – cl(A) and τ_G – int(A) will respectively denote the closure and interior of A in (X, τ_G). Similarly, whenever we say that a subset A of a space X is open (or closed), it will mean that A is open (or closed) in (X, τ). For open and closed sets with respective to any other topology on X, eg. τ_G we shall write τ_G - open and τ_G – closed. The collection of all open neighborhoods of a point X in (X, τ) will be denoted by τ(x). (X, τ, G) denotes a topological space (X, τ) with a grill G.

Definition 2.2: A subset A of a topological space (X, τ) is called
1. b open if $A \subseteq int(cl(A)) \cup cl(int(A))$
2. b^*g closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is b open
3. $(b^*g)^*$ closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is b^*g open
4. θ closed if A = θ cl(A) where $\theta\, cl(A) = \{x \in X : cl(U) \cap, A \neq \emptyset \forall\, U \in \tau \text{ and } X \epsilon U\}$
5. δ closed if A= δ cl(A) where $\delta\, cl(A) = \{X \epsilon X, int(cl(U) \cap A \neq \emptyset, \forall\, U \in \tau \text{ and } X \epsilon U\}$

The complements of the above mentioned closed sets are respective open sets.

Definition 2.3: A function f: (X, τ) → (y, σ) is called
1. continuous if $f^{-1}(V)$ is open in X, for every $V \in \sigma$
2. τ_G continuous if $f^{-1}(V)$, τ_G is open in X, for every $V \in \sigma$

3. $(b^*g)^*$ continuous if $f^{-1}(V)$, is $(b^*g)^*$ open in X, for every $V \in \sigma$
4. θ continuous if $f^{-1}(V)$, is θ open in X, for every $V \in \sigma$
5. δ continuous if $f^{-1}(V)$, is δ open in X, for every $V \in \sigma$

Definition 2.4: A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is called
1.Closed if f(A) is closed in Y, for every closed set A of X
2.τ_G closed if f(A) is τ_G closed in Y,for every closed set A of X
3.$(b^*g)^*$ closed if f(A) is $(b^*g)^*$ closed in Y, for every closed set A of X
4.θ closed if f(A) is θ closed in Y, for every closed set A of X
5.δ closed if f(A) is δ closed in Y, for every closed set A of X
Definition 2.5: A function F : $(X, \tau) \rightarrow (Y, \sigma)$ is called
1. Open if f(U) is open in Y, for every $U \in \tau$
2. τ_G open if f(U) isτ_G open in Y, for every $U \in \tau$
3. $(b^*g)^*$ open if f(U) is $(b^*g)^*$ open in Y for every $U \in \tau$
4. θ open if f(U) is θ open in Y for every $U \in \tau$
5. δ open if f(U) is δ open in Y for every $U \in \tau$

Theorem 2.6: [7] Let (X, τ) be a topological space and G be a grill on X. Then for any $A, B \subseteq X$ the following hold

(a) $A \subseteq B \Rightarrow \phi(A) \subseteq \phi(B)$

(b) $\phi(A \cup B) = \phi(A) \cup \phi(B)$

(c) $\phi(\phi(A) \subseteq \phi(A) = cl(\phi(A)) \subseteq cl(A)$

3. $G_{(b*g)*}$ Closed Sets

Definition 3.1: A subset A of (X,τ,G) is called $G_{(b*g)*}$ closed if $\emptyset(A) \subseteq U$ whenever $A \subseteq U$ and U is b^*g open in X.
Theorem 3.2: Let (X, τ, G) be a grill topological space
1. Every closed set in X is $G_{(b*g)*}$ closed
2. Every τ_G closed set is $G_{(b*g)*}$ closed
3. Every non member in G is $G_{(b*g)*}$ closed
4. Every $(b^*g)^*$ closed set is $G_{(b*g)*}$ closed
5. Every θ closed set is $G_{(b*g)*}$ closed
6. Every δ closed set is $G_{(b*g)*}$ closed

Proof :
(1) Let A be closed in X. Then cl(A) = A. Let $A \subseteq U$, where U is b^*g open $\emptyset(A) \subseteq cl\, A = A \subseteq U$. Hence A is $G_{(b*g)*}$ closed.
(2) Let A be τ_G closed. Then $\emptyset(A) \subseteq A$. Let $A \subseteq U$ where U is b*g open $\emptyset(A) \subseteq A \subseteq U$. Hence A is $G_{(b*g)*}$ closed
(3) Let $A \notin G$. Let $A \subseteq U$ where U is b*g open then $\emptyset(A) = \emptyset \subseteq U$. Hence A is $G_{(b*g)*}$ closed
(4) Let A be $(b^*g)^*$ closed Let $A \subseteq U$, where U is b*g open. $\emptyset(A) \subseteq cl(A) \subseteq U$ Hence A is $G_{(b*g)*}$ closed
(5) Let A be θ closed. Then $\theta cl(A) = A$. Let $A \subseteq U$, where U is b*g open $\emptyset(A) \subseteq cl(A) \subseteq \theta cl(A) = A \subseteq U$. Hence A is $G_{(b*g)*}$ closed
(6) Let A be δ closed. Then $\delta\, cl\, A = A$ Let $A \subseteq U$, where U is b*g open $\emptyset(A) \subseteq cl\, A \subseteq \delta\, cl\, A = A \subseteq U$. Hence A is $G_{(b*g)*}$ closed.
The converse of the above statements need not be true can be seen from the following examples.

Example 3.3: Let X={a, b, c} $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ $G = \{\{a, c\}, X\}$ {a, c} is $G_{(b*g)*}$ closed but not closed

Example 3.4: Let X={a, b, c} $\tau = \{\emptyset, \{a, b\}, X\}$$G = \{\{b\}, \{a, b\}, \{b, c\}, X\}$ {b, c} is $G_{(b*g)*}$ closed but not $\tau_G$closed

Example 3.5: Refer example 3.3
{a, c} is $G_{(b*g)*}$ closed but not a non member of G.

Example 3.6: Refer example 3.3. {a, c} is $G_{(b*g)*}$ closed but not (b*g)* closed

Example 3.7: Refer example 3.3. {a, c} is $G_{(b*g)*}$ closed but not θ closed

Example 3.8: Refer example 3.3 {a, c} is $G_{(b*g)*}$ closed but not δ closed

Lemma 3.9: Let (X, τ) be a space and G be a grill on X. If A $\subseteq$ X is
τ_G – dense in itself, then
$\phi(A) = cl\ \phi(A) = \tau_G - cl(A) = cl(A)$

Theorem 3.10: Let (X, τ) be a topological space and G be a grill on X. Then for A ⊆ X, A is $G_{(b*g)*}$ closed iff τ_G – cl (A) ⊆ U and U is b*g open.

Proof: Suppose A is $G_{(b*g)*}$ closed then $\emptyset(A) \subseteq U \Longrightarrow A \cup \emptyset(A) \subseteq U$. Therefore τ_G – cl(A) ⊆ U, A ⊆ U and U is b*g open. Conversely, τ_G – cl(A) ⊆ U, A ⊆ U and U is b*g open. Therefore $A \cup \emptyset(A) \subseteq U \Longrightarrow \emptyset(A) \subseteq U$. Hence A is $G_{(b*g)*}$ closed.

Theorem 3.11: Let G be a grill on a space (X, τ). If A is τ_G – dense is itself and $G_{(b*g)*}$ closed, then A is (b*g)* closed.

Proof: Let A be τ_G – dense in itself, then by Lemma 3.9 $\emptyset(A) = cl(A)$. Since A is $G_{(b*g)*}$ closed $\emptyset(A) \subseteq U$ when U is b*g open in X and A ⊆ U. Therefore cl(A) ⊆ U when U is b*g open in X and A ⊆ U. Hence A is (b*g)* closed.

Theorem 3.12: For any grill G on a space (X, τ), the following are equivalent
a) Every subset of X is $G_{(b*g)*}$ closed
b) Every b*g open subset of (X, τ) is τ_G closed

Proof: (a) ⟹ b let A be b*g open in (X, τ). Then by (a). A is $G_{(b*g)*}$ closed so that $\emptyset(A) \subseteq A$. Therefore A is τ_G closed.
(b) ⟹ (a) Let A ⊆ X and U be b*g open in (X, τ) such that A ⊆ U. Then by (b), $\emptyset(U) \subseteq U$. Also, $A \subseteq U \Longrightarrow \emptyset(A) \subseteq \emptyset(U) \subseteq U$. Therefore A is $G_{(b*g)*}$ closed.

Theorem 3.13: Let (X, τ) be a topological space and G be a grill on X and A, B be subsets of X such that $A \subseteq B \subseteq \tau_G - cl(A)$. If A is $G_{(b*g)*}$ closed, then B is $G_{(b*g)*}$ closed.

Proof: Suppose B ⊆ U and U is b*g open in X. Since A is $G_{(b*g)*}$ closed.

$$\emptyset(A) \subseteq U \Longrightarrow \tau_G - cl(A) \subseteq U \rightarrow (1)$$

Now $A \subseteq B \subseteq \tau_G - cl(A)$ which implies $\tau_G - cl(A) \subseteq \tau_G - cl(B) \subseteq \tau_G - cl(A)$
Therefore $\tau_G - cl(A) = \tau_G - cl(B)$
Therefore by (1) $\tau_G - cl(B) \subseteq U$. Hence B is $G_{(b*g)*}$ closed.

Corollary 3.14: τ_G – closure of every $G_{(b*g)*}$ closed set is $G_{(b*g)*}$ closed.

Theorem 3.15: Let G be a grill on a space (X, τ) and A, B be subsets of X such that $A \subseteq B \subseteq \emptyset(A)$. If A is $G_{(b*g)*}$ closed, then A and B are (b*g) closed.

Proof: Let $A \subseteq B \subseteq \emptyset(A)$ .Then $A \subseteq B \subseteq \tau_G - cl(A)$. By theorem 3.13, B is $G_{(b*g)*}$ closed. Again $A \subseteq B \subseteq \emptyset(A) \Longrightarrow \emptyset(A) \subseteq \emptyset(B) \subseteq \emptyset(\emptyset(A)) \subseteq \emptyset(A)$. This implies that $\emptyset(A) = \emptyset(B)$. By theorem 3.11, A and B are (b*g) closed.

Theorem 3.16: Let G be a grill on a space (X, τ). Then A subset A of X is $G_{(b*g)*}$ open iff $F \subseteq \tau_G - int(A)$ whenever F ⊆ A and F is (b*g) closed.

Proof: Let A be $G_{(b*g)*}$ open set and F ⊆ A where F is b*g closed. Then $X - A \subseteq X - F$. This implies that $\emptyset(X - A) \subseteq \emptyset(X - F) = X - F$. Hence $\tau_G - cl(X - A) \subseteq X - F$ which implies $F \subseteq \tau_G - int(A)$.

Conversely, $F \subseteq \tau_G - int(A)$, $\tau_G -\ cl(X - A) \subseteq X - F\ \emptyset(X - A) \subseteq X - F$. Hence A is $G_{(b*g)*}$ open.

4. $G_{(b*g)*}$ Continuous Function:

Definition 4.1: A function $F: (X, \tau, G) \longrightarrow (Y, \tau)$ is said to be $G_{(b*g)*}$ continuous (resp. (b*g)* continuous). if f^{-1}(V) is $G_{(b*g)*}$ open. (resp. (b*g)* open) for each V ∈ σ.

Theorem 4.2:

1. Every continuous function is $G_{(b*g)*}$ continuous
2. Every τ_G continuous function is $G_{(b*g)*}$ continuous
3. Every (b*g)* continuous function is $G_{(b*g)*}$ continuous
4. Every θ continuous function is $G_{(b*g)*}$ continuous
5. Every δ continuous function is $G_{(b*g)*}$ continuous

Proof: Obvious

Converse of the above statements need not be true can be seen from the following examples:

Example 4.3: Refer Example 3.4

Define f : $(X, \tau, G) \longrightarrow (X, \tau)$ by f(a) = c, f(b) = b, f(c) = c f is $G_{(b*g)*}$ continuous but not continuous as f^{-1} [{a, b}]={b} is not open.

Example 4.4: Let $X = \{a, b, c\}$, $\tau = \{\ \emptyset, \{a\}, X\}$ G ={{a}, {b}, {c,}, {a, b}, {a, c}, {b, c}, X}

Define f: $(X, \tau, G) \longrightarrow (X, \tau)$ by f(a) = b, f(b) = c, f(c) = a, f is $G_{(b*g)*}$ continuous but not τ_G continuous as $f^{-1}(\{a\})$ = {c} is not τ_G open.

Example 4.5: Refer Example 3.3

Define f by f(a) = b, f(b) = a, f(c) = c, f is $G_{(b*g)*}$ continuous but not (b*g)* continuous as $f^{-1}(\{a\})$ = {b} is not (b*g)* open.

Example 4.6: Take the previous example

f is $G_{(b*g)*}$ continuous but not θ continuous as $f^{-1}(\{a\} = \{b\}$ is not θ open.

Example 4.7: Take the previous example f is $G_{(b*g)*}$ continuous but not δ continuous as $f^{-1}(\{a\})$ = {b} is not δ open.

Definition 4.8: A function f : $(X, \tau) \longrightarrow (Y, \sigma, G)$ is said to be $G_{(b*g)*}$ closed if f(A) is $G_{(b*g)*}$ closed in Y, for every closed set A of X.

Theorem 4.9:

1. Every closed function is $G_{(b*g)*}$ closed
2. Every τ_G closed function is $G_{(b*g)*}$ closed
3. Every (b*g)* closed function is $G_{(b*g)*}$ closed
4. Every θ closed function is $G_{(b*g)*}$ closed
5. Every δ closed function is $G_{(b*g)*}$ closed

Proof: Obvious

Converse of the above statements need not be true can be seen from the following examples.

Example 4.10: Refer example 3.4

Define f : $(X, \tau) \longrightarrow (X, \tau, G)$ by f(a) = a, f(b) = b, f(c) = b, f is $G_{(b*g)*}$ closed but not closed as f({c}) = {b} is not closed.

Example 4.11 Refer example 4.4

Define f : $(X, \tau) \longrightarrow (X, \tau, G)$ by f(a) = c, f(b) = a, f(c) = b, f is $G_{(b*g)*}$ closed but not τ_G closed as f({b, c}) = {a, b} is not τ_G closed.

Example 4.12: Refer example 3.3

Define f: $(X, \tau) \longrightarrow (X, \tau, G)$ by f(a) = b, f(b) = a, f(c) =c, f is $G_{(b*g)*}$ closed but not (b*g)* closed as f({b, c}) = {a, c} is not (b*g)* closed.

Example 4.13: Take the previous example
f is $G_{(b*g)*}$ closed but not θ closed as f({b, c}) = {a, c} is not θ closed.

Example 4.14: Take the previous example
f is $G_{(b*g)*}$ closed but not δ closed as f({b, c}) = {a, c} is not δ closed.

Theorem 4.15: If f: $(X, \tau) \rightarrow (Y, \sigma)$ is closed and g : $(Y, \sigma) \rightarrow (Z, \eta, G)$ is $G_{(b*g)*}$ closed, then g o f : $(X, \tau) \rightarrow (Z, \eta, G))$ is $G_{(b*g)*}$ is closed.

Theorem 4.16: A map f : X → Y is $G_{(b*g)*}$ closed if and only if for each subset S of Y and each open set U of X such that $f^{-1}(S) \subseteq U$, there is a $G_{(b*g)*}$ open subset V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof: Let f be $G_{(b*g)*}$ closed. Let S⊆ Y and U be an open set of X such that f^{-1} (S) ⊆ U.
X - U is closed in X. f(X -U) is $G_{(b*g)*}$ closed in Y. $V = Y - f(X - U)$ is $G_{(b*g)*}$ open in Y $f^{-1}(V)=X - f^{-1}(f(X - U)) \subseteq X - (X - U) = U$
Conversely, let F be closed in X $f^{-1}(f(F^c)) \subset F^c$ and F^c open in X.
By assumption, there exists a $G_{(b*g)*}$ open subset V of Y such that $f(F^c) \subseteq v$ and $f^{-1}(V) \subseteq F^c$.
This implies $F \subseteq (f^{-1}(V))^{c,}$
Hence $V^c \subseteq (f(F^c))^c = f(F) \subseteq f(f^{-1}(V))^c \subseteq v^c$ so, $f(F) = V^c$ which is $G_{(b*g)*}$ closed.

Definition 4.17: Let X and Y be topological spaces. A map f : X→ Y is called $G_{(b*g)*}$ open map if the image of every open set of X is $G_{(b*g)*}$ open in Y.

Theorem 4.18: For any bijection map f: X -> Y the following are equivalent

1. $f^{-1}: Y \longrightarrow X$ is $G_{(b*g)*}$ continuous map
2. f is $G_{(b*g)*}$ open map
3. f is $G_{(b*g)*}$ closed map

Proof: (1) ⇒ (2) Let U be open in X. $(f^{-1})^{-1}$ (U) is $G_{(b*g)*}$ open in Y. That is f(U) is $G_{(b*g)*}$ open in Y. (2) ⇒ (3) Let F be a closed set of X. Then F^c is open in X. By assumption $f(F^c)$ is $G_{(b*g)*}$ open in Y $f(F^c) = (f(F))^c$ is $G_{(b*g)*}$ open in Y, f(F) is $G_{(b*g)*}$ closed in Y.
(3) ⇒ (1) Let F be closed in X f(F) is $G_{(b*g)*}$ closed in Y. $f(F) = (f^{-1})^{-1}$ (F) is $G_{(b*g)*}$ closed in Y. Hence f^{-1} is $G_{(b*g)*}$ continuous map.

Definition 4.19: Let (X, τ) be a topological space and (Y, σ, G) be a grill topological space. A function f : $(X, \tau) \longrightarrow (Y, \sigma, G)$ is said to be $G_{(b*g)*}$ open (resp. $G_{(b*g)*}$ closed), if for each $V \in \tau$, f(V) is $G_{(b*g)*}$ open (resp. $G_{(b*g)*}$ closed) in (Y, σ, G).

Theorem 4.20:

1. Every open function is $G_{(b*g)*}$ open
2. Every τ_G open function is $G_{(b*g)*}$ open
3. Every (b*g)* open function is $G_{(b*g)*}$ open
4. Every θ open function is $G_{(b*g)*}$ open
5. Every δ open function is $G_{(b*g)*}$ open

Proof: Obvious
Converse of the above statements needs not be true can be seen from the following examples.

Example 4.21: Refer example 3.4
Define f : $(X, \tau) \longrightarrow (X, \tau, G)$ by f(a)=b, f(b) =b, f(c) = c, f is $G_{(b*g)*}$ open but not open as f({a, b}) = {b} is not open.

Example 4.22: Refer example 4.4
Define f : $(X, \tau) \rightarrow (X, \tau, G)$ by f(a) = c, f(b) = b, f(c) =a, f is $G_{(b*g)*}$ open but not τ_G open as f({a}) = {c} is not τ_G open.

Example 4.23: Refer example 3.3
Define $f : (X, \tau) \rightarrow (X, \tau, G)$ by f(a)= b, f(b)= b, f(c) = c, f is $G_{(b*g)*}$ open but not $G_{(b*g)*}$ open as f({a}) = {b} is not $G_{(b*g)*}$ open.

Example 4.24: Refer example 4.23
f is $G_{(b*g)*}$ open but not θ open as f({a}) = {b} is not θ open.

Example 4.25: Refer example 4.23
f is $G_{(b*g)*}$ open but not δ open as f({a}) = {b} is not δ open.

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