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A Comparison between the Zeroes of the Solutions of One Quadrature Solvable Vekua Equation with a Sturm Approach

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***ABSTRACT**: The Vekua equation is an areolar differential equation from a complex function, which cannot be solved in general case. Its origin is from a practice problem from the theory of elasticity. In the paper are considered the zeroes of one type of Vekua equation,* $\frac{\partial W}{\partial \bar{z}} = A(z,\bar{z})W$ *, which can be solved with quadratures, with the Sturm approach to the zeroes of the regular differential equations* $u = \operatorname{Re} W$ *and* $v = \operatorname{Im} W$ *.*
***KEYWORDS**: Areolar derivative, Areolar equation, Vekua equation, zeroes of differential equation, Sturm theory.*

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I. INTRODUCTION

The equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \qquad (1)$$

where $A = A(z)$, $B = B(z)$ and $F = F(z)$ are given complex functions from a complex variable $z \in D \subseteq$ is the well known Vekua equation [1] according to the unknown function $W = W(z) = u + iv$. The derivative on the left side of this equation has been introduced by G.V. Kolosov in 1909 [2]. During his work on a problem from the theory of elasticity, he introduced the expressions

$$\frac{1}{2}[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})] = \frac{\hat{d}W}{dz} \qquad (2)$$

and

$$\frac{1}{2}[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})] = \frac{\hat{d}W}{d\bar{z}} \qquad (3)$$

known as operator derivatives of a complex function $W = W(z) = u(x,y) + iv(x,y)$ from a complex variable $z = x + iy$ and $\bar{z} = x - iy$ corresponding. The operating rules for this derivatives are completely given in the monograph of Г. Н.Положий [3] (page18-31). In the mentioned monograph are defined so cold operator integrals

$$\hat{\int} f(z)dz \text{ and } \hat{\int} f(z)d\bar{z}$$

from $z = x + iy$ and $\bar{z} = x - iy$ corresponding (page 32-41). As for the complex integration in the same monograph is emphasized that it is assumed that all operator integrals can be solved in the area D.

In the Vekua equation (1) the unknown function $W = W(z)$ is under the sign of a complex conjugation which is equivalent to the fact that $B = B(z)$ is not identically equaled to zero in D. That is why for (1) the quadratures that we have for the equations where the unknown function $W = W(z)$ is not under the sign of a complex conjugation, stop existing.

This equation is important not only for the fact that it came from a practical problem, but also because depending on the coefficients A, B and F the equation (1) defines different classes of generalized analytic functions. For example, for $F = F(z) \equiv 0$ in D the equation (1) i.e.

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W}$$

which is called canonical Vekua equation, defines so cold generalized analytic functions from fourth class; and for $A \equiv 0$ and $F \equiv 0$ in D, the equation (1) i.e. the equation $\frac{\hat{d}W}{d\bar{z}} = B\overline{W}$ defines so cold generalized analytic functions from third class or the (r+is)-analytic functions [3], [4].

Those are the cases when $B \neq 0$. But if we put $B \equiv 0$, we get the following special cases. In the case $A \equiv 0$, $B \equiv 0$ and $F \equiv 0$ in the working area $D \subseteq$ the equation (1) takes the following expression $\frac{\hat{d}W}{d\overline{z}} = 0$ and this equation, in the class of the functions $W = u(x,y) + iv(x,y)$ whose real and imaginary parts have unbroken partial derivatives u'_x, u'_y, v'_x and v'_y in D, is a complex writing of the Cauchy - Riemann conditions. In other words it defines the analytic functions in the sense of the classic theory of the analytic functions. In the case $B \equiv 0$ in D is the so cold areolar linear differential equation [3] (page 39-40) and it can be solved with quadratures.

II. MAIN PART AND DISCUSSION

Let's put $z = x + iy, W = u + iv, A(z,\overline{z}) = a(x,y) + ib(x,y)$, in

$$\frac{\partial W}{\partial \overline{z}} = A(z,\overline{z})W \tag{4}$$

and
$$\frac{\partial W}{\partial \overline{z}} = A(z,\overline{z})\overline{W} \tag{5}$$

so, we get the equations

$$\frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = (a+ib)(u+iv) = au - bv + i(bu + av) \tag{4´}$$

$$\frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = (a+ib)(u-iv) \tag{5´}$$

or, if we separate the real from the imaginary part, we get the following corresponding systems:

$$\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 2(au - bv) \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= 2(bu + av) \end{aligned} \tag{6}$$

and

$$\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 2(au + bv) \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= 2(bu - av) \end{aligned} \tag{7}$$

The two systems (6) and (7) are linear and very similar. So, it is expected to have similar difficulties and properties in the process of their solution. But it is not like that at all. The equation (4) can be solved with quadratures , i.e.

$$W_1(z,\overline{z}) = u_1 + iv_1 \equiv C(z)e^{\hat{\int} A(z,\overline{z})d\overline{z}} \tag{8}$$

and the equation (5) cannot be solved with quadratures. If in (8) we separate the real and the imaginary part, we have that

$$u_1 + iv_1 = (\alpha + i\beta)e^{\int(a+ib)(dx-idy)} = (\alpha + i\beta)e^{\int(adx+bdy)}e^{i\int(bdx-ady)} =$$

$$= e^{\int(adx+bdy)}(\alpha + i\beta)\left(\cos\int(bdx - ady) + i\sin\int(bdx - ady)\right)$$

where from we get that

$$\begin{aligned} u_1 &= e^{\int(adx+bdy)}\left[\alpha\cos\int(bdx - ady) - \beta\sin\int(bdx - ady)\right] \\ v_1 &= e^{\int(adx+bdy)}\left[\beta\cos\int(bdx - ady) + \alpha\sin\int(bdx - ady)\right] \end{aligned} \tag{9}$$

so, we can formulate the following theorem:

Theorem 1: *The Vekua equation (4) or the corresponding system of partial equations (6) has a general solution given with the exact quadratures (9), where* $a = a(x,y), b = b(x,y)$ *are given continuous functions and*

$$\alpha = \alpha(x,y) = \operatorname{Re} C(z),$$

$$\beta = \beta(x,y) = \operatorname{Im} C(z)$$

where $C(z)$ *is an arbitrary analytic function from* z *, i.e.:*

$$\alpha'_x = \beta'_y, \quad \alpha'_y = -\beta'_x$$

For the system (7), which is similar to the system (6) but with different order of the coefficients $a = a(x, y), b = b(x, y)$, a theorem like this does not exists, but we can consider the solutions $u_2 = \mathrm{Re}\,W, v_2 = \mathrm{Im}\,W$ in the form of possible iterations. Since a system of partial equations of first order is equivalent to one equation of second order, the idea is to compare them.

A) Comparison between a quadrature solvable system of partial equations and the zeroes of their solutions using Sturm theory.

Let's consider only the equations (6). We would like to see if in them are hidden ordinary differential equations, continuous on the Ox-axis, for $y = 0$. Then

$$u(x) = u(x,0), \quad v(x) = v(x,0)$$

therefore $\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = 0$ so if we put in the system (6)

$$a(x, y)\Big|_y = a(x,0) = a(x)$$

and $$b(x, y)\Big|_y = b(x,0) = b(x)$$

now, $\frac{\partial u}{\partial x} = \frac{du}{dx}, \quad \frac{\partial v}{\partial x} = \frac{dv}{dx}$ so

$$\frac{du}{dx} = 2\left[a(x)u - b(x)v\right] \tag{10}$$

$$\frac{dv}{dx} = 2\left[b(x)u + a(x)v\right] \tag{11}$$

We can eliminate one of the functions if we differentiate, and we get

$$\frac{d^2u}{dx^2} = 2\left[a'u + a\frac{du}{dx} - b'v - b\frac{dv}{dx}\right] \tag{12}$$

and if we express v from (10)

$$v = \frac{a(x)}{b(x)}u - \frac{1}{2b}\frac{du}{dx}. \tag{13}$$

Substituting (11) and (13) in (12), then we get an ordinary linear differential equation of second order, i.e.

$$\frac{d^2u}{dx^2} = 2a'u + 2a\frac{du}{dx} - 2b'v - 2b\frac{dv}{dx} =$$

$$= 2a'u + 2a\frac{du}{dx} - 2b'\left(\frac{a}{b}u - \frac{1}{2b}\frac{du}{dx}\right) - 2b\cdot 2\left[bu + av\right] =$$

$$= \left[2a + \frac{b'}{b}\right]\frac{du}{dx} + \left[2a' - \frac{2b'}{b}a\right]u - 4b^2u - 4ab\cdot\frac{2au - u'}{2b} =$$

$$= \left[2a + \frac{b'}{b} + 2a\right]\frac{du}{dx} + \left[2a' - \frac{2b'}{b}a - 4b^2 - 4a^2\right]u$$

or $$u'' - \left[4a + \frac{b'}{b}\right]u' - \left[2a' - \frac{2ab'}{b} - 4\left(a^2 + b^2\right)\right]u = 0$$

or $$u'' - \left[\frac{b'}{b} + 4a\right]u' + \left[4\left(a^2 + b^2\right) + \frac{2ab'}{b} - 2a'\right]u = 0. \tag{14}$$

With the substitution $u = e^{-\frac{1}{2}\int\left[\frac{b'}{b} + 4a\right]dx}\cdot z(x)$, the equation (14) can be reduced to a canonical form

$$z'' + \Phi(x)z = 0 \tag{15}$$

where $$\Phi(x) = 4\left(a^2 + b^2\right) + \frac{2ab'}{b} - 2a' - \frac{1}{2}\left[\frac{b'}{b} + 4a\right]' - \frac{1}{4}\left[\frac{b'}{b} + 4a\right]^2 \tag{16}$$

and
$$a(x) = \operatorname{Re} A(z,\bar{z})\big|_{y=0}$$
$$b(x) = \operatorname{Im} A(z,\bar{z})\big|_{y=0}$$

According to the Sturm theory and its theorems, if

1) $\Phi(x) > 0$ on $(0,+\infty)$ and

2) $\Phi(x)$ is big enough to make oscillations, i.e. $\int_0^{+\infty} \Phi(x)\,dx$ to diverge, and the equation (14) by u along the $x-$ axis $(y=0)$ has oscillatory solutions. So, follows

Theorem 2: *The real part of the solution of a quadrature solvable Vekua equation (4) has solutions $u(x,0)$ along the $x-$ axis $(y=0)$, which fulfill an ordinary differential equation (14) which depends on $A(z,\bar{z})\big|_{y=0}$ i.e. from $a(x,0)$ and $b(x,0) \neq 0$. Under the condition the canonical equation (15) to have a positive coefficient (16) along the whole half axis $x \geq 0$ and the integral $\int_0^{+\infty} \Phi(x)\,dx$ to diverge, $u(x,0)$ has an infinite many zeroes along the $x-$ axis.*

This is regarding the zeroes of $u = \operatorname{Re} W$ for $y=0$, i.e. $u(x,0)=0$. But, we need also the common zeroes of $u(x,0)$ and $v(x,0)$.

B) Zeroes of the imaginary part $v(x,0)=0$, for $y=0$.

If now in the system of equations (10)+(11) we eliminate $u(x)$, in a similar fashion, we will get an ordinary differential equation for $v(x)$. If we differentiate the equation (11)

$$\frac{d^2v}{dx^2} = 2\left[b'u + b\frac{du}{dx} + a'v + a\frac{dv}{dx}\right] \tag{17}$$

and if we eliminate $u(x)$ from (11), and $\frac{du}{dx}$ from (10), we get that

$$u = \frac{1}{b}\left[\frac{1}{2}\frac{dv}{dx} - a(x)v\right] \quad \text{and} \quad \frac{du}{dx} = 2\left[a(x)\frac{1}{b}\left[\frac{1}{2}\frac{dv}{dx} - a(x)v\right] - bv\right] = \frac{a}{b}\frac{dv}{dx} - 2\frac{a^2}{b}v - 2bv$$

and if we substitute $u(x)$ and $\frac{du}{dx}$ in (17) we get that:

$$\frac{d^2v}{dx^2} = 2b'\frac{1}{b}\left[\frac{1}{2}\frac{dv}{dx} - av\right] + 2b\left[\frac{a}{b}\frac{dv}{dx} - 2\frac{a^2}{b}v - 2bv\right] + 2a'v + 2a\frac{dv}{dx}$$

$$\frac{d^2v}{dx^2} = \left[\frac{b'}{b} + 2a + 2a\right]\frac{dv}{dx} + \left[-2a\frac{b'}{b} - 4a^2 - 4b^2 + 2a'\right]v$$

or
$$\frac{d^2v}{dx^2} - \left[\frac{b'}{b} + 4a\right]\frac{dv}{dx} + \left[2a\frac{b'}{b} + 4\left(a^2+b^2\right) - 2a'\right]v = 0 \tag{18}$$

If we compare A) and B), i.e. the equations (14) for u and (18) for v, we can see that they are identical equations. Therefore, they have the same fundamental system of integrals, i.e.

$$u(x) = C_1u_1 + C_2u_2$$
$$v(x) = Au_1 + Bu_2 \tag{19}$$

only the chosen particular integral can be different, which depends on the initial conditions. Therefore we have the following

Theorem 3: *The Vekua equation (4) along the Ox-axis for $y=0$, fulfills has the same differential equations both for the real and for the imaginary part.*

That means from $W = u + iv$, where $u = v$ we have $W = u(1+i)$ and for $W = 0$ it is enough that $u = 0$ or $v = 0$. Then the solution is

$$W = u + iv = u(1+i) = v(1+i) = 0\cdot(1+i) = 0$$

i.e. we have only the trivial solution of the linear equation (4). This corresponds to the considered in [5].

We have the same from the quadrature solution (9) : $u_1 = v_1 = 0$ gives that

$$\alpha \cos \int (b dx - a dy) - \beta \sin \int (b dx - a dy) = 0$$

$$\beta \cos \int (b dx - a dy) + \alpha \sin \int (b dx - a dy) = 0$$

where from

$$tg \int (b dx - a dy) = \frac{\alpha (x, y)}{\beta (x, y)}$$

$$tg \int (b dx - a dy) = \frac{\beta (x, y)}{\alpha (x, y)}$$

i.e. $\frac{\alpha}{\beta} = \frac{\beta}{\alpha}$ or $\alpha^2 = \beta^2$.

But α, β are parts of a constant $C(z)$: $C(z) = \alpha + i\beta$ which should be analytic function from z . So,

$$\alpha (x, y) = \pm \beta (x, y)$$

and the Cauchy - Riemann conditions hold, so

$$\alpha'_x = \beta'_y, \quad \alpha'_y = -\beta'_x$$

and along the Ox -axis is $\alpha'_y = 0, \beta'_y = 0$ so we get $\alpha'_x = 0, \beta'_x = 0$ i.e. $\alpha + i\beta = const$, or $C(z) = const$ and the solution is only the trivial, i.e.

$$W_1 = 0 \cdot e^{\hat{\int} A(z, \bar{z}) d\bar{z}} = 0 .$$

III. A COMPARISON BETWEEN THE EXACT ZEROES ONLY OF $u(x,0)$ AND ONLY OF $v(x,0)$ WITH THE APROXIMATE STURM ZEROES

Now, we will consider the formula (9) in order to find the zeroes of the solutions of the equation (4), found with quadratures:

$$\alpha \cos \int (b dx - a dy) - \beta \sin \int (b dx - a dy) = 0$$
$$\beta \cos \int (b dx - a dy) + \alpha \sin \int (b dx - a dy) = 0 \qquad (9)$$

As we consider the discussion only along the x -axis, i.e. for $y = 0$ we have that $dy = 0$ and $\alpha = \alpha(x), \beta = \beta(x), a = a(x), b = b(x)$ so,

$$\alpha (x) \cos \int b(x) dx - \beta (x) \sin \int b(x) dx = 0$$

$$\beta (x) \cos \int b(x) dx + \alpha (x) \sin \int b(x) dx = 0$$

which is possible only for $\alpha = \beta$. It remains on the x -axis to be

$$\cos \int b(x) dx = \sin \int b(x) dx$$

and $\cos \lambda$ is equal to $\sin \lambda$ only in the points $\lambda = \frac{\pi}{4} + k\pi , k = 0,1,2,\cdots$

i.e. $\int b(x) dx = \frac{\pi}{4} + k\pi = const$

and we cannot conclude anything else. But, the ordinary differential equations, which are the same both for u and v , i.e.

$$u'' - \left[\frac{b'}{b} + 4a \right] u' + \left[4\left(a^2 + b^2\right) + \frac{2ab'}{b} - 2a' \right] u = 0 \qquad (14)$$

and whose final form is $z'' + \Phi(x) z = 0$ (15)

where $\Phi(x) = 4\left(a^2 + b^2\right) + \frac{2ab'}{b} - 2a' - \frac{1}{2}\left[\frac{b'}{b} + 4a\right] - \frac{1}{4}\left[\frac{b'}{b} + 4a\right]^2$ (16)

According to the Sturm theorems if

1) $\Phi(x) > 0$

2) $\int_0^{+\infty} \Phi(x)\,dx$ diverges,

then they have Sturm oscillatory solutions

$$z_1 = \cos_{\Phi(x)} x \tag{20}$$

$$z_2 = \sin_{\Phi(x)} x \tag{21}$$

given with the series of iterations

$$z_1 = 1 - \iint \Phi + \iint \Phi \iint \Phi - \iint \Phi \iint \Phi \iint \Phi + \cdots \tag{22}$$

$$z_2 = x - \iint x\Phi(x)\,dx^2 + \iint \Phi\,dx^2 \iint x\Phi\,dx^2 - \cdots \tag{23}$$

where, as we see from (16), $\Phi(x)$ depends from two continuous functions $a(x,y)$ and $b(x,y)$, therefore the function $\Phi(x)$ itself is continuous $\Phi = \Phi(a,b)$.

IV. CONCLUSION

With (20), (21) and (16) one trigonometry of second order is determined, $T_y^{II}\{a(x)\}$, whose base depends on two functions a and b for which the conditions 1) and 2) hold. The zeroes of the functions $\cos_{\Phi(x)} x$ and $\sin_{\Phi(x)} x$ are in the solutions of the equations

$$x\sqrt{\Phi(x)} = n\pi,\ n = 0,1,2,\cdots \tag{24}$$

and $$x\sqrt{\Phi(x)} = (2k-1)\frac{\pi}{2},\ k = 1,2,\cdots \tag{25}$$

and the sinusoidal solution $\sin_\Phi x \approx \frac{\sin\left(x\sqrt{\Phi}\right)}{\sqrt{\Phi}} \to 0,\quad \Phi \to \infty$, and the cosinusoidal solution $\cos_\Phi x \approx \cos\left(x\sqrt{\Phi}\right)$ remains limited.

But, as the equations (24) and (25) never overlap, because they have same left hand side, and always different right hand side, i.e. $n\pi$ is a full multiple of π, and $(2k-1)\frac{\pi}{2}$ is an odd multiple of $\frac{\pi}{2}$, and it is never whole. Therefore, the zeroes only of u and only of v are always different. We can formulate the following

Theorem 4: *Only the real part $u(x,y)$ of the solution W of the equation*

$$\frac{\partial W}{\partial \bar{z}} = A(z,\bar{z})W$$

and only the imaginary part $v(x,y)$ of the solution W can have infinite number of zeroes along the x-axis, but neither of them overlap, so that there are not common zeroes, except the trivial solution $u = v = 0$.

V. EXAMPLES

Example 1: Let's consider the formulas (14) and (18) on one elementary example, the equation

$$\frac{\partial W}{\partial \bar{z}} = (3z + 2\bar{z})W$$

where $A(z,\bar{z}) = 3z + 2\bar{z} = 2(z+\bar{z}) + z = 2\cdot 2x + z = 4x + z$. The solution can be found with quadratures, i.e.

$$W(z,\bar{z}) = C(z)e^{\int^{a}(3z+2\bar{z})d\bar{z}} = C(z)e^{3z\bar{z}+2\frac{\bar{z}^2}{2}} = C(z)e^{3z\bar{z}+\bar{z}^2} =$$

$$= C(z)e^{3\left(x^2+y^2\right)}\cdot e^{\left(x^2-y^2-2ixy\right)} = C(z)e^{4x^2+2y^2}\cdot e^{-2ixy} = C(z)e^{4x^2+2y^2}\cdot\left[\cos(2xy) - i\sin(2xy)\right]$$

and since $\cos(2xy)$ and $\sin(2xy)$ are never equal to zero at the same time, except the trivial solution $W = 0$, other zeroes of the solution may appear only in isolated zeroes of the analytic coefficient $C(z)$.

Let's see now the differential equation for $u(x,0)$ and $v(x,0)$ along the Ox-axis. That is the equation (one same equation for both of them):

$$u'' - \left(\frac{b'}{b} + 4a\right)u' + \left[4\left(a^2+b^2\right) + 2a\frac{b'}{b} - 2a'\right]u = 0$$

where $A(z,\overline{z}) = a + ib = 3z + 2\overline{z} = 3(x+iy) + 2(x-iy) = 5x + iy$

$$a(x,y) = 5x, \quad b(x,y) = y \quad \Rightarrow \quad a(x,0) = 5x, \quad b(x,0) = 0 \quad \Rightarrow \quad a' = 5, \quad b' = 0$$

and since $b \equiv 0, \quad b' = 0$ along the Ox -axis $(y = 0)$, that is why the equation is as follows

$$bu'' - (b' + 4ab)u' + b \cdot 4\left(a^2 + b^2 + 2ab' - 2a'b\right)u = 0$$

and it does not give a result for $b = 0$. But, we have a quadrature solution.

Example 2: Let's take into consideration not so elementary case where $b'(x) \neq 0$. Let

$$b(x,y) = e^x + y, a(x,y) = x \qquad \Rightarrow \quad b(x,0) = e^x \quad ,b'(x) = e^x, \quad a'(x) = 1$$

that means we have chosen $A(z,\overline{z}) = x + i\left(e^x + y\right) = a + ib = \frac{z+\overline{z}}{2} + i\left(e^{\frac{z+\overline{z}}{2}} + \frac{z-\overline{z}}{2i}\right)$, so we have a Vekua equation

$$\frac{\partial W}{\partial \overline{z}} = \left[\left(\frac{z+\overline{z}}{2}\right) + ie^{\frac{z+\overline{z}}{2}} + \frac{z-\overline{z}}{2}\right]W = \left[z + ie^{\frac{z+\overline{z}}{2}}\right]W$$

and the equation (14) is

$$u'' - \left(\frac{e^x}{e^x} + 4x\right)u' + \left[4\left(x^2 + e^{2x}\right) + 2x\frac{e^x}{e^x} - 2 \cdot 1\right] = 0$$

or $$u'' - (1 + 4x)u' + \left[4\left(x^2 + e^{2x}\right) + 2x(x - i)\right]u = 0$$

which may have Sturm zeros. The final form $z'' + \Phi(x)z = 0$ has a coefficient

$$\Phi(x) = 4\left(x^2 + e^{2x}\right) + 2(x-1) + \frac{1}{2}(1+4x)' - \frac{1}{4}(1+4x)^2 =$$

$$= 4x^2 + 4e^{2x} + 2x - 2 + \frac{1}{2} - \frac{1}{4}\left(1 + 4x + 16x^2\right) = 4e^{2x} + x - 2 + \frac{1}{4} = 4e^{2x} + x - \frac{9}{4} > 0$$

for $x > 0$. Therefore, $\Phi(x) > 0$, and the equation $z'' + \left(4e^{2x} + x - \frac{9}{4}\right)z = 0$ has oscillatory solutions

$$z_1 = \cos_\phi x$$

$$z_2 = \sin_\phi x$$

and from the substitution

$$u = e^{-\frac{1}{2}\int(1+ux)dx} \cdot z = e^{-\frac{x}{2}-x^2} \cdot z$$

we have solutions for u

$$u_1 = e^{-x^2-\frac{x}{2}} \cdot \cos_\phi x$$

$$u_2 = e^{-x^2-\frac{x}{2}} \cdot \sin_\phi x$$

in the form of Sturm functions.

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