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A Novel Application of Cubic Equation

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ABSTRACT
A glass of water is poured into a bowl which is a spherical cap. The process of finding the depth of water in the spherical cap evolves a cubic equation amenable to its solution in closed form. Given the volume of water in a bucket of shape as a truncated portion of a right circular cone, a cubic equation is encountered. Cubic equations are also tackledled in some algebraic problems that are solved with insight and subtlety.

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I. INTRODUCTION

Many Mathematicians[2] discussed theory of cubic equations and biquadratic equations in textbooks of Algebra and in articles published by them.Herein is presented formation of a cubic equation in realistic sense and thereafter a popular solution to the equation.A cubic equation with variable x are of the following forms:

x^3+p=0 x^3+px+q=0, x^3+rx^2+px+q=0, x^3+rx^2+x =0 which with constants p,q,r have at least one real root. Sivaraman[1] in his article incorporated history of equations including cubic equations. The above third and fourth equations by proper substitution can be converted to the above second form. In this article we shall encounter a cubic equation of the fourth form amenable to its solution. We also innovate cubic equations in course of dealing with complicated problems of projectile motion/ ballistics and general mensuration.Zucker[2] (2008) dwelled on cubic equation at the irreducible case. Some cubic equations are derived out of some problems of algebra and are solved.

VOLUME OF SPHERICAL CAP SLICED FROM A H0LLOW SPHERE.

Let a liquid be placed in a spherical cap from a cylindrical glass of radius a and height H and the depth of the liquid thus collected in the cap sliced from a sphere of radius R be h. To find the volume of the liquid thus contained in it let us consider a circular space of radius r and thickness dx occupied by the liquid, at a distance x from the lowest point of the spherical cap.

Then by geometry, one gets

$r^2 = R^2 - (R-x)^2$=2Rx$-x^2$ (1)

Elementary volume of the liquid contained in the above circular space is

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dv=$\pi r^2 dx$ ==(2Rx$-x^2)dx$ (2)

integrating which from x=0 to x=h, is obtained the volume of the liquid in the cap:

$V=\int_0^h \pi(2Rx - x^2)dx \ = \pi\left(Rh^2 - \frac{h^3}{3}\right)$ (3)

Since this liquid is poured from the cylinder, $\pi\left(Rh^2 - \frac{h^3}{3}\right) = \pi a^2 H$

Or, $-h^3 + 3Rh^2 - 3a^2H = 0$ (4)

which is a cubic equation in h and can be solved as follows:

Equation (4) can be rewritten as

$(R-h)^3 - R^3 + 3R^2h - 3a^2H = 0$

In order to solve this cubic equation let us put y=R−h and adopt Cardan's method so that

$y^3 - 3R^2y + 2R^3 - 3a^2H = 0$

With $2R^3 - 3a^2H = -b$ (5)

$y^3 - 3R^2y - b = 0$ (6)

Let $y=m^{\frac{1}{3}} + n^{\frac{1}{3}}$ (7)

Or, $y^3 - 3m^{\frac{1}{3}}n^{\frac{1}{3}}y - (m+n) = 0$ (8)

Comparing equation (8) with (6) we get

$m^{\frac{1}{3}}n^{\frac{1}{3}} = R^2$ and m+n=b (9)

$(m-n)^2 = (m+n)^2 - 4mn = b^2 - 4R^6$

$m-n=\sqrt{b^2-4R^6}$ m>n (10)

$m=\frac{b+\sqrt{b^2-4R^6}}{2}$, $n=\frac{b-\sqrt{b^2-4R^6}}{2}$ (11)

Hence in view of (7), we get

$y=R-h=(\frac{b+\sqrt{b^2-4R^6}}{2})^{\frac{1}{3}}+(\frac{b-\sqrt{b^2-4R^6}}{2})^{\frac{1}{3}}$

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Or, $h=R-\{(\frac{b+\sqrt{b^2-4R^6}}{2})^{\frac{1}{3}}+\frac{b-\sqrt{b^2-4R^6}}{2})^{\frac{1}{3}}\}$ (12)

which gives the depth of the liquid in the hemispherical bowl of radius R and the height of the empty space in it is obviously

$k=(\frac{b+\sqrt{b^2-4R^6}}{2})^{\frac{1}{3}}+\frac{b-\sqrt{b^2-4R^6}}{2})^{\frac{1}{3}}$ (13)

Now let us tackle cubic equation in order to find solution to a projectile motion.

FORMATION OF THE PROJECTILE PROBLEM

An enemy ship is sailing with a uniform speed v in a straight line path parallel to the bank of a river and from a gun positioned on the opposite bank canons are fired at the ship. As soon as the line of sight of the gunman with the ship makes an angle $\theta<90^0$ with the straight line path of the ship while the distance the between the gun man and the ship is a, ie the length of the line of sight , a canon is fired with a velocity u at an angle α to the horizontal and hits the ship. Find the range and the time of flight of the shot.

EQUATIONS OF TIME OF FLIGHT

Let h_1 be height of the ship and t the time of flight ie time taken to hit the ship. By dynamics and geometry, we get the range R_1 :

$R_1^2=(a\cos\theta+vt)^2+(a\sin\theta)^2=a^2+2avt\cos\theta+(vt)^2$ (14)

$h_1=(u\sin\alpha)t-\frac{1}{2}gt^2$ (15)

Owing to projectile motion, the range is also given by

$(u\cos\alpha)t=R_1$ (16)

Eliminating R_1,α among the above three equations, we have

$a^2+2avt\cos\theta+(vt)^2+(h_1+\frac{1}{2}gt^2)^2=(ut)^2$

Or $(\frac{1}{2}g)^2t^4+(gh_1-u^2+v^2)t^2\ +2avt\cos\theta+a^2+h_1^2\ =0$

which can be rewritten as

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$t^4+2at^2+2bt+c=0$ (17)

where $\frac{gh_1-u^2+v^2}{(\frac{1}{2}g)^2}=2a>0$, $\frac{2av\cos\theta}{(\frac{1}{2}g)^2}=2b>0$, $\frac{a^2+h_1^2}{(\frac{1}{2}g)^2}=c>0$ (18)

Equation (17) can be adjusted in the form:

$t^4+2a_1t^2+a_1^2-2(a_1-a)\{t^2-\frac{bt}{(a_1-a)}+(\frac{b}{2(a_1-a)})^2\}+c-a_1^2+\frac{b^2}{2(a_1-a)}=0$ (19)

$(t^2+a_1)^2-2(a_1-a)\{t-\frac{b}{2(a_1-a)}\}^2+\ c-a_1^2+\frac{b^2}{2(a_1-a)}=0$ (20)

We have incorporated a_1 in (17) so as to acquire equation (20),which calls for determining a_1 in terms of the other known constants appearing in (20) and as such

$(t^2+a_1)^2=2(a_1-a)\{t-\frac{b}{2(a_1-a)}\}^2$ (21)

and $c-a_1^2+\frac{b^2}{2(a_1-a)}=0$ (22)

Equation (22) can be turned into the form

$2c(a_1-a)-2a_1^2(a_1-a)+b^2=0$

$a_1^3-aa_1^2-ca_1+ca-b^2/2=0$ (23)

which is a cubic equation in $a_1$and can be put in simpler form to solve it by putting in (23)

$a_1=x-\frac{a}{3}$ (24)

$(x+\frac{a}{3})^3-a(x+\frac{a}{3})^2-c\left(x+\frac{a}{3}\right)+ca-\ b^2/2=0$

Or,$x^3-\frac{(3a^2+9c)x}{9}+(\frac{a^3}{27}+\frac{2}{3}ca-\frac{b^2}{2})=0$ (25)

which can be solved in the same line as in the earlier text,equations (6) to (11). Solving (21) is obtained the time of flight :

$t^2 + a_1 = \pm\sqrt{2(a_1 - a)}\ \{\ t - \frac{b}{2(a_1-a)}\}$ $\quad a_1 > a \quad$ (26)

Or, $t^2 - \sqrt{2(a_1 - a)}t + a_1 + \frac{b}{\sqrt{2(a_1-a)}} = 0$

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Or, $t = \frac{\sqrt{2(a_1-a)} \pm \sqrt{2(a_1-a) - 4\{a_1 + \frac{b}{\sqrt{2(a_1-a)}}\}}}{2} > 0$ (27)

which gives two positive values of t as two times of hitting the moving target by using the given and evaluated values of the constants involved. The other two values of t from (26) are are found to be negative,because from (22) $a_1 > \frac{b^2}{2(a_1-a)}$. Thus we have come across a cubic equation while exploring a complicated realistic projectile motion striking a target. However, employing this value of t in (14) and (15) we are able to find the range and angle of projection. In view of the constant height of the target and its time- varying distance from the firing position, if the velocity u projection is chosen sufficiently high then angle of projection has to be determined for success of the mission and vice versa. In the preceding design we have taken into consideration the height of the ship only to pose a cubic equation. This is somewhat redundant , because height of this target is insignificant in comparison to its distance from the firing position. Nonetheless, with the zero height of the target we cannot get rid of the cubic equation in the process of solution to the problem. Hence the above analysis can be amended in practice by neglecting the height of the ship ie by putting h_1=o. It is observed that neglecting height of the target(h_1) and taking $\theta = 0$ in the equation (14) of the range ie when the line of sight of target at the time of firing is perpendicular to the path of the moving target, we can rule out involvement of any cubic equation, bringing forth simple tractable bi-quadratic equation convertible to a quadratic equation. In this situation the relevant equation of motion is presented as

$a^2 + +(vt)^2 + (\frac{1}{2}gt^2)^2 = (ut)^2$ (27)

Or, $(\frac{1}{2}gt^2)^2 - (u^2 - v^2)t^2 + a^2 = 0$

$t^2 = 2\{\frac{(u^2-v^2) \pm \sqrt{(u^2-v^2)^2 - a^2g^2}}{g^2}\}$

Or, $t = \sqrt{2\{\frac{(u^2-v^2) \pm \sqrt{(u^2-v^2)^2 - a^2g^2}}{g^2}\}}$ $\quad (u^2 - v^2)^2 > a^2g^2$ and u>v (28)

which can be equated to the time of flight to find the angle of projection α:

$\sin\alpha = \sqrt{\frac{(u^2-v^2) \pm \sqrt{(u^2-v^2)^2 - a^2g^2}}{2u^2}}$ (29)

Equation (29) can be recast to obtain projection velocity u as an explicit function of sin α and reveales that choosing the same angle α of projection four different values of

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projection velocity u correspond to four different times of flight to successfully hit the moving target. Obviously the line of sight of the moving target ,when hit, makes an angle β with the initial line of sight is given by

$\tan\beta = \frac{vt}{a} = v\ \sqrt{\frac{(u^2-v^2) \pm \sqrt{(u^2-v^2)^2 - 4a^2g^2}}{a^2g^2}}$ (30)

ANOTHER EXAMPLE OF CUBIC EQUATION IN MENSURATION

A bucket , made of truncated portion of a right circular cone of radius r, is filled with water of depth h_2 .This bucketful of water is poured into a rectangular tank of length a and breadth b. The depth of water in the tank is measured as c. Height of the cone that is to be calculated gives a cubic equation: If h be the height of the original cone, the volume of water in the bucket satisfies the equation

$\frac{1}{3}\pi r^2 h - \frac{1}{3}\pi(h - h_2)r^2\frac{(h-h_2)^2}{h^2} = abc$

Or, $\frac{1}{3}\pi r^2\frac{h^3-(h-h_2)^3}{h^2} = abc$

Or, $(h_2^3 - 3hh_2^2 + 3h^2h_2) = abc\frac{3h^2}{\pi r^2}$

Or, $3h^2\left(h_2 - \frac{abc}{\pi r^2}\right) - 3hh_2^2 + h_2^3 = 0$ (31)

Which is a quadratic equation in 'h',ie, knowing the depth of the bucket the height h of the original cone can be determined by solving this equation. In other words,(31) is a a simple cubic equation in 'h_2'.Observing previous lines of (31) is obtained

$(h - h_2)^3 = h^2(h - abc\frac{3}{\pi r^2})$

Or, $h_2 = h - \sqrt[3]{h^2(h - abc\frac{3}{\pi r^2})}$ (32)

In other words, the problem can be projected: The bucket is put under a tap connected to right circular or rectangular- shape water tank and is filled with water by opening the tap.The bucketful of water is related to fall of water level in the water tank.

CUBIC EQUATION IN A PROBLEM OF KINEMATICS

A train is to travel from station A to station B, distance S apart. The train leaves station A from rest with an acceleration proportional to time. After reaching a position C,it moves for tme t_1.

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with the uniform speed that it acquires there. Thereafter the train goes with a retardation to come to rest at station B,taking time t_2.Find the distance between the two stations.

If v is the velocity of the train and s the distance described at any instant of time t before reaching the position C,

$\frac{d^2s}{dt^2} = \frac{dv}{dt} = 12kt$

$\frac{ds}{dt} = v$ (33)

where 12k is the constant of proportionality.

Solving this differential equation with the initial conditions: Initially at t=0, s=0 and v=o, we get

$v = 6kt^2$ and $s = 2kt^3$ (34)

If v_1 and s_1 be the velocity and distance covered by the train at time T, relation (34) gives

$v_1 = 6kT^2$ and $s_1 = =2kT^3$ (35)

The distance s_2 traversed with uniform velocity there after in time t_1 is given by

$s_2 = 6kT^2t_1$ (36)

With initial velocity v_1 obtained from (35), and final velocty 0 in time t_2, the distance s_3 covered in time t_2, is given by

$s_3 = \frac{v_1^2}{2f}$ (37)

where the deceleration f to stop the train after describing the above distance is to be eliminated with given time t_2 and hence,

$0 = v_1 - ft_2$ (38)

Thus eliminating v_1 and f among (35), (37) and (38) is obtained s_3 as a function of T:

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$s_3 = \frac{v_1^2}{2f} = \frac{v_1 . v_1}{2f} = \frac{6kT^2 ft_2}{2f} = 3kT^2t_2$ (39)

Adding (35), (36) and (39), we get the total distance described, ie, the distance S between the two stations as

$S = 2kT^3 + 6kT^2t_1 + 3kT^2t_2$

Or, $T^3 + 3T^2(2t_1 + t_2)/2 - \frac{S}{2k} = 0$

Let us put $(2t_1 + t_2)\frac{3}{2} = p$ (40)

$T^3 + 3pT^2 - \frac{S}{2k} = 0$

Substituting $y = T + P$ (40.1)

in (40) resulting in

$(y - p)^3 + 3p(y - p)^2 - \frac{S}{2k} = 0$

leading to a standard cubic equation as encountered in the preceding design:

$y^3 - 3yp^2 + (2p^3 - \frac{S}{2k}) = 0$ (41)

which can be solved to get a real root, ultimately value of T.

Suppose $y = q^{\frac{1}{3}} + r^{\frac{1}{3}}$ (42)

Or, $y^3 - 3y(qr)^{\frac{1}{3}} - (q + r) = 0$ (43)

Comparing (43) with (41), one gets the relationships

$qr = p^6$ and $(q + r) = \frac{S}{2k} - 2p^3$ (44)

which ratifies that q and r are the roots of the quadratic equation

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$x^2 - \left(\frac{S}{2k} - 2p^3\right)x + p^6 = 0$ (45)

Therefore, in consequence of relationships (40.1),(42) and (46),

$$T=\left(\frac{\left(\frac{S}{2k}-2p^3\right)+\sqrt{\left(\frac{S}{2k}-2p^3\right)^2-4p^6}}{2}\right)^{\frac{1}{3}}+\left(\frac{\left(\frac{S}{2k}-2p^3\right)-\sqrt{\left(\frac{S}{2k}-2p^3\right)^2-4p^6}}{2}\right)^{\frac{1}{3}}-p \quad - \qquad (46)$$

where p is determined by (40).

EXAMPLE OF CUBIC EQUATION IN ANOTHER PROBLEM OF KINEMATICS

The preceding problem of this type is modified as follows:The train traveling from station A from rest with uniform acceleration f reaches a position C after a time T and thereafter moves for a time t_1 with the uniform velocity acquired at C .At the end of time t_1 the train exerts a brake effecting a uniform retardation and comes to rest at station B in time $t_{2.}$ Find the distance between the two stations.

The distance traveled by the train reaching position C followed by the uniform velocity gained for the given time is

$$s_1 + s_2 = \frac{1}{2} fT^2 + fTt_1 \qquad (47)$$

With initial velocity fT the train afterwards moves with unniform retardation, say, F and comes to rest at station B in time t_2 covering the distance s_3 such

that $s_3 = \frac{(fT)^2}{2F} = \frac{fT.Ft_2}{2F}$

Or, $s_3 = \frac{fT.t_2}{2}$ (48)

Adding (47) and (48) is found the total distance described by the train ie the distance between the two stations:

$$S=\frac{1}{2} fT^2 + fTt_1 + \frac{fT.t_2}{2} \qquad (49)$$

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which is quadratic equation in 'T' .But if the brake is applied generating time-varying deceleration, cubic equation in 'T' occurs:

In that case if s is the distance described and v the velocity attained at any instant of time t reckoning the initial velocity FT, (k=constant)

$$v=\frac{ds}{dt}; \qquad \frac{d^2s}{dt^2} = \frac{dv}{dt} = -kt \qquad (50)$$

Solution to equation (50) yields

$$v=\frac{ds}{dt} = Ft - \frac{kt^2}{2} \qquad (51)$$

$$s= F.\frac{t^2}{2} - \frac{kt^3}{6} \quad . \qquad (52)$$

If T is the time taken to cover the distance $s_{3,}$ then in consequence of (52),

$$s_3 = F.\frac{T^2}{2} - \frac{kT^3}{6} \qquad (52)$$

CUBIC EQUATION POSED IN AN ALGEBRAIC PROBLEM

Cubic equation in a different form appears in course of solving two unusual simultaneous equations

$$x^2 + y^2 = a, \qquad x^3 + y^3 = b \qquad (53)$$

Cubing the first of (53) and squaring the second of the same, is obtained

$$(x^2 + y^2)^3 = x^6 + y^6 + 3ax^2y^2 = a^3 \qquad (54)$$

$$(x^3 + y^3)^2 = x^6 + y^6 + 2x^3y^3 = b^2 \qquad (55)$$

Subtracting (54) from (55) and replacing xy by z we get a cubic equation

$$2x^3y^3 - 3ax^2y^2 = b^2 - a^3 \qquad b>a$$

Or, $2z^3 - 3az^2 - (b^2 - a^3) = 0$ (56)

With insight a and b have numerical values. a=b leads to

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$$z^2(2z-3a) - a^2(1-a) = 0 \qquad (57)$$

We can solve (57) coupled with (53) adopting methods utilized in the previous text.Now let us assign some numerical values to a and b in equation(53) to find integer solutions so that

$$x^2 + y^2 = 13 \qquad (58)$$

$$x^3 + y^3 = 35 \qquad (59)$$

In the light of equations (56),(58) and (59) is obtained a cubic equation

$$2z^3 - 39z^2 + 972 = 0 \qquad (60)$$

We can look for a trial solution to equation (60) by putting z=1,2,3, 4,5,6 and see that the equation is satisfied by z=6.In order to find the other real roots, if any, we rewrite the equation:

$$2z^3 - 39z^2 + 972 = 2z^2(z-6) - 27z(z-6) - 162(z-6)$$

Or,$(2\,z^2 - 27z - 162)(z - 6)$ (61)

$2\,z^2 - 27z - 162 = 0$ (62)

0r, z=$\frac{27\pm\sqrt{27^2+8\times27\times6}}{4} = \frac{27\pm\sqrt{27(27+8\times6)}}{4} = \frac{27\pm\sqrt{27\times75}}{4}$

$z = 18$ or $-\frac{9}{2}$ xy=18 or$-\frac{9}{2}$ (63)

which , combined with (58) yield imaginary values of x and y

On the otherhand from (61), z=xy=6 (64)

which because of (58) leads to x=3and y=2 or vice-versa.

Let us take a comparatively simpler version of the above problem to create a cubic equation:

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xy =6 (65)

$x^3 + y^3 = 35$ (66)

which can be combined as

$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

Or, putting x+y=a (67)

one gets an as usual cubic equation

$a^3 - 18a - 35 = 0$ (68)

which can be solved by above (m,n) method or by trial method if it admits of integer root. So as to seek for integer solution, (68) is satisfied by a=5, whereby (68) implies

$a^2(a - 5) + 5a(a - 5) + 7(a - 5) = 0$

Or, $(a - 5)(a^2 + 5a + 7)$ =0 (69)

From (69), $(a^2 + 5a + 7)$ =0 gives two imaginary roots

a=$\frac{-5\pm\sqrt{-3}}{2}$ and one real root a=x+y=6 (70)

This is combined with (65) to find x=3 or2 and y=2 or 3.

For the sake of exhibiting cubic equation either in x or in y we can consider the two equations as

$x^2 + y = a$ and xy=b (71)

Next in closing, let us introduce cubic equation in "Arithmetic Progression".

We know the sum of the squares of n natural numbers is ge

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S=$\frac{n(n+1)(2n+1)}{6}$ (72)

If S is given,(72) suggests a cubic equation in "n".Now we put n=6 to find the value of 6S.Then we can have the relevant cubic equation from (72)

$$2n^3 + 3n^2 + n = 546 \qquad (73)$$

Now the task arises how to find the value of n by solving equation (73).Though it is a tenth/ eleventh standard problem, we shall solve it at first by trial and factorization method and thereafter by comparison method as utilized above.(73) is arranged as

$$2n^2(\mathrm{n}-6) + 15n(\mathrm{n}-6) + 91(\mathrm{n}-6) = 0$$

$$(\mathrm{n}-6)\ (2n^2 + 15n + 91) = 0$$

Or, $(\mathrm{n}-6) = 0$ or, $n^2 + 15n + 91 = 0$

Or, n=6 and from the other quadratic equation we get

n=$\frac{-15\pm\sqrt{225-364}}{2} = \frac{-15\pm\sqrt{-139}}{2}$, which is an imaginary root.

Equation (73) is converted to standard form eliminating n^2 by substituting

n=m-- $\frac{1}{2}$: (74)

$$2(\mathrm{m}-\tfrac{1}{2})^3 + 3(\mathrm{m}-\tfrac{1}{2})^2 + (\mathrm{m}-\tfrac{1}{2}) = 546$$

Or, $\mathrm{m}^3 - m/4 - 273 = 0$ (75)

For the sake of obtaining its solution in comparison method,let us put

$$\mathrm{m} = \mathrm{p}^{\frac{1}{3}} + \mathrm{q}^{\frac{1}{3}} \qquad (76)$$

Cubing which one gets

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$$\mathrm{m}^3 - (p+q) - 3\mathrm{p}^{\frac{1}{3}}\mathrm{q}^{\frac{1}{3}}\,\mathrm{m} = 0 \qquad (77)$$

Comparing (77) with (75) one gets

$$3\mathrm{p}^{\frac{1}{3}}\mathrm{q}^{\frac{1}{3}} = \frac{1}{4} \text{ and } \mathrm{p}+\mathrm{q} = 273 \qquad (78)$$

$$(p-q)^2 = (p+q)^2 - 4pq = 273^2 - \frac{1}{27\times 16} \qquad (79)$$

Combining (78) and (79) is obtained

p=$\frac{273+\sqrt{273^2-\frac{1}{27\times16}}}{2}$ and q=$\frac{273-\sqrt{273^2-\frac{1}{27\times16}}}{2}$ or vice- versa (80)

0r, $\mathrm{m} = \left(\frac{273+\sqrt{273^2-\frac{1}{27\times16}}}{2}\right)^{\frac{1}{3}} + \left(\frac{273-\sqrt{273^2-\frac{1}{27\times16}}}{2}\right)^{\frac{1}{3}}$ (81)

which is supposed to reduce to m=13/2 ie n=6

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