International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321- 4767 P-ISSN: 2321-4759
www.ijmsi.org Volume 9 Issue 1 || January, 2021 || PP 29-34

Compound Distribution of Poisson-Weibull

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ABSTRACT
In this paper an attempt is made to present a new compound distribution with Poisson and Weibull distributions for studying the consumer behavior. Its properties like, mean, variance, moments, skewness, kurtosis, characteristic function, parameters are derived. It can be used for studying the consumer behavioral distribution for varying.
KEYWORDS: *Poisson, Weibull, Geometric, Hyper Geometric, Consumer Behavior.*

Date of Submission: 21-02-2021 Date of Acceptance: 04-03-2021

I. INTRODUCTION

A few distributions are used to study the behavior of consumers in marketing research analysis. Specifically, it can be used to study the behavior of repeated purchase. The Poisson–Weibull distribution has tremendous applications in socio–economical aspects like marketing science, financial engineering and business administration etc. We are derived a new life distribution, and also its particular nature of the form to estimate the repeated purchases of consumers.

Definition: A random variable X is said to follow Poisson distribution with parameter λ, if it possesses the probability law

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!};\ \text{where } x = 0, 1, 2, \ \ldots$$

Definition: A random variable X is said to follow Weibull distribution with parameter λ, if it possesses the probability law

$$P(X;a,c) = \left[\frac{c}{a}\right]\left[\frac{x}{a}\right]^{c-1}.\exp\left\{-\left[\frac{x}{a}\right]^{c}\right\};\ x > 0$$

Definition: A random variable X is said to follow Poisson – Weibull distribution with parameters (a, c), if it possesses the probability law

$$P(x) = \frac{a^{x}}{x!}A_p;\ \text{Where } A_p = \sum_{p=0}^{\infty}\frac{(-a)^p}{p!}\Gamma\left[\frac{x+p}{c}+1\right]$$

II. POISSON–WEIBULL COMPOUND DISTRIBUTION

The parameter '*c'* is shape (or slope) and '*a'* is the scale (or characteristic life) parameter. In general, for varying values of the shape parameter may have some marked insights into the behavior of the distribution i.e., the shape parameter is a dimension less that result the distribution equations to reduce to the other distributions.

i) As c < 1 indicates that the purchase rate decreases over the time. This happens if the consumers are showing less attention towards the product i.e., the consumers are less likely to purchase.

ii) As c = 1 indicates that the purchase rate is constant over a time i.e., the consumers are showing their interest to purchase.

iii) As c > 1 indicates that the purchasing rate is increases with the time i.e., the consumers are more likely to purchase.

Since, the scale parameter dictates when in time, a given proportion of the population will purchase at a given time period.

Note: When c = 1, the Weibull distribution leads to exponential and *1/a* = η can be viewed as purchase rate.

III. PROPERTIES OF POISSON – WEIBULL DISTRIBUTION

The properties of the compound distribution of Poisson-Weibull are presented below.

1. The characteristic function of Poisson-Weibull distribution $\phi_X(t)$ is

$$\phi_x(t) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(-a)^r (r/c)! \binom{r}{n} e^{n(it)}}{r!}; \quad r, n \geq 0$$

2. The moment generating function of Poisson-Weibull distribution $M_x(t)$ is

$$M_x(t) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(-a)^r (r/c)! \binom{r}{n} e^{nt}}{r!}; \quad r, n \geq 0$$

3. The r^th^ non central moment of Poisson-Weibull $\mu_r{}'$ is

$$E(x^r) = \sum_{r=0}^{r} (-1)^{r+1} \frac{a^r (r/c)!}{(r-1)!} \left\{ 1 - \binom{r-1}{1} 2^{r-1} + \binom{r-1}{2} 3^{r-1} - \binom{r-1}{31} 4^{r-1} + \ldots \right\}$$

4. The first four central moments can be evaluated as

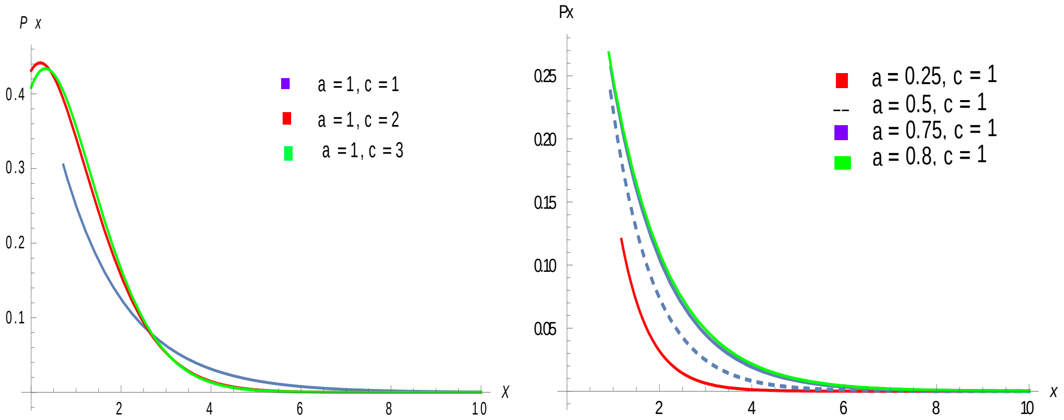
$$\mu_1' = a\left(\frac{1}{c}\right)!$$

$$\mu_2 = a^2\left(\frac{2}{c}\right)! + a\left(\frac{1}{c}\right)!\left\{1 - a\left(\frac{1}{c}\right)!\right\}$$

$$\mu_3 = a^3\left\{\left(\frac{3}{c}\right)! - 3\left(\frac{2}{c}\right)!\left(\frac{1}{c}\right)! + 2\left[\left(\frac{1}{c}\right)!\right]^3\right\} + 3a^2\left\{\left(\frac{2}{c}\right)! - \left[\left(\frac{1}{c}\right)!\right]^2\right\} + a\left(\frac{1}{c}\right)!$$

$$\mu_4 = a^4\left\{\left(\frac{4}{c}\right)! - 4\left(\frac{3}{c}\right)!\left(\frac{1}{c}\right)! + 6\left(\frac{2}{c}\right)!\left[\left(\frac{1}{c}\right)!\right]^2 - 3\left[\left(\frac{1}{c}\right)!\right]^4\right\} + 6a^3\left\{\left(\frac{3}{c}\right)! - 2\left(\frac{2}{c}\right)!\left(\frac{1}{c}\right)! + \left[\left(\frac{1}{c}\right)!\right]^3\right\} + a^2\left\{7\left(\frac{2}{c}\right)! - 4\left[\left(\frac{1}{c}\right)!\right]^2\right\} + a\left(\frac{1}{c}\right)!$$

5. The probability plot of Poisson – Weibull distribution for varying parameters is given by



i) density function for varying 'c' ii) density function for varying '*a*'

Fig. 1 (i) & (ii) Poisson – Weibull distribution curvature for varying parameters

Note: The moments of Poisson-Weibull distribution its behavior for various parameters and its asymptotic nature can be observed.

Table: Moments of Poisson - Weibull distribution for varying parameters

c	a	μ_1'	μ_2	μ_3	μ_4	β_1	β_2
1	1	1	2	6	40	4.5	10
	2	2	6	30	330	4.166	9.166
	3	3	12	84	1308	4.083	9.083
	4	4	20	180	3620	4.05	9.05
2	1	0.886	1.1	1.592	10.112	1.901	8.34
	2	1.77	2.63	4.849	31.738	1.291	4.585
	3	2.658	4.59	10.146	90.465	1.064	4.293
	4	3.544	6.978	17.861	200.667	0.938	4.12
3	1	0.892	0.998	1.214	4.651	1.483	4.667
	2	1.785	2.207	3.095	19.578	0.891	4.018
	3	2.678	3.626	5.678	49.459	0.675	3.759
	4	3.571	5.257	8.995	99.702	0.556	3.607
4	1	0.906	0.971	1.098	4.178	1.318	4.431
	2	1.812	2.071	2.577	16.41	0.747	3.824
	3	2.719	3.301	4.426	39.166	0.544	3.59
	4	3.625	4.66	6.638	75.194	0.435	3.462
5	1	0.918	0.962	1.048	3.992	1.233	4.31
	2	1.836	2.013	2.348	15.117	0.675	3.729
	3	2.754	3.153	3.884	34.956	0.481	3.517
	4	3.672	4.381	6.351	65.933	0.378	3.428
10	1	0.951	0.96	0.993	3.827	1.099	4.115
	2	1.903	1.955	2.057	13.692	0.566	3.58
	3	2.854	2.972	3.194	30.028	0.388	3.4
	4	3.813	3.967	4.402	53.292	0.31	3.38

6. The Poisson-Weibull distribution exhibits its asymptotic behavior towards the normality.
   a. As fixing the parameter '*a*' and incrementing the parameter *c* then variance will be minimum i.e., $\forall a = k$ (constant, say) then there exists a parameter *c* such that as *c* incrementing then variance of Poisson – Weibull distribution will be minimum.
   b. As fixing one parameter and incrementing another parameter of Poisson – Weibull distribution then it tends to Normal distribution i.e.
      i. Fix '*a*' and as '*c*' incrementing such that $\beta_1 \to 0$ and $\beta_2 \to 3$ i.e., $\forall a = k'$ (constant, say) then there exists a parameter *c*; as $c \to \infty$ then Poisson – Weibull distribution will be tending to Normal distribution.
      ii. Fix *c* and as '*a*' incrementing such that $\beta_1 \to 0$ and $\beta_2 \to 3$ i.e. $\forall a = k''$ (constant, say) then there exists a parameter '*a*'; as $a \to \infty$ then Poisson – Weibull distribution will be tends to Normal distribution.
7. When $c = 1$ in Weibull distribution, the Poisson-Weibull distribution tends to Geometric distribution with $p\left(=\frac{1}{1+a}\right)$.
8. When c = 2 in Weibull distribution, the resulting Poisson-Weibull distribution is

$$P(x) = \frac{a^x}{x!}[Gamma(1+\frac{x}{2})Hypergeometric1F1(1+\frac{x}{2},\frac{1}{2},\frac{a^2}{4}) - a.Gamma(\frac{3+x}{2})Hypergeometric1F1(\frac{3+x}{2},\frac{3}{2},\frac{a^2}{4})];\ Re(x) > -2$$

9. Let X follows Poisson with parameter λ, and λ follows Weibull distribution with two parameters (*a*, *c*) then the Bayesian Poisson-Weibull function is

$$P(\lambda/X) = \left[\frac{c}{a}\right]\left[\frac{\lambda}{a}\right]^{x+c-1}\left[\frac{e^{[-\lambda+\left(\frac{\lambda}{a}\right)^{c}]}}{\sum_{p=0}^{\infty}\left[\frac{(-a)^{p}}{p!}.\left(\frac{x+p}{c}\right)!\right]}\right]$$

Remarks:

1. The moments of Poisson –Weibull with c = 1 are

$$\mu_1' = a$$

$$\mu_2 = a(1+a)$$

$$\mu_3 = 2a^3 + 3a^2 + a$$

$$\mu_4 = 9a^4 + 18a^3 + 10a^2 + a$$

2. The generating functions of Poisson – Weibull with c = 1 are

The Moment Generating function, $M_X(t) = \left[1 + a\left(1 - e^{t}\right)\right]^{-1}$

The Characteristic function, $\phi_X(t) = \left[1 + a\left(1 - e^{it}\right)\right]^{-1}$ and

The Probability Generating function, $P_X(t) = \left[1 + a(1-s)\right]^{-1}$

3. The cumulative distribution function of Poisson Weibull with c = 1is

$$F(x) = P(X \le s) = 1 - \left(\frac{a}{1+a}\right)^{s}$$

4. The Skewness and Kurtosis of Poisson Weibull with *c* = 1 are

$\beta_1 = \frac{(1+2a)^2}{a(1+a)}$ and $\beta_2 = \frac{9a^2+9a+1}{a(1+a)}$. The excess kurtosis is $\frac{6a^2+6a+1}{a(1+a)}$

5. Let X_1 and X_2 be the independent random variables each having Poisson Weibull distribution with *c* = 1 then the conditional distribution of $X_1/(X_1+X_2)$ or $X_2/(X_1+X_2)$ is uniform.

6. When *c* = 1, the Bayesian estimation of Poisson – Weibull distribution is given by

$$P(\lambda/X) = \frac{e^{-\lambda}\lambda^{x}}{x!}.\left(1+(1/a)\right)^{x+1}e^{-\left(\frac{\lambda}{a}\right)} = P(X/\lambda).\left(1+(1/a)\right)^{x+1}e^{-\left(\frac{\lambda}{a}\right)}.$$

7. When c = 2, the moments are given by

$$\mu_1' = \frac{a}{2}\sqrt{\pi}$$

$$\mu_2 = \frac{a^2}{4}(3-\pi) + \frac{a}{2}\sqrt{\pi}$$

$$\mu_3 = \frac{a^3}{4}(\pi-3)\sqrt{\pi} - \frac{3a^2}{4}\pi + \frac{a}{2}\sqrt{\pi}$$

$$\mu_4 = \frac{a^4}{16}(32-3\pi^2) + \frac{3a^3}{4}(\pi-2)\sqrt{\pi} + a^2(7-\pi) + \frac{a}{2}\sqrt{\pi}$$

8. When c = 2, the Bayesian Poisson-Weibull distribution is

$$P(\lambda / X) = \left(\frac{2}{a}\right)\left(\frac{\lambda}{a}\right)^{x+1}\left[\frac{e^{-\left(\lambda+\frac{\lambda}{a}\right)^2}}{\left[\begin{array}{l} Gamma(1+\frac{x}{2})Hypergeometric1F1(1+\frac{x}{2},\frac{1}{2},\frac{a^2}{4}) - \\ aGamma(\frac{3+x}{2})Hypergeometric1F1(\frac{3+x}{2},\frac{3}{2},\frac{a^2}{4}) \end{array}\right]}\right]; \mathrm{Re}(x) > -2$$

9. The Poisson – Weibull distribution is not in a condensed form and it is of the form with hypo hyper geometrical in nature. If we are restricting the parameter *c*, we get the closed expressions.
10. When $c > 1$ the distribution function tends to generalized hyper geometric distributional forms. The Poisson – Weibull distribution tends to Normal distribution as limiting case of the parameters dually.
11. The Poisson – Weibull distribution is used to ascertain the behavior of consumers in marketing activities for the specified parameters.
12. The purchase of a product in successive intervals over the period of time follows a Poisson distribution with average purchase rate of consumer λ, where the parameter λ varies from consumer to consumer in the long run among the whole population and follows Weibull distribution with parameters (*a*, c = 1), the consequence of Compound distribution results a Poisson – Weibull model which tends to Negative binomial distribution.

EXAMPLE: Ehrenberg (1959) studied consumers repeated purchase of non-durable specified brand of item in frequently bought products for the brand or pack – size and data collected on the number of units purchased (X) by the householder and the number of households (frequency) purchased among the 2000 household consumers over a period of 26 weeks

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13
F	1612	164	71	47	28	17	12	12	5	7	6	3	3	5

X	14	15	16	17	18	19	20	21	22	23	24	25	26
F	0	0	0	2	0	0	1	0	2	0	0	1	2

For this data set, mean = 0.636, variance = 4.502504, mode = 0.526, C.V = 3.33 and β_1 = 0.0514 etc. We estimated the theoretical frequencies by using recurrence relation (*) and also tested for the goodness of fit by using Chi – square statistic and Theils U statistic. The estimated parameters are a = 6.0794, r = 0.104615 p = 0.14125 and q = 0.85875. $P(0) = p^r$ and $P(x + 1) = \{(r + x) / (x + 1)\}.\ q.\ P(x)$. The expected repeated purchases of consumer in successive time points at distant apart are

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13
E	1630	146	70	42	28	20	14	11	8	6	5	4	3	3

X	14	15	16	17	18	19	20	21	22	23	24	25	26
E	2	2	1	1	1	1	1	1	0	0	0	0	0

We estimated and obtained statistics such as mean = 0.626, variance = 4.106, Mode = 0.523, C.V = 3.23 and β_1 = 0.0508, which state that the expected frequencies are closely associated with the original data. The test for goodness of fit is done in bi aspects with chi – square and Theils U statistic. The Theils U statistic value is 0.008 (which is closer to zero) which concludes that Poisson – Weibull distribution strongly fits and gives equality association with the observed distribution in estimating the repeated purchases of consumer, and also χ^2 calculated value is 6.726 and its tabulated value of χ^2 for 12 d.f. is 21.026 (χ^2 cal < χ^2 tab) or (p = 0.91568 >

0.05 which accept that the fitted model, and gives good approximation). Hence, we can conclude that Poisson – Weibull distribution is applied in marketing science.

Note: When $c = 2$, the Poisson – Weibull distribution tends to Hyper-geometric distribution. The Hyper-geometric distribution is used to assess the brand leader of the market in oligopolistic market environment. Whenever $c = 2$, Poisson – Weibull distribution is used to test the brand loyalty among the same product line. Let us consider a basic example for describe the situation to apply Poisson – Weibull distribution in examining the loyalty of brand. In a brand loyalty contest, a panel of N number of customers is selected to judge the leader of brand in the market. At least two brands are considered. Let us say Brand A, and Brand B, where the opinion of customers got divided such that M numbers of customers were in favor of Brand A where remaining $N – M$ number of customers were in favor of Brand B. A random sample of n ($\leq N$) customers were drawn from the panel, then we have to find out what is the probability that out of n customers, m number of customers are in favor of Brand A.

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Kumaraswamy Kandukuri and Bhatracharyulu N Ch "Compound Distribution of Poisson-Weibull." *International Journal of Mathematics and Statistics Invention (IJMSI)*, vol. 09(01), 2021, pp. 29-34.