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Growth Rates of the Functions Formed by the Composition of Polynomials and Meromorphic Functions

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ABSTRACT*: In this paper, we try to derive some relations in connection with order, lower order, L-order, L-lower order, L*-order, L*-lower order of the meromorphic functions and its composite functions with polynomials* **.**
KEYWORDS*: Meromorphic function, polynomials, composite functions, order, lower order, slowly changing functions, L-order, L*-order.*

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I. INTRODUCTION

Let f be a meromorphic function and g be an entire function defined in ℂ, the set of all finite complex numbers. The maximum modulus function corresponding to entire g is defined as

$$M_g(r) = \max \{|g(z)|:|z| = r\}.$$

$M_f(r)$ cannot be defined for meromorphic function f, as f is not analytic. In this situation, one may define another function $T_f(r)$, known as Nevanlinna's Characteristic function of f, which is playing the same role as maximum modulus. All the standard notations and definitions in the theory of entire and meromorphic functions which are available in **[4]** and **[1]**.

II. PRELIMINARIES (DEFINITIONS AND LEMMAS)

In this connection we just recall the following definitions and lemmas which are relevant:

Definition 2.1 The order ρ_f and lower order λ_f of a meromorphic function f is defined by

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}$$

Sato (1963) defined the generalized order and generalized lower order of an entire function.

Definition 2.2 The generalized order $\rho_f^{[m]}$ and generalized lower order $\lambda_f^{[m]}$ of a meromorphic function f is defined by

$$\rho_f^{[m]} = \limsup_{r\to\infty} \frac{\log^{[m-1]} T_f(r)}{\log r} \text{ and } \lambda_f^{[m]} = \liminf_{r\to\infty} \frac{\log^{[m-1]} T_f(r)}{\log r}$$

Let L≡L(r) be a positive continuous function increasing slowly i.e., L(ar)~L(r) as $r \to \infty$ for every positive constant a. Singh et.al. (1977) defined it in the following way:

Definition 2.3[3]A positive continuous function L(r) is called a slowly changing function if for ε(>0),

$$\frac{1}{k^\varepsilon} \le \frac{L(kr)}{L(r)} \le k^\varepsilon \text{ for } r \ge r(\varepsilon) \text{ and uniformly for } k(\ge 1).$$

If further, L(r) is differentiable, the above condition is equivalent to $\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0.$

Definition 2.4[2] The L-order $\rho_f^{\,L}$ and the L-lower order $\lambda_f^{\,L}$ of a meromorphic function f are defined as follows:

$$\rho_f^{\ L} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[rL(r)\right]} \text{ and } \lambda_f^{\ L} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[rL(r)\right]}$$

Definition 2.5 The generalized L-order $\rho_f^{[m]L}$ and the generalized L-lower order $\lambda_f^{[m]L}$ of a meromorphic function f are defined as follows:

$$\rho_f^{[m]L} = \limsup_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log\left[rL(r)\right]} \text{ and } \lambda_f^{[m]L} = \liminf_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log\left[rL(r)\right]}$$

Definition 2.6[2]The L^*-order $\rho_f^{\ L^*}$ and the L^* -lower order $\lambda_f^{\ L^*}$ of a meromorphic function f are defined as

$$\rho_f^{\ L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{\ L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

Definition 2.7 The generalized L^*-order $\rho_f^{[m]L^*}$ and the generalized L^* -lower order $\lambda_f^{[m]L^*}$ of a meromorphic function f are defined as

$$\rho_f^{[m]L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[m]L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log\left[re^{L(r)}\right]}$$

Definition 2.8 A polynomial function P(z) of degree n is defined by $P(z) = c_0 + c_1 z + c_2 z^2 + \cdots. + c_n z^n$, $c_n \neq 0$.

Lemma 2.1 [1] If P(u) is a polynomial of degree p and f(z) is a meromorphic function, then

$$T(r; P(f(z))) = pT(r; f(z)) + O(1)$$

III. MAIN RESULTS

In this section we present the main results of the paper.

Theorem 3.1 If f(z) be a meromorphic function and P(u) is a polynomial of degree p, then

$$\rho_{P\circ f} = \rho_f \text{ and } \lambda_{P\circ f} = \lambda_f.$$

Proof In view of Lemma 2.1, for a sequence of values of r tending to infinity,

$$T(r; P(f(z))) \approx pT(r; f(z))$$

$$\text{i.e., } \log T(r; P(f(z))) = \log T(r; f(z)) + O(1)$$

$$\text{So, } \rho_{P\circ f} = \limsup_{r\to\infty} \frac{\log T(r, P\circ f)}{\log r}$$

$$= \limsup_{r\to\infty} \frac{\log T(r; P(f(z)))}{\log r}$$

$$= \limsup_{r\to\infty} \frac{\log T(r; f(z)) + O(1)}{\log r}$$

$$= \limsup_{r\to\infty} \left(\frac{\log T(r; f(z))}{\log r} + \frac{O(1)}{\log r} \right)$$

$$= \limsup_{r\to\infty} \frac{\log T(r; f(z))}{\log r} \left(\because r\to\infty \Rightarrow \frac{O(1)}{\log r} \to 0 \right)$$

$$= \rho_f$$

$$\text{Similarly, } \lambda_{P\circ f} = \liminf_{r\to\infty} \frac{\log T(r, P\circ f)}{\log r} = \lambda_f.$$

Theorem 3.2 (Generalized case) : If f(z) be a meromorphic function and P(u) is a polynomial of degree p, then

$$\rho_{P\circ f}^{[m]} = \rho_f^{[m]} \text{ and } \lambda_{P\circ f}^{[m]} = \lambda_f^{[m]}.$$

Proof In view of Lemma 2.1, for a sequence of values of r tending to infinity,

$$T(r; P(f(z))) \approx pT(r; f(z))$$

$$\text{i.e., } \log T(r; P(f(z))) = \log T(r; f(z)) + O(1)$$

$$\text{i.e., } \log T(r; P(f(z))) \approx \log T(r; f(z))$$

$$\text{i.e., } \log^{[m-1]} T(r; P(f(z))) = \log^{[m-1]} T(r; f(z)) + O(1)$$

$$\begin{aligned} So,\ \rho^{[m]}_{P\circ f} &= \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, P\circ f)}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r; P(f(z)))}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r; f(z)) + O(1)}{\log r} \\ &= \limsup_{r\to\infty} \left(\frac{\log^{[m-1]} T(r; f(z))}{\log r} + \frac{O(1)}{\log r} \right) \\ &= \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r; f(z))}{\log r} \left(\because r\to\infty \Rightarrow \frac{O(1)}{\log r} \to 0 \right) \\ &= \rho^{[m]}_f \end{aligned}$$

$$\text{Similarly, } \lambda^{[m]}_{P\circ f} = \liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, P\circ f)}{\log r} = \lambda^{[m]}_f.$$

Theorem 3.3 If f(z) be a meromorphic function and P(u) is a polynomial of degree p, then

$$\rho^{L}_{P\circ f} = \rho^{L}_{f} \text{ and } \lambda^{L}_{P\circ f} = \lambda^{L}_{f}.$$

Proof In view of Lemma 2.1, for a sequence of values of r tending to infinity,

$$T(r; P(f(z))) \approx pT(r; f(z))$$

$$\text{i.e., } \log T(r; P(f(z))) = \log T(r; f(z)) + O(1)$$

$$\begin{aligned} \text{So, } \rho^{L}_{P\circ f} &= \limsup_{r\to\infty} \frac{\log T(r, P\circ f)}{\log\left[rL(r)\right]} \\ &= \limsup_{r\to\infty} \frac{\log T(r; P(f(z)))}{\log\left[rL(r)\right]} \\ &= \limsup_{r\to\infty} \frac{\log T(r; f(z)) + O(1)}{\log\left[rL(r)\right]} \\ &= \limsup_{r\to\infty} \frac{\log T(r; f(z))}{\log\left[rL(r)\right]} + 0 \\ &= \rho^{L}_f \end{aligned}$$

$$\text{Similarly, } \lambda^{L}_{P\circ f} = \liminf_{r\to\infty} \frac{\log T(r, P\circ f)}{\log\left[rL(r)\right]} = \lambda^{L}_{f}.$$

The following theorem can be proved in the line of Theorem 3.2 with help of Definition 2.5, so the proof is omitted.

Theorem 3.4 (Generalized case) : If f(z) be a meromorphic function and P(u) is a polynomial of degree p, then

$$\rho^{[m]L}_{P\circ f} = \rho^{[m]L}_{f} \text{ and } \lambda^{[m]L}_{P\circ f} = \lambda^{[m]L}_{f}.$$

Theorem 3.5 If f(z) be a meromorphic function and P(u) is a polynomial of degree p, then

$$\rho_{P\circ f}^{L*} = \rho_f^{L*} \text{ and } \lambda_{P\circ f}^{L*} = \lambda_f^{L*}.$$

Proof In view of Lemma 1, for a sequence of values of r tending to infinity,

$$T(r; P(f(z))) \approx pT(r; f(z))$$

$$\text{i.e., } \log T(r; P(f(z))) = \log T(r; f(z)) + O(1)$$

$$\text{So, } \rho_{P\circ f}^{L*} = \limsup_{r\to\infty} \frac{\log T(r, P\circ f)}{\log\left[re^{L(r)}\right]}$$

$$= \limsup_{r\to\infty} \frac{\log T(r; P(f(z)))}{\log\left[re^{L(r)}\right]}$$

$$= \limsup_{r\to\infty} \frac{\log T(r; f(z)) + O(1)}{\log\left[re^{L(r)}\right]}$$

$$= \limsup_{r\to\infty} \frac{\log T(r; f(z))}{\log\left[re^{L(r)}\right]} + 0$$

$$= \rho_f^{L*}$$

$$\text{Similarly, } \lambda_{P\circ f}^{L*} = \liminf_{r\to\infty} \frac{\log T(r, P\circ f)}{\log\left[re^{L(r)}\right]} = \lambda_f^{L*}.$$

The following theorem can be proved in the line of Theorem 3.2 with help of Definition 2.7, so the proof is omitted.

Theorem 3.6(Generalized case) : If f(z) be a meromorphic function and P(u) is a polynomial of degree p, then

$$\rho_{P\circ f}^{[m]\,L*} = \rho_f^{[m]\,L*} \text{ and } \lambda_{P\circ f}^{[m]\,L*} = \lambda_f^{[m]\,L*}.$$

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