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Numerical Solution of Delay Differential Equations by two and three Pointblock Simpson's Method

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ABSTRACT: In this paper, two and three points block Simpson's methods were considered for the solution of first order delay differential equations (DDEs). An accurate formula in [1]will be implemented for the solution of delay argument. The continuous formulations of the methods were derived through the multi-step collocation by matrix inversion technique in which their discrete schemes were deduced from them to form a block. The P-stability analysis of the block methods were carried out. The performance of the block methods were measured by solving some problems and compare them with other existing ones in terms of accuracy.
KEYWORDS: Delay Argument, Matrix inversion technique, P-stability Analysis and Simpson's Method

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I. INTRODUCTION

Delay differential equations arise in many different areas of science and engineering. If an action is to be made based on an assessment of the current state of a system and if some time is necessary to process the information, the action will not be taken instantaneously but rather a delay will arise. This delay is best incorporated in differential equations by making the action a function of past rather than of instantaneous values of the independent variables.

Delay differential equations are similar to ordinary differential equations (ODEs), except that they involve past values of the independent variables. Because of this, rather than needing an initial value to be fully specified, DDEs require input of an initial function. In this paper we considered DDEs of the form:

$$\begin{aligned} y'(t) &= f\,(\mathrm{t}, \mathrm{y}, y(t-\tau)), \quad t \ge t_0, \tau > 0 \\ y(t) &= \varphi(t) \qquad\qquad\quad t \le t_0 \end{aligned} \tag{1}$$

where $\varphi(t)$ is the initial function, $\tau(t, y(t))$ is the delay, $t - \tau(t, y(t))$ is the delay argument and value of $y(t - \tau(t, y(t)))$ is the solution of the delay argument. The delay is called constant delay if it is a constant, it is also called time dependent delay if it is a function of independent variable and it is called state dependent delay if it is a function of independent variable and dependent variable.

Delay differential equation is known as retarded delay differential equation (RDDE) if the delay depends on dependent variable and it is also known as neutral delay differential equation (NDDE) when the delay depends on derivative of the independent variable. Most researchers like [2, 3,4,5] etc. have used various families of Runge-kutta methods and Interpolation techniques to solve DDEs, while some like [6,7,8] and [9] applied linear multi-step methods (LMMs) and Nordsieck's interpolation technique for the numerical solution DDEs. In this paper, the three cases of delay will be considered in solving some problems for RDDEs type and we only concerned with LMMs in which a Simpson's Method for a step number *k = 2* and *3* will be used to solve first order DDEs.

II. DERIVATION TECHNIQUES

Derivation of Multistep Collocation Method

In [10], a *k*-step multistep collocation method with *m* collocation points was obtained as

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f(x, y(x)) \tag{2}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients of the method defined as

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i \text{ for } j = \{0,1,...,t-1\} \quad (3) \qquad h\beta_j(x) = \sum_{i=0}^{t+m-1} h\beta_{j,i+1} x^i \text{ for } j = \{0,1,...,m-1\} \quad (4)$$

where $x_0, \ldots, x_{m-1}$ are the m collocation points and x_{n+j}, j = 0, 1, 2, …, t – 1 are the t arbitrarily chosen interpolation points.
To get $\alpha_j(x)$ and $\beta_j(x)$, [10] arrived at a matrix equation of the form

$$DC = I \qquad (5)$$

where I is the identity matrix of dimension $(t+m)\times(t+m)$ while D and C are matrices defined as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & \cdots & (t+m-1)x_0^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{m-1} & \cdots & (t+m-1)x_{m-1}^{t+m-2} \end{bmatrix} \qquad (6)$$

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m} \end{bmatrix} \qquad (7)$$

It follows from (5) that the columns of $C = D^{-1}$ give the continuous coefficients of the continuous scheme (2).

Derivation of Continuous Formulation of Simpson's Methods for k = 2

Here, the number of interpolation points, t = 1 and the number of collocation points m = 3. Therefore, (2) becomes:

$$y(x) = \alpha_0(x)y_n + h(\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2})$$

The matrix D in (6) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 0 & 1 & 2x_n & 3x_n^2 \\ 0 & 1 & 2x_n + 2h & 3(x_n+h)^2 \\ 0 & 1 & 2x_n + 4h & 3(x_n+2h)^2 \end{bmatrix} \qquad (8)$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple 18 from which the following continuous scheme is obtained using (2)

$$\begin{aligned} y(x) = y_n &+ \left(-x_n\left(\frac{x_n}{4h} + \frac{2x_n^2}{6h^2}\right) + x\left(\frac{3x_n}{2h} + \frac{x_n^2}{2h} - \frac{x^2(3h+2x_n)}{4h^2} + \frac{x^3}{6h^2}\right)\right) f_n \\ &+ \left(\frac{1}{3}\frac{x_n^2(x_n+3h)}{h^2} - \frac{x x_n(x_n+2h)}{h^2} + \frac{x^2(x_n+h)}{h^2} - \frac{1}{3}\frac{x^3}{h^2}\right) f_{n+1} \\ &+ \left(-\frac{x_n^2(3h+2x_n)}{12h^2} + \frac{x x_n(x_n+h)}{2h^2} - \frac{x^2(h+2x_n)}{4h^2} + \frac{x^3}{6h^2}\right) f_{n+2} \qquad (9) \end{aligned}$$

Evaluating and simplifying (9) at $x = x_{n+1}$ and $x = x_{n+2}$, the following discrete schemes are obtained:

$$\begin{aligned} y_{n+1} &= y_n + \frac{5}{12}hf_n + \frac{2}{3}hf_{n+1} - \frac{1}{12}hf_{n+2} \\ y_{n+2} &= y_n + \frac{1}{3}hf_n + \frac{4}{3}hf_{n+1} + \frac{1}{3}hf_{n+2} \qquad (10) \end{aligned}$$

Derivation of Continuous Formulation of Simpson's Methods for k = 3

Here, also the number of interpolation points, t = 1 and the number of collocation points m = 4.
Therefore, (2) becomes: $y(x) = \alpha_0(x)y_n + h(\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3})$
The matrix D in (6) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_n+2h & 3(x_n+h)^2 & 4(x_n+h)^3 \\ 0 & 1 & 2x_n+4h & 3(x_n+2h)^2 & 4(x_n+2h)^3 \\ 0 & 1 & 2x_n+6h & 3(x_n+3h)^2 & 4(x_n+3h)^3 \end{bmatrix} \quad (11)$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple 18 from which the following continuous scheme is obtained using (2)

$$\begin{aligned} y(x) = y_n + \frac{1}{6h^3}\Bigg(& -x_n\left(6h^3+5h^2x_n+2hx_n^2+\frac{x_n^3}{4}\right)+x\left(6h^3+11h^2x_n+6hx_n^2+x_n^3\right) \\ & -x^2\left(5h^2+6hx_n+2x_n^2+x^3(x_n+2h)-\frac{x^4}{4h^3}\right)f_n \\ & +\Bigg(\frac{x_n^2(6h^2+3hx_n+x_n^2)}{2h^3}-\frac{xx_n(3h^2+2hx_n+x_n^2)}{12h^3}+\frac{x^2(3h^2+5hx_n+x_n^2)}{4h^3} \\ & -\frac{x^3(h+x_n)}{6h^3}+\frac{x^4}{4h^3}\Bigg)f_{n+1} \\ & +\Bigg(\frac{xx_n(3h^2+4hx_n+x_n^2)}{h^3}-\frac{x_n^2(9h^2+8hx_n+x_n^2)}{12h^3}-\frac{x^2(h^2+4hx_n+x_n^2)}{h^3} \\ & +\frac{x^3(4h+3x_n)}{3h^3}-\frac{x^4}{4h^3}\Bigg)f_{n+2} \\ & +\Bigg(\frac{x_n^2(h^2+hx_n+x_n^2)}{6h^3}-\frac{xx_n(2h^2+3hx_n+x_n^2)}{h^3}+\frac{x^2(2h^2+6hx_n+3x_n^2)}{2h^3} \\ & -\frac{x^3(x_n+h)}{h^3}+\frac{x^4}{4h^3}\Bigg)f_{n+3} \end{aligned} \quad (12)$$

Evaluating and simplifying (12) at $x = x_{n+1}$, $x = x_{n+2}$ and $x = x_{n+3}$, the following discrete schemes are obtained:

$$y_{n+1} = y_n + \frac{3}{8}hf_n + \frac{19}{24}hf_{n+1} - \frac{5}{24}hf_{n+2} + \frac{1}{24}hf_{n+3}$$

$$y_{n+2} = y_n + \frac{1}{3}hf_n + \frac{4}{3}hf_{n+1} + \frac{1}{3}hf_{n+2}$$

$$y_{n+3} = y_n + \frac{3}{8}hf_n + \frac{9}{8}hf_{n+1} + \frac{9}{8}hf_{n+2} + \frac{3}{8}hf_{n+3} \quad (13)$$

III. P-STABILITY ANALYSIS

In this section, the P-stability analysis of the methods will be illustrated using the following test equation.

$$\begin{aligned} y'(t) &= \lambda y(t) + \mu y(t-\tau), & t > t_0 \\ y(t) &= \varphi(t), & t \le t_0 \end{aligned} \quad (14)$$

where λ, μ are complex coefficients and h is the step size.

Then from the discrete schemes in (10)

let $W_1 = \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix}$, $Y_1 = \begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix}$, and $Z_{1,1} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}$ Since, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$ and

$$C_{1,1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{12} \\ \frac{4}{3} & \frac{1}{3} \end{pmatrix}$$

we obtain, $A_1W_1 = B_1Y_1 + h\sum_{j=1}^{2} C_{1,j}Z_{1,j}$ (15)

Also from the discrete schemes in (13),

let$W_2 = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix}$, $Y_2 = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$, $Z_{1,2} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}$ and $Z_{2,2} = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$ since, $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$ and $C_{2,1} =$
$\begin{pmatrix} \frac{19}{24} & -\frac{5}{24} & \frac{1}{24} \\ \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} \end{pmatrix}$

we obtain, $A_2 W_2 = B_2 Y_2 + h\sum_{j=1}^{2} C_{2,j} Z_{2,j}$ (16)

According to [11], the P-polynomials are obtained by applying (15) and (16) to (14). Thus the P-stability polynomials for the discrete schemes in (10) and (13) are given respectively by

$P_1(\lambda) = \det\left[(A_1 - H_1 C_{1,2})\lambda^{2+r} - (B_1 - H_1 C_{1,1})\lambda^{1+r} - H_2 \sum_{j=1}^{2} C_{1,j} \lambda^j \right]$ and

$P_2(\lambda) = \det\left[(A_2 - H_1 C_{2,2})\lambda^{2+r} - (B_1 - H_1 C_{2,1})\lambda^{1+r} - H_2 \sum_{j=1}^{2} C_{2,j} \lambda^j \right]$

where $H_1 = h\lambda$, $H_2 = h\mu$ and $r \in$ then we plot the P- stability regions for r =1 for the schemes (10) and (13) are shown in Figure. 1 and 2

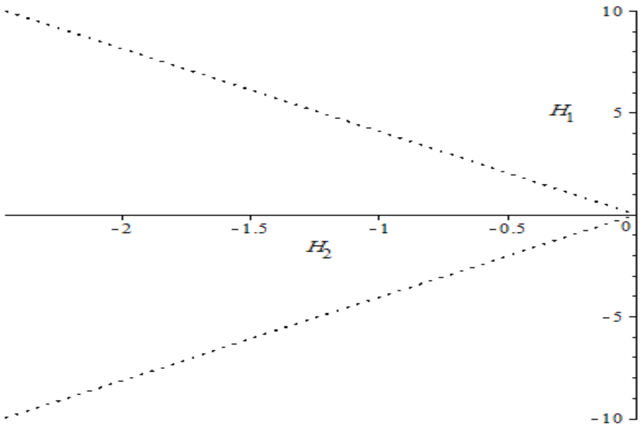

Figure.1 The P-stability region of the schemes in (10)

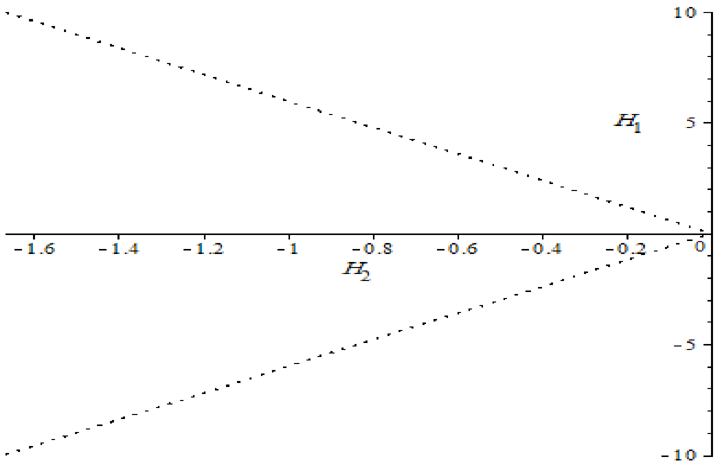

Figure.2 The P-stability region of the schemes in (13)

IV. NUMERICAL RESULT

In order to study the efficiency of the derived methods, we present some numerical results for the following problems:

Problem 1

$y'(t) = -y(t) - \frac{\pi}{2}e^{-1}y(t-1),\ 0 \le t \le 3$

$y(t) = e^{-t}\sin(\frac{\pi}{2}t), t \le 0$

Exact solution $y(t) = e^{-t}\sin(\frac{\pi}{2}t)$

Problem 2

$y'(t) = -y(t-1+e^{-t}) + \sin(t-1+e^{-t}) + \cos(t),\ 0 \le t \le 3$

$y(t) = \sin(t), t \le 0$

Exact Solution $y(t) = \sin(t)$

Problem 3

$y'(t) = \cos(t)(y(y(t)-2))\ 0 \le t \le 3$

$y(t) = 1, t \le 0$

Exact Solution $y(t) = 1 + \sin(t)$

The above problems were also solved using two and three point block Simpson's methods with the formula in [1] to approximate the delay argument are given in the Table 1 to 3.

V. NOTATIONS

h Step size
NS Total number of steps taken
*ME*Maximum Error
RBBDF Reformulated Block BDF method for step number $k = 3$ in[1]
*RBBDF** Reformulated Block BDF method for step number $k = 4$ in [1]
2BSM 2-Point Block Simpson's Methods
3BSM 3-Point Block Simpson's Methods

The maximum error *ME* is a highest value of the absolute error for total number of steps taken.

Table 1. Comparison between 2BSM and 3BSM using Problem 1

h	METHOD	NS	ME
10^{-2}	2BSM	150	2.57E-09
	3BSM	100	1.95E-09
10^{-3}	2BSM	150	3.23E-10
	3BSM	100	2.33E-10

Table 2. Comparison between RBBDF and SPS using Problem 2

h	METHOD	NS	ME
10^{-2}	RBBDF*	150	1.61E-07
	RBBDF	100	1.54E-07
	2BSM	150	3.20E-10
	3BSM	100	2.60E-10
10^{-3}	RBBDF*	1500	1.28E-08
	RBBDF	1000	2.58E-09
	2BSM	1500	3.72E-11
	3BSM	1000	1.97E-11

Table 3. Comparison between RBBDF and SPS using Problem 3

h	METHOD	NS	ME
10^{-2}	RBBDF*	150	2.16E-07
	RBBDF	100	2.96E-08
	2BSM	150	2.94E-09
	3BSM	100	2.82E-09
10^{-3}	RBBDF*	1500	2.14E-08
	RBBDF	1000	2.27E-09
	2BSM	1500	3.23E-10
	3BS	1000	1.72E-10

VI. Conclusion

This paper considered three numerical examples to test the efficiency of our two derived methods. It was observed that the results obtained from 3BSM performed better than 2BSM with the exact solutions and the error analysis shows that the method was found to be more efficient in terms of accuracy when compared with other methods like RBBDF in [1]. It is concluded that Block Simpson's Methods are more suitable for the solution of the first order delay differential equations.

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