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Generating the Subgroup Representations and Actions of Finite Groups on Signal Space

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ABSTRACT: *This paper aimed at determining all subgroups representations ofthe Symmetric group S_5 up to Isomorphism using Sylow's theorem and Lagrange's theorem. It was vividly described and derived 156 subgroups of S_5 and their conjugacy class size and Isomorphism class. The generated representations are used as actions on signal space which produced output for every corresponding input signal. Hence, the subgroup representations act on the signal space by conjugation. The derived subgroups can be used to determine the number of Fuzzy subgroups of the symmetric group S_5 for further research.*
Keywords: *Finite groups, subgroups, representations, group action, signal space.*

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I. INTRODUCTION

Given any set X= $\{x_1,...,x_n\}$,the set Sym(X) of all permutations of X is a group under composition, and the subset Alt(X) of even permutations of X is a group under composition. Since the elements of X are in definite order, we think of Sym(X) as S_n and Alt(X) as A_n. The Dihedral group $D_n$is considered as a group of permutations of a regular n-gon, since the rigid motions of the vertices determine the new position of then-gon. Hence, theSymmetric Groups S_n, Alternating Groups A_n and Dihedral Groups $D_n$for $n \geq 3$, all behave as permutations on certain sets. If the vertices of the n-gon is labeled in a definite manner by the numbers from 1 to n,thenD_n can be viewedas a subgroup of S_n.

Let G be a group (finite or infinite) and let X be a set.Then an action of G on X can be defined as a function$G \times X \to X$ denoted by $(g, x) \to g \cdot x$such that$e \cdot x = x$ and $(gh) \cdot x= g \cdot (h \cdot x)$.In fact, an action of G onXis equivalent to a group homomorphism (also a **representation**) $\rho: G \to A(X)$. Equivalently,given an action $G \times X \to X$, define a group homomorphism $\rho: G \to A(X)$ by the rule $\rho(g) = \sigma: X \to X$, where $\sigma(x) = g \cdot x$and given a representation (called a group homomorphism) $\rho: G \to A(X)$, define an action $G \cdot X \to X$ by the rule $g \cdot x= \rho(g)(x)$.The basic idea of group action is that the elements of the group are viewed as permutations of a set in such a way that composition of the corresponding permutations matches multiplication in the original group.

II. PRELIMINARIES

Definition 2.1: Let G be a group and let N be a proper normal subgroup of G. Then N is called maximal subgroup of G if there does not exists any proper normal subgroup M of G such that $N \leq M \leq G$ [1].
Definition 2.2: A subgroup N of G is said to be a normal subgroup of G if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$ [2].
Definition 2.3: A homomorphism $\varphi: G \to K$ from a group G to a group K is a function with the property that $\varphi(g_1 * g_2) = \varphi(g_1) * \varphi(g_2)$ for all $g_1, g_2 \in G$, where $*$ denotes the group operation on G and on K [3].
Definition 2.4: An isomorphism $\varphi: G \to K$ between two groups G and K is a homomorphism that is also a bijection mapping G onto K. Two groups G and K are isomorphic if there exists an isomorphism mapping G onto K, written as $G \cong K$. While an automorphism is an isomorphism mapping a group onto itself [4].
Theorem 2.5: (Lagrange's Theorem) If G is a finite group and H is a subgroup of G, then order of H is a divisor of order of G [5].
Theorem 2.6: If G is a finite group and $x \in G$, then order of x is a divisor of order of G [2].
Theorem 2.7: (Cauchy's Theorem) Let G be a finite group and let p be a prime number that divides the order of G. Then G contains an element of order p [6].
Theorem 2.8:(Cayley)**:** Every finite group G can be embedded in a symmetric group[6].
Theorem 2.9: (The First Sylow Theorem) Let G be a finite group and let $|G| = p^n m$ where $n \geq 1$, p is a prime number and $(p, m) = 1$. Then G contains a subgroup of order p^k for each k where $1 \leq k \leq n$ [7].
Definition 2.10: Let G be a finite group and let $|G| = p^n m$ where $n \geq 1$, p is a prime number and $(p, m) = 1$. The subgroup of G of order p^n is called the sylow p−subgroup of G [8].

Theorem 2.11: Let G be a group of order pq, where p and q are distinct primes and p < q. Then G has only one subgroup of order q. This subgroup of order q is normal in G [8].
Definition 2.12: A non-trivial group G is said to be simple if the only normal subgroups of G are the whole of G and the trivial subgroup {e} whose only element is the identity element e of G [6].
Definition 2.13: Let X be a non-empty set and G be a group. A left action of G on the set X is a defined as a map $G\times X \to X$ given by $(g, x) \to gx$ such that
i. $i\cdot x = x$ for all $x\in X$ and
ii. $(g_1g_2)x = g_1(g_2x)$ for all $x\in X$ and $g_1, g_2\in G$.
Under these considerations, the set X is called a G-set.
Theorem 2.14: The action of any group G on a set X is the same as group homomorphism from G to Sym(X), the group of permutations of X.
Definition 2.15: (Signal Space): If signal can be represented by n-tuple, then it can be treated in much the same way as n-dimensional vector space. Hence, the n-dimensional Euclidean space is called Signal space.

III. METHODOLOGY

In this section, the method used in generating the subgroups representations of a finite group S_n, n = 5 is presented. Let G = S_5.Then the one-headed group Gis the group of permutations of the set S = {1, 2, 3, 4, 5}, i.e., the set of all bijections $\sigma : S \to S$ defined by $\sigma(a_i) = a_j; i, j \le 5$. The collection of all such bijections give rise to a group of order 120 as follows:
$G = \{i, \rho_1, \rho_2, ..., \rho_{10}, \sigma_1, \sigma_2, ..., \sigma_{20}, \tau_1, \tau_2, ..., \tau_{30}, \gamma_1, \gamma_2, ..., \gamma_{15}, \beta_1, \beta_2, ..., \beta_{24}, \delta_1, \delta_2, ..., \delta_{20}\}$.
The elements are listed as follows:
i = (1) = the identity permutation;
ρ_1 = (4 5), ρ_2 = (3 5), ρ_3 = (3 4), ρ_4 = (2 5), ρ_5 = (2 3), ρ_6 = (2 4), ρ_7 = (1 5), ρ_8 = (1 4), ρ_9 = (1 3), ρ_{10} = (1 2);
σ_1 = (1 2 3), σ_2 = (1 3 2), σ_3 = (1 2 4), σ_4 = (1 4 2), σ_5 = (1 2 5), σ_6 = (1 5 2), σ_7 = (1 3 4), σ_8 = (1 4 3), σ_9 = (1 4 5), σ_{10} = (1 5 4), σ_{11} = (1 3 5), σ_{12} = (1 5 3), σ_{13} = (2 3 4), σ_{14} = (2 4 3), σ_{15} = (2 3 5), σ_{16} = (2 5 3), σ_{17} = (2 4 5), σ_{18} = (2 5 4), σ_{19} = (3 4 5), σ_{20} = (3 5 4);
τ_1 = (2 3 4 5), τ_2 = (2 5 4 3), τ_3 = (2 3 5 4), τ_4 = (2 4 5 3), τ_5 = (2 4 3 5), τ_6 = (2 5 3 4), τ_7 = (1 2 3 4), τ_8 = (1 4 3 2), τ_9 = (1 2 3 5), τ_{10} = (1 5 3 2), τ_{11} = (1 2 4 3), τ_{12} = (1 3 4 2), τ_{13} = (1 2 4 5), τ_{14} = (1 5 4 2), τ_{15} = (1 2 5 3), τ_{16} = (1 3 5 2), τ_{17} = (1 2 5 4), τ_{18} = (1 4 5 2), τ_{19} = (1 3 4 5), τ_{20} = (1 5 4 3), τ_{21} = (1 3 5 4), τ_{22} = (1 4 5 3), τ_{23} = (1 3 2 4), τ_{24} = (1 4 2 3), τ_{25} = (1 3 2 5), τ_{26} = (1 5 2 3), τ_{27} = (1 4 3 5), τ_{28} = (1 5 3 4), τ_{29} = (1 4 2 5), τ_{30} = (1 5 2 4);
γ_1 = (2 4)(3 5), γ_2 = (2 5)(3 4), γ_3 = (2 3)(4 5), γ_4 = (1 3)(2 4), γ_5 = (1 3)(2 5), γ_6 = (1 4)(2 3), γ_7 = (1 4)(2 5), γ_8 = (1 5)(2 3), γ_9 = (1 5)(2 4), γ_{10} = (1 4)(3 5), γ_{11} = (1 5)(3 4), γ_{12} = (1 2)(3 4), γ_{13} = (1 2)(3 5), γ_{14} = (1 3)(4 5), γ_{15} = (1 2)(4 5);
β_1 = (1 2 3 4 5), β_2 = (1 3 5 2 4), β_3 = (1 4 2 5 3), β_4 = (1 5 4 3 2), β_5 = (1 2 3 5 4), β_6 = (1 3 4 2 5), β_7 = (1 5 2 4 3), β_8 = (1 4 5 3 2), β_9 = (1 2 4 5 3), β_{10} = (1 4 3 2 5), β_{11} = (1 5 2 3 4), β_{12} = (1 3 5 4 2), β_{13} = (1 2 4 3 5), β_{14} = (1 4 5 2 3), β_{15} = (1 3 2 5 4), β_{16} = (1 5 3 4 2), β_{17} = (1 2 5 4 3), β_{18} = (1 5 3 2 4), β_{19} = (1 4 2 3 5), β_{20} = (1 3 4 5 2), β_{21} = (1 2 5 3 4), β_{22} = (1 5 4 2 3), β_{23} = (1 3 2 4 5), β_{24} = (1 4 3 5 2);
δ_1 = (1 2 3)(4 5), δ_2 = (1 3 2)(4 5), δ_3 = (1 2 4)(3 5), δ_4 = (1 4 2)(3 5), δ_5 = (1 2 5)(3 4), δ_6 = (1 5 2)(4 5), δ_7 = (1 3 4)(2 5), δ_8 = (1 4 3)(2 5), δ_9 = (1 4 5)(2 3), δ_{10} = (1 5 4)(2 3), δ_{11} = (1 3 5)(2 4), δ_{12} = (1 5 3)(2 4), δ_{13} = (1 5)(2 3 4), δ_{14} = (1 5)(2 4 3), δ_{15} = (1 4)(2 3 5), δ_{16} = (1 4)(2 5 3), δ_{17} = (1 3)(2 4 5), δ_{18} = (1 3)(2 5 4), δ_{19} = (1 2)(3 4 5), δ_{20} = (1 2)(3 5 4);
Now, the order of an element x of a group G is the least positive integer n for which $x^n = i$, the identity element of G.Here,x^n represents $x\cdot x\cdot x\cdot \ldots \cdot x$n-times. Then writing the elements ofG in the form x^n, we classify them according to their order as follows:Note that the order of each $x\in G$ divides the order of G (Theorem 2.6).

Table 1: Order of elements of G

Order	Elements	Formula Calculating Element Order
1	I	LCM{1}
2	$\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8, \rho_9, \rho_{10}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}$	LCM{2,1}
3	$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}, \sigma_{20}$	LCM{3,1}
4	$\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{14}, \tau_{15}, \tau_{16}, \tau_{17}, \tau_{18}, \tau_{19}, \tau_{20}, \tau_{21}, \tau_{22}, \tau_{23}, \tau_{24}, \tau_{25}, \tau_{26}, \tau_{27}, \tau_{28}, \tau_{29}, \tau_{30}$	LCM{4,1}
5	$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \beta_{17}, \beta_{18}, \beta_{19}, \beta_{20}, \beta_{21}, \beta_{22}, \beta_{23}, \beta_{24}$	LCM{5,1}
6	$\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}, \delta_{12}, \delta_{13}, \delta_{14}, \delta_{15}, \delta_{16}, \delta_{17}, \delta_{18}, \delta_{19}, \delta_{20}$	LCM{2,3}

3.1SUBGROUP PRESENTATIONS OF G

According to Lagrange's theorem (Theorem 2.5), the order of any non-trivial subgroup of G divides the order of G. Hence, allthe subgroup representations of Gare determinedand their isomorphism class as analysed by Samaila, 2013 [9].

Obviously, the only subgroup of G of order 1 is the trivial subgroup $G_1 = \{i\}$, whose only element is the identity element.

3.1.1 Subgroups of order 2

Let H be arbitrary subgroup of Gof order 2. Since 2 is a prime number, H is cyclic. Hence, H is generated by an element of G of order 2. Thus all subgroups of G of order 2, isomorphic to the cyclic group Z_2 are:

$H_k = \{i, \rho_j : 1 \leq j \leq 10\} = \langle\rho_j\rangle$; $2 \leq k \leq 11$, (for each j, $H_k \cong S_2$), and

$H_k = \{i, \gamma_j : 1 \leq j \leq 15\} = \langle\gamma_j\rangle$; $12 \leq k \leq 26$, (subgroups generated by double transpositions in S_5).

3.1.2 Subgroups of order 3

Subgroups of G of order 3 are generated by the elements of G of order 3. Thus, these subgroups of order 3, isomorphic to the cyclic group Z_3 are

$L_k = \{i, \sigma_j, \sigma_{j+1} : \sigma_j^{-1} = \sigma_{j+1}; 1 \leq j \leq 19\} = \langle\sigma_j\rangle = \langle\sigma_{j+1}\rangle$; $27 \leq k \leq 36$.

Note that if $\sigma_j^{-1} = \sigma_{j+1}$, then j = j+2 for the next k. L_k is cyclic since 3 is prime.

3.1.3 Subgroups of order 4

Let M be arbitrary subgroup of G of order 4. Then by Theorem 2.5, elements of M must have order 1, 2 or 4. Hence if M consists of elements of order 4, then M is generated by an element of order 4. Thus, we obtained

$M_k = \{i, \tau_j, \gamma_{(j+1)/2}, \tau_{j+1} : \tau_j^{-1} = \tau_{j+1}; j = 1, 3, ..., 29\} = \langle\tau_j\rangle = \langle\tau_{j+1}\rangle$; $37 \leq k \leq 51$.

There are also subgroups of G of order 4 generated by pair of disjoint transpositions in G as follows:

$M_{52} = \{i, \rho_2, \rho_6, \gamma_1\}$, $M_{53} = \{i, \rho_3, \rho_4, \gamma_2\}$, $M_{54} = \{i, \rho_1, \rho_5, \gamma_3\}$, $M_{55} = \{i, \rho_6, \rho_9, \gamma_4\}$, $M_{56} = \{i, \rho_4, \rho_9, \gamma_5\}$, $M_{57} = \{i, \rho_5, \rho_8, \gamma_6\}$, $M_{58} = \{i, \rho_4, \rho_8, \gamma_7\}$, $M_{59} = \{i, \rho_5, \rho_7, \gamma_8\}$, $M_{60} = \{i, \rho_6, \rho_7, \gamma_9\}$, $M_{61} = \{i, \rho_2, \rho_8, \gamma_{10}\}$, $M_{62} = \{i, \rho_3, \rho_7, \gamma_{11}\}$, $M_{63} = \{i, \rho_3, \rho_{10}, \gamma_{12}\}$, $M_{64} = \{i, \rho_2, \rho_{10}, \gamma_{13}\}$, $M_{65} = \{i, \rho_1, \rho_9, \gamma_{14}\}$, $M_{66} = \{i, \rho_1, \rho_{10}, \gamma_{15}\}$.

Furthermore, 5 other subgroups of G of order 4 are generated by double transpositions on four elements, i.e. M_k for $67 \leq k \leq 71$.

3.1.4 Subgroups of order 5

Let N be a subgroup of S_5 of order 5. Since 5 is a prime number, the subgroup N is cyclic and is generated by an element of S_5 of order 5. Hence, there are 6 such subgroups given by

$N_k = \{i, \beta_j, \beta_{j+1}, \beta_{j+2}, \beta_{j+3}\} = \langle\beta_j\rangle = \langle\beta_{j+1}\rangle = \langle\beta_{j+2}\rangle = \langle\beta_{j+3}\rangle$; $72 \leq k \leq 77$.

3.1.5 Subgroups of order 6

If P is an arbitrary subgroup of G of order 6, then we generates from the δ'^s, the following subgroups, isomorphic to the cyclic group Z_6;

$P_k = \{i, \delta_{2j-1}, \sigma_{2j}, \rho_j, \sigma_{2j-1}, \delta_{2j} : \delta_{2j}^{-1} = \delta_{2j-1}, \sigma_{2j}^{-1} = \sigma_{2j-1}; 1 \leq j \leq 10\}$
$= \langle\delta_{2j}\rangle = \langle\delta_{2j-1}\rangle$; $78 \leq k \leq 87$.

Again by Sylow's theorem (Theorem 2.9), since $6 = 2 * 3$, other subgroups of G of order 6 can be generated from the product of the elements of G of order 2 with those elements of order 3. i.e. ρ'^s and σ'^s given by

$\{i, \sigma_1, \alpha_1, \sigma_2, \delta_1, \delta_2\}$, $\{i, \sigma_3, \alpha_2, \sigma_4, \delta_3, \delta_4\}$, ..., $\{i, \sigma_{19}, \alpha_{10}, \sigma_{20}, \delta_{19}, \delta_{20}\}$.

Hence,

$P_k = \{i, \sigma_{2j-1}, \alpha_j, \sigma_{2j}, \delta_{2j-1}, \delta_{2j} : \sigma_{2j}^{-1} = \sigma_{2j-1}; \delta_{2j}^{-1} = \delta_{2j-1}; 1\leq j\leq 10\}$; $88 \leq k \leq 97$.

Also, S_3 is obviously a subset of S_5.Thus, there are subgroups generated by each of the following set of elements:

(1 2 3), (1 2 4), (1 2 5), (1 3 4), (1 3 5), (1 4 5), (2 3 4), (2 3 5), (2 4 5) and (3 4 5).

Hence, we generate 10 such subgroups of S_5 of order 6, isomorphic to S_3 i.e. P_k such that $98 \leq k \leq 107$.

3.1.6 Subgroups of order 8

Since 8 is a multiple of 2 and 4, elements of the subgroup of order 8 must have orders 2 or 4, except the identity. Consider the set of permutations

Q = {i, (2 3 4 5), (2 5 4 3), (2 4)(3 5), (2 4), (3 5), (2 3)(4 5), (2 5)(3 4)}, i.e.

$Q = \{i, \tau_1, \tau_2, \gamma_1, \rho_6, \rho_2, \gamma_3, \gamma_2\}$

Obviously, this is a subgroup of S_5 of order 8. To see this, let us construct a multiplication table of $Q \times Q$ as follows.

Table 2: Multiplication table of Q ×Q.

$*$	i	τ_1	τ_2	γ_1	ρ_6	ρ_2	γ_3	γ_2
i	i	τ_1	τ_2	γ_1	ρ_6	ρ_2	γ_3	γ_2
τ_1	τ_1	γ_1	i	τ_2	γ_3	γ_2	ρ_2	ρ_6
τ_2	τ_2	i	γ_1	τ_1	γ_2	γ_3	ρ_6	ρ_2
γ_1	γ_1	τ_2	τ_1	i	ρ_2	ρ_6	γ_2	γ_3
ρ_6	ρ_6	γ_2	γ_3	ρ_2	i	γ_1	τ_2	τ_1
ρ_2	ρ_2	γ_3	γ_2	ρ_6	γ_1	i	τ_1	τ_2
γ_3	γ_3	ρ_6	ρ_2	γ_2	τ_1	τ_2	i	γ_1
γ_2	γ_2	ρ_2	ρ_6	γ_3	τ_2	τ_1	γ_1	i

Clearly, from Table 2 above, the set Q is a subgroup of S_5 of order 8. By constructing such subgroups from the combinations of τ_j's, γ_j's and ρ_j's, 15 subgroups of S_5 of order 8, isomorphic to the Dihedral group D_8 are obtained. i.e. Q_k such that $108 \le k \le 122$.

3.1.7 Subgroups of order 10

Let R be arbitrary subgroup of S_5 of order 10. Now, consider the elements (1 2 3 4 5) of order 5 and the transposition (2 5)(3 4) of order 2 (since $10 = 2 * 5$). Then

$R_k = \langle(1\ 2\ 3\ 4\ 5), (2\ 5)(3\ 4)\rangle = \langle\beta_1, \gamma_2\rangle$

= {i, (1 2 3 4 5), (2 5)(3 4), (1 3 5 2 4), (1 4 2 5 3), (1 5)(2 4), (1 4)(2 3), (1 5 4 3 2), (1 3)(4 5), (1 2)(3 5)}. i.e.

$R_k = \{i, \beta_1, \gamma_2, \beta_2, \beta_3, \gamma_9, \gamma_6, \beta_4, \gamma_{14}, \gamma_{13}\}$

is a subgroup of S_5 of order 10. By constructing similar subgroups, 6 subgroups of S_5 of order 10, isomorphic to the Dihedral group D_5 are obtained. i.e. R_k, $123 \le k \le 128$.

3.1.8 Subgroups of order 12

Since $12 = 2^2 * 3$, the direct product of S_2 and S_3 in S_5 is a subgroup of S_5. Hence, if T is a subgroup of S_5 of order 12, then

$T = \{i, \rho_{10}, \alpha_1, \sigma_1, \sigma_2, \rho_5, \delta_1, \rho_9, \gamma_{15}, \gamma_3, \delta_2, \gamma_{14}\}$

is a subgroup of S_5 of order 12. Hence,10 such subgroups of order 12. i.e. T_k; $129 \le k \le 138$, isomorphic to the direct product of S_2 and $S_3$are obtained.

Similarly, A_4 is obviously a subgroup of S_5, and each of the elements (1 2 3 4), (1 2 3 5), (1 2 4 5), (1 3 4 5) and (2 3 4 5) generate A_4. Thus, there are 5 such subgroups i.e. T_k; $139 \le k \le 143$ isomorphic to A_4.

3.1.9 Subgroups of order 20

The composition of elements of S_5 of order 5 with those elements of order 4 formed subgroups of S_5 of order 20 ($20 = 5 * 4$). Hence, if U is an arbitrary subgroup of S_5 of order 20, then

$U = \langle(1\ 2\ 3\ 4\ 5), (2\ 3\ 5\ 4)\rangle$

is a subgroup generated by two elements β_1 and τ_3. By considering similar elements, 6 such subgroups of order 20 are obtained. i.e. U_k; $144 \le k \le 149$, isomorphic to D_{10}.

3.1.10 Subgroups of order 24

Each of the following subset of S_5 consisting of four elements generates subgroup of S_5 of order 24. i.e. (1 2 3 4), (1 2 3 5), (1 2 4 5), (1 3 4 5) and (2 3 4 5). Hence, if V is any arbitrary subgroup of S_5 generated by any of the above elements, then V is a subgroup of order 24, i.e. V_k; $150 \le k \le 154$, isomorphic to S_4.

3.1.11 Subgroup of order 60

The only subgroup of S_5 of order 60 is the alternating group A_5, consisting of all the even permutations in S_5. Such subgroup is unique. Hence,

$A_5 = \langle(1\ 2\ 3\ 4\ 5), (1\ 2\ 3)\rangle = \langle\beta_1, \sigma_1\rangle$, i.e.

$A_5 = \{i, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}, \sigma_{20}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \beta_{17}, \beta_{18}, \beta_{19}, \beta_{20}, \beta_{21}, \beta_{22}, \beta_{23}, \beta_{24}\}$.

3.1.12 Subgroup of order 120

Every group is a subgroup of itself. Hence, the whole group S_5 is a subgroup of S_5 of order 120.

There are seven conjugacy classes corresponding to the unordered partitions of {1, 2, 3, 4, 5}. Now since cycle type determine conjugacy class and the length of a cycle is found to be the number of elements in that cycle, we notice that any conjugate of a k-cycle is again a k-cycle [10]. This is also supported by the following theorem:

Theorem 9: The conjugacy classes of any S_n are determined by cycle type. That is, if σ has cycle type (k_1, k_2, …, k_l), then any conjugate of σ has cycle type (k_1, k_2, …, k_l), and if γ is any other element of S_n with cycle type (k_1, k_2, …, k_l), then σ is conjugate to γ[10].
For the proof of Theorem 9, see [11]. We therefore use this information to derive the following table, classifying the size of conjugacy class of elements of S_5.

Table 4: Size of Conjugacy classes of elements of S_5

Element	**Partition**	**Verbal description of cycle type**	**Representative element with the cycle type**	**Size of conjugacy class**	**Formula Calculating Size of Con-jugacy Class**
i	1 + 1 + 1 + 1 + 1	five fixed points	(1) the identity element	1	$\frac{5!}{(1^5)(5!)}$
ρ_j	2 + 1 + 1 + 1	transposition: one 2-cycle, three fixed point	(1 2)	10	$\frac{5!}{[(2^1)(1!)][(1^3)(3!)]}$
γ_j	2 + 2 + 1	double transposition: two 2-cycles, one fixed point	(1 2)(3 4)	15	$\frac{5!}{[(2^2)(2!)][(1^1)(1!)]}$
σ_j	3 + 1 + 1	one 3-cycle, two fixed points	(1 2 3)	20	$\frac{5!}{[(3^1)(1!)][(1^2)(2!)]}$
δ_j	3 + 2	one 3-cycle, one 2-cycle	(1 2 3)(4 5)	20	$\frac{5!}{[(3^1)(1!)][(2^1)(1!)]}$
τ_j	4 + 1	one 4-cycle, one fixed point	(1 2 3 4)	30	$\frac{5!}{[(4^1)(1!)][(1^1)(1!)]}$
β_j	5	one 5-cycle	(1 2 3 4 5)	24	$\frac{5!}{(5^1)(1!)}$
Total				**120**	**5!**

The sum of the conjugacy classes is equal to the order of the group S_5. The center of a group G is defined to be the set of those elements that commute with every other element of G, given by Z(G) = {x : xg = gx for all g∈G}. Observed that the center of S_5 is the trivial subgroup { i}, consisting of the identity permutation. Hence, S_5 is centreless. S_5 is also almost simple group since it contains a centralizer-free simple normal subgroup, i.e. A_5. The Alternating group A_5 is simple. Hence, A_5 is the unique maximal normal subgroup of S_5.

IV. THE ACTIONS OF GROUP REPRESENTATIONS ON SIGNAL SPACE

Let $(X,\pi)=\{\phi_1,\phi_2,...,\phi_n\}$ be a signal space where π is the representative of the functions over X and let σ∈S_n withn≥ 2. Then the group S_n acts on the space Xby permuting its elements as follows:

$$(\sigma\cdot\pi)(\phi_1,\phi_2,...,\phi_n)=\pi(\phi_{\sigma(1)},\phi_{\sigma(2)},...,\phi_{\sigma(n)}) \quad 4.1.$$

Every elementσ∈S_n satisfy $\phi_i \mapsto \phi_{\sigma(i)}$ in $\pi(\phi_1,\phi_2,...,\phi_n)$.

Lemma 4.1: The function defined in Equation 4.1 is a group action of S_n on the signal space X.
Proof: Obviously, $i\cdot\pi=\pi$. Next, we show that $\sigma\cdot(\delta\cdot\pi)=(\sigma\delta)\cdot\pi$ for all $\sigma,\delta\in S_n$. Now,

$$(\sigma\cdot(\delta\cdot\pi))(\phi_1,\phi_2,...,\phi_n)=(\delta\cdot\pi)(\phi_{\sigma(1)},\phi_{\sigma(2)},...,\phi_{\sigma(n)})$$
$$=\pi(\phi_{\sigma(\delta(1))},\phi_{\sigma(\delta(2))},...,\phi_{\sigma(\delta(n))})$$
$$=\pi(\phi_{(\sigma\delta)(1)},\phi_{(\sigma\delta)(2)},...,\phi_{(\sigma\delta)(n)})$$
$$=((\sigma\delta)\cdot\pi)(\phi_{(1)},\phi_{(2)},...,\phi_{(n)}).$$

Hence, the result follows.

Lemma 4.2: Let $\sigma, \delta \in S_n$ and $(X, \pi) = \{\phi_1, \phi_2, ..., \phi_n\} \in \Re^n$ be a signal space. Then $\pi_\sigma \circ \pi_\delta = \pi_{\delta\sigma}$.

Proof: Let $\sigma, \delta \in S_n$ and $w = (\varphi_1, \varphi_2, ..., \varphi_n) \in \Re^n$ be arbitrary element of X.

Then

$$\begin{aligned}
\pi_\sigma \circ \pi_\delta(w) &= \pi_\sigma(\pi_\delta(w)) \\
&= \pi_\sigma(\pi_\delta(\varphi_1, \varphi_2, ..., \varphi_n) \\
&= \pi_\sigma(\varphi_{\delta(1)}, \varphi_{\delta(2)}, ..., \varphi_{\delta(n)}) \\
&= \pi_\sigma(\lambda_1, \lambda_2, ..., \lambda_n) \text{ where } \lambda = \varphi_{\delta(i)} \\
&= (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, ..., \lambda_{\sigma(n)}) \\
&= (\varphi_{\delta(\sigma(1))}, \varphi_{\delta(\sigma(2))}, ..., \varphi_{\delta(\sigma(n))}) \\
&= (\varphi_{(\delta\sigma)(1)}, \varphi_{(\delta\sigma)(2)}, ..., \varphi_{(\delta\sigma)(n)}) \\
&= \pi_{\delta\sigma}(\varphi_1, \varphi_2, ..., \varphi_n) \\
&= \pi_{\delta\sigma}(w).
\end{aligned}$$

Since w is arbitrary, the result is true for all w∈X.

Lemma 4.3: With G = S_n, let X be defined as above and Y be a signal space not necessarily isomorphic to X. Let $\xi(X, Y)$ be the collection of all maps from X to Y, i.e. $\xi : X \to Y$. Then the action of G on $\xi(X, Y)$ is given by the rule

$$(\pi_g \xi)(w) = \xi(gw).$$

4.2

Proof: Obviously, gw is the action of g∈G on w∈Xand $\pi_g \xi$ is a function from X to Y. To find a group action of G on $\xi(X, Y)$, note that G acts on X from the left. Now, replace g with g^{-1} in Equation 4.2 and set $(g \cdot \xi)(w) = \xi(g^{-1}w)$, then

$$\begin{aligned}
(g_1 \cdot (g_2 \cdot \xi))(w) &= (g_2 \cdot \xi)(g_1^{-1}w) \\
&= \xi(g_2^{-1}(g_1^{-1}w)) \\
&= \xi((g_2^{-1}g_1^{-1})w) \\
&= \xi((g_1 g_2)^{-1}w) \\
&= ((g_1 g_2) \cdot \xi)(w).
\end{aligned}$$

Hence, $g_1 \cdot (g_2 \cdot \xi) = (g_1 g_2) \cdot \xi$ is a group action of G on $\xi(X, Y)$.

Example 4.4: Suppose G = S_n and choosen = 5. Then G = S_5 and $X = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\} \in \Re^5$.

Now, from

$$(\sigma \cdot (\delta \cdot \pi))(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (\delta \cdot \pi)(\phi_{\sigma(1)}, \phi_{\sigma(2)}, \phi_{\sigma(3)}, \phi_{\sigma(4)}, \phi_{\sigma(5)})$$

and Section 3,

$$\begin{aligned}
(\sigma_3 \cdot (\delta_1 \cdot \pi))(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) &= (\delta_1 \cdot \pi)(\phi_{(4)}, \phi_{(1)}, \phi_{(3)}, \phi_{(2)}, \phi_{(5)}) \\
&= (\phi_{(3)}, \phi_{(4)}, \phi_{(1)}, \phi_{(5)}, \phi_{(2)}).
\end{aligned}$$

But $\sigma_3 \cdot \delta_1 = \delta_{18}$ in Gand

$$(\delta_{18} \cdot \pi)(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (\phi_{(3)}, \phi_{(4)}, \phi_{(1)}, \phi_{(5)}, \phi_{(2)}).$$

Hence, $(\sigma_3 \cdot (\delta_1 \cdot \pi))(w) = ((\sigma_3 \delta_1) \cdot \pi))(w) = (\delta_{18} \cdot \pi)(w)$.

Example 4.5: Again, let $w = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in \Re^5$. Then $\pi_\sigma \circ \pi_\delta(w) = \pi_{\delta\sigma}(w)$.

To see this, pick $\tau_2, \gamma_{15} \in G$. Then

$$\pi_{\tau_2} \circ \pi_{\gamma_{15}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) = \pi_{\gamma_{15}}(\varphi_1, \varphi_3, \varphi_4, \varphi_5, \varphi_2)$$
$$= (\varphi_3, \varphi_1, \varphi_4, \varphi_2, \varphi_5).$$

Now, $\tau_2 \cdot \gamma_{15} = \tau_{11}$ in Gand

$\pi_{\tau_{11}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) = (\varphi_3, \varphi_1, \varphi_4, \varphi_2, \varphi_5)$.

Thus, $\pi_{\tau_2} \circ \pi_{\gamma_{15}}(w) = \pi_{\tau_{11}}(w)$.

Now, the elements of the group G act on the signal space X as functions. Hence ingeneral, if X is any signal space and G is any subgroup of S_X, then X is a G-set under the group action

$$(\sigma, \phi(t)) \mapsto \sigma(\phi(t))$$

for all $\sigma \in G$ and $\phi(t) \in X$.

Lemma 4.6: Let |G| = n such that G≅X. Then G acts on X by the left regular representation given by $(\sigma, \phi(t)) \mapsto \pi_\sigma(\phi(t)) = \sigma\phi(t)$.

Proof: Since π_σ is a left multiplication,

$$i \cdot \phi(t) = \pi_i \phi(t) = i\phi(t) = \phi(t)$$

where i is the identity element of G. Also,

$$(\sigma\delta) \cdot \phi(t) = \pi_{\sigma\delta}\phi(t) = \pi_\sigma \pi_\delta \phi(t) = \pi_\sigma(\delta\phi(t)) = \sigma \cdot (\delta \cdot \phi(t)).$$

This established the result.

Lemma 4.7: Let |G| = n such that G≅X and let Kbe a subgroupof G.Then X is aK-set under conjugation. i.e., an action $K \times X \to X$ of K on X defined by

$$(\delta, \phi(t)) \mapsto \delta(\phi(t))\delta^{-1}$$

for all $\delta \in K$ and $\phi(t) \in X$.

Proof: Clearly, the first axiom for group action is satisfied. Next, observed that

$$(\sigma\tau, \phi(t)) = \sigma\tau(\phi(t))(\sigma\tau)^{-1}$$
$$= \sigma\tau(\phi(t))(\tau^{-1}\sigma^{-1})$$
$$= \sigma(\tau(\phi(t))\tau^{-1})\sigma^{-1}$$
$$= (\sigma, (\tau, \phi(t)))$$

which shows that the second condition is also satisfied as required.

Example 4.8: Signals are regarded as functions on some discrete groups, usually identified with the group of integers Z, or its subgroups Z_p of integer's modulo p, i.e. $\varphi : Z_p \to Z_p$. Let p = 2. Then Z_p is isomorphic to S_2. Now, a binary symmetric channel is described as a model consisting of a transmitter which is capable of sending a binary signal together with a receiver.

Then one possible coding scheme is to send a signal several times so as to compare the received signals with one another. Suppose that the signal to be encoded is (1 1 0 1 0 0) into a binary 4n-tuple, let $\sigma \in S_n$. Then $\sigma : Z_2 \to Z_2$ encode (1 1 0 1 0 0) into a binary 4n-tuple as

$$(110100) \mapsto (110100110100110100110100).$$

The decoded signal depends on the function $\sigma \in S_n$. The function σis also required tobe one-to-one in order that two signals will not be encoded into the same image.

V. CONCLUSION

From abstract point of view, the symmetric group S_n is generally not a nilpotent group since it has no central series. Thus, the group S_5 is centerless and one-headed since the Alternating group A_5 is its unique maximal normal subgroup. The generated subgroup representations of S_5appear to be useful in signal processing andfor studying its Fuzzy subgroups. It is therefore concluded that the subgroup representations of S_n play an important role in signal processing.Other properties such as isomorphism classes of Sylow subgroups and the corresponding Sylow numbers and fusion systems, extended automorphism group and the lattice structure of S_5 that are not treated in this article, are recommended for further studies.

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