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Moments of Extended Erlang-Truncated Exponential Distribution based on $k-$th Lower Record Values and Characterizations

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ABSTRACT: *Recently, a new lifetime distribution, called extended Erlang-truncated exponential distribution was introduced by Okorie et al. (2017). In this paper, we present explicit expressions as well as some recurrence relations for single and product moments of $k-th$ lower record values from this distribution. The results are deduced for moments of lower record values. Further, conditional expectation, recurrence relations for single moments and truncated moments are used to characterize this distribution. At the end, we also carry out some computational work.*

KEYWORDS - *Order statistics, $k-th$ lower record values, extended Erlang-truncated exponential distribution, moments, recurrence relations, truncated moments and characterization.*

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I. INTRODUCTION

A random variable X is said to have extended Erlang-truncated exponential distribution (Okorie *et al.* (2017)) if its probability density function (pdf) is given by

$$f(x)=\alpha\beta(1-e^{-\lambda})\,e^{-\beta(1-e^{-\lambda})x}(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha-1}, \quad x\geq 0, \quad \alpha, \beta, \lambda>0 \tag{1}$$

with the distribution function (df)

$$F(x)=(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha}, \quad x\geq 0, \quad \alpha, \beta, \lambda>0. \tag{2}$$

It is easily seen that

$$\alpha\beta(1-e^{-\lambda})\,F(x)=(e^{\beta(1-e^{-\lambda})x}-1)\,f(x), \tag{3}$$

where α , β are shape parameters and λ is scale parameter.

The extended Erlang-truncated exponential distribution have several characteristics like (has tractable pdf whose shape is either decreasing or unimodal) and the failure rate function is characterized by decreasing, constant and increasing shapes that make it more sophisticated for modeling data sets from various life time characteristic especially those with early failure time characteristic. This distribution demonstrates a more reasonable fit than the other competitive distributions [Okorie *et al.* (2017)].

Let $\{X_n, \ n\geq 1\}$ be a sequence of independent identically distributed (iid) random variables with df $F(x)$ and pdf $f(x)$. The $j-$th order statistic of a sample $X_1, X_2, \ldots, X_n$ is denoted by $X_{j:n}$. For a fixed $k\geq 1$, we define the sequence $\{L_k(n), \ n\geq 1\}$ of $k-$th lower record times of $\{X_n, \ n\geq 1\}$ as follows:

$$L_k(1)=1$$

$$L_k(n+1)=\min\{j>L_k(n): X_{k:L_k(n)+k-1}>X_{k:j+k-1}\}.$$

The sequence $\{Z_n^{(k)}, \ n\geq 1\}$ with $Z_n^{(k)}=X_{k:L_k(n)+k-1}$, $n=1,2,\ldots$, is called the sequence of $k-$th lower record values of $\{X_n, \ n\geq 1\}$. For convenience, we shall also take $Z_0^{(k)}=0$. Note that for $k=1$ we

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*Corresponding author e-mail: aruke@rediffmail.com have $Z_n^{(1)} = X_{L(n)}$, $n \geq 1$, i.e. the record values of $\{X_n, \; n \geq 1\}$. Then the *pdf* of $Z_n^{(k)}$ and the joint *pdf* of $Z_m^{(k)}$ and $Z_n^{(k)}$ are as follows:

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x), \quad n \geq 1, \tag{4}$$

$$f_{Z_m^{(k)}, Z_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!}[-\ln F(x)]^{m-1} \frac{f(x)}{F(x)}$$
$$\times [\ln F(x) - \ln F(y)]^{n-m-1}[F(y)]^{k-1} f(y), \quad y < x, \; 1 \leq m < n, \; n \geq 2, \tag{5}$$

respectively, [Pawlas and Szynal (1998)].

The conditional *pdf* of $Z_n^{(k)}$ given $Z_m^{(k)} = x$, is

$$f_{Z_n^{(k)} | Z_m^{(k)}}(y | x) = \frac{k^{n-m}}{(n-m-1)!}[\ln F(x) - \ln F(y)]^{n-m-1} \left(\frac{F(y)}{F(x)} \right)^{k-1} \frac{f(y)}{F(x)}, \quad y < x. \tag{6}$$

For some recent developments on $k-$th lower record values with special reference to those arising from generalized extreme value, Gumble, inverse Pareto, inverse generalized Pareto, inverse Burr, inverse Weibull, power, uniform, Frechet and Dagum distributions, see Pawlas and Szynal (1998, 2000), Bieniek and Szynal (2002) and Kumar (2016), respectively. In this paper we mainly focus on the study of $k-$th lower record values arising from the extended Erlang-truncated exponential distribution.

II. RELATIONS FOR SINGLE MOMENTS

In this section, we derive explicit expressions and recurrence relations for single moments of $k-$th lower record values from the extended Erlang-truncated exponential distribution.

Theorem 2.1. For the extended Erlang-truncated exponential distribution as given in (2). Fix a positive integer k, for $n \geq k \geq 1$ and $j = 0, 1, \ldots,$

$$E(Z_n^{(k)})^j = \frac{(\alpha k)^n}{[\beta(1-e^{-\lambda})]^j} \sum_{p=0}^{\infty} \frac{z_p(j)}{(j+p+\alpha k)^n}. \tag{7}$$

Proof. From (4), we have

$$E(Z_n^{(k)})^j = \frac{k^n}{(n-1)!} \int_0^{\infty} x^j [-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\, dx. \tag{8}$$

Setting $t = [F(x)]^{1/\alpha}$ in (8), we find that

$$E(Z_n^{(k)})^j = \frac{(\alpha k)^n}{(n-1)![\beta(1-e^{-\lambda})]^j} \int_0^1 [-\ln(1-t)]^j (-\ln t)^{n-1} t^{\alpha k-1} dt. \tag{9}$$

On using the logarithmic expansion

$$[-\ln(1-t)]^j = \left(\sum_{p=0}^{\infty} \frac{t^p}{p} \right)^j = \sum_{p=0}^{\infty} Z_p(j) t^{j+p}, \tag{10}$$

where $Z_p(j)$ is the coefficient of t^{p+j} in the expansion of $\left(\sum_{p=0}^{\infty} \frac{t^p}{p} \right)^j$ [see Balakrishnan and Cohen (1991), p-44], (9) can be rewritten as

$$E(Z_n^{(k)})^j = \frac{(\alpha k)^n}{(n-1)![\beta(1-e^{-\lambda})]^j} \sum_{p=0}^{\infty} Z_p(j) \int_0^1 (-\ln t)^{n-1} t^{j+p+\alpha k-1} dt. \tag{11}$$

The result given in (7) can be proved in view of Gradshetyn and Ryzhik (2007, p-551), by noting that

$$\int_0^1 [-\ln x]^{\mu-1} x^{\nu-1} dx = \frac{\Gamma\mu}{\nu^{\mu}}. \tag{12}$$

Corollary 2.1. The explicit expression for single moments of lower record values from the extended Erlang-truncated exponential distribution has the form

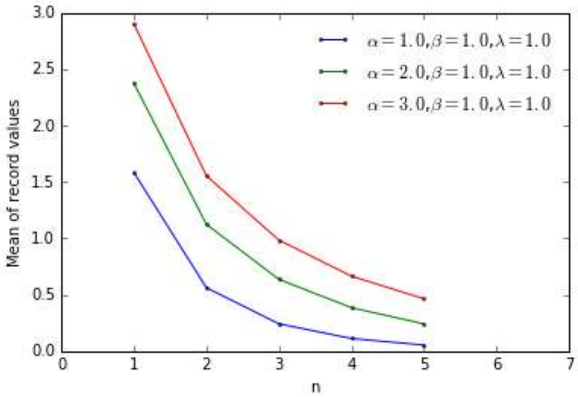
$$E(X^{j}_{L(n)}) = \frac{\alpha^{n}}{[\beta(1-e^{-\lambda})]^{j}} \sum_{p=0}^{\infty} \frac{Z_{p}(j)}{(\alpha + j + p)^{n}}\,. \tag{13}$$

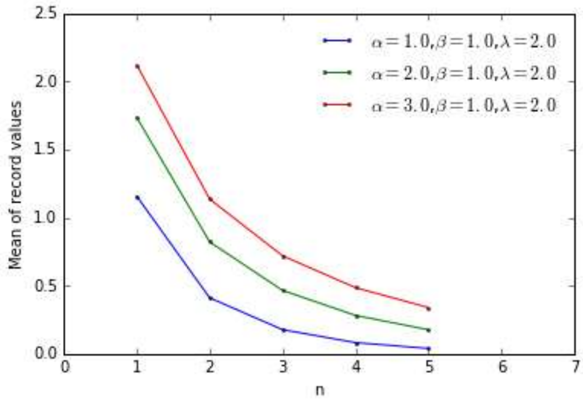
Expression (13) can be used to obtain the means of lower record values from extended Erlang-truncated exponential distribution for arbitrary chosen values of α, β, λ and various sample size $n=1,2,\ldots,5$. Some numerical computations for are given in Table 2.1.

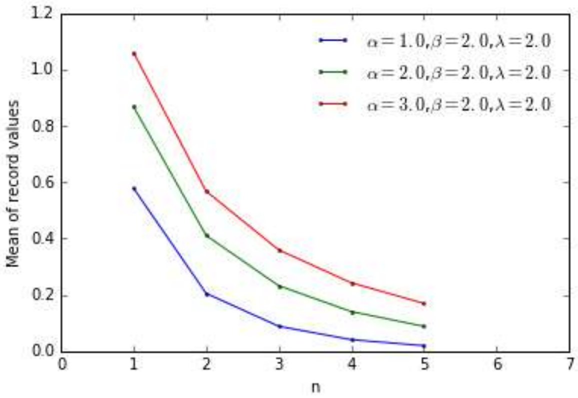
Table 2.1. Means of lower record values

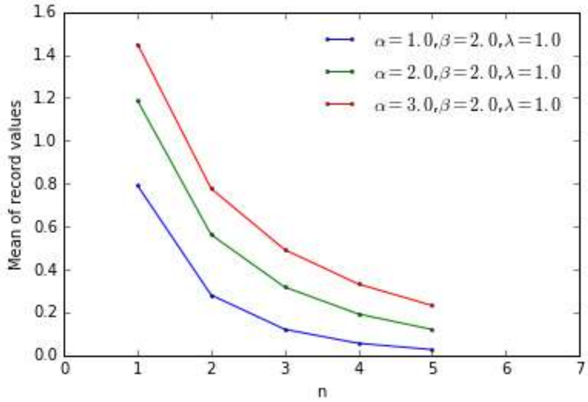
n	$\beta=1,\ \lambda=1$			$\beta=1,\ \lambda=2$		
	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=1$	$\alpha=2$	$\alpha=3$
1	1.581819	2.372649	2.899816	1.156402	1.734545	2.119935
2	0.561706	1.123412	1.553286	0.410640	0.821280	1.135544
3	0.242057	0.635803	0.983492	0.176958	0.464809	0.718990
4	0.111823	0.384924	0.664101	0.081749	0.281402	0.485496
5	0.053404	0.241211	0.463878	0.039042	0.176339	0.339122
n	$\beta=2,\ \lambda=1$			$\beta=2,\ \lambda=2$		
	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=1$	$\alpha=2$	$\alpha=3$
1	0.790909	1.186324	1.449908	0.578201	0.867272	1.059967
2	0.280853	0.561706	0.776643	0.205320	0.410640	0.567772
3	0.121028	0.317902	0.491746	0.088479	0.232405	0.359495
4	0.055911	0.192462	0.332050	0.040875	0.140701	0.242748
5	0.026702	0.120605	0.231939	0.019520	0.088169	0.169561

Behaviours of the means of record statistics from the extended Erlang truncated exponential distribution for $n=5$ and different values of parameters are presented in following figures.









Now, we obtain the recurrence relations for single moments of $k-$th lower record values from the extended Erlang-truncated exponential distribution in the following theorem.

Theorem 2.2. For the distribution given in (2) and $n \geq k \geq 1$, $j=0,1,\ldots$,

$$E(Z_{n}^{(k)})^{j} = E(Z_{n-1}^{(k)})^{j} + \frac{j}{\alpha\beta(1-e^{-\lambda})k}[E(Z_{n}^{(k)})^{j-1} - E(\phi(Z_{n}^{(k)}))], \tag{14}$$

where

$$\phi(x) = x^{j-1} e^{\beta(1-e^{-\lambda})x}.$$

Proof. In view of Bieniek and Szynal (2002), note that

$$E(Z_n^{(k)})^j - E(Z_{n-1}^{(k)})^j = -\frac{jk^{n-1}}{(n-1)!}\int_{-\infty}^{\infty} x^{j-1}[-\ln F(x)]^{n-1}[F(x)]^k\, dx. \quad (15)$$

On using (3) in (15) and rearranging the resulting expression, which gives (14).

Corollary 2.2. The recurrence relations for single moments of lower record values from the extended Erlang-truncated exponential distribution has the form

$$E(X_{L(n)}^j) = E(X_{L(n-1)}^j) + \frac{j}{\alpha\beta(1-e^{-\lambda})}[E(X_{L(n)}^{j-1}) - E(\phi(X_{L(n)}))].$$

where

$$\phi(x) = x^{j-1} e^{\beta(1-e^{-\lambda})x}.$$

III. RELATIONS FOR PRODUCT MOMENTS

In this section, we present explicit expressions and recurrence relations for product moments of $k-$th lower record values from the extended Erlang-truncated exponential distribution.

Theorem 3.1. For the distribution given in (2). Fix a positive integer $k \geq 1$, for $1 \leq m \leq n-1$ and $i,\ j = 0,1,\ldots$,

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = \frac{(\alpha k)^n}{[\beta(1-e^{-\lambda})]^{i+j}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{z_p(j) z_q(i)}{[p+\alpha(j+k)]^{n-m}[i+p+q+\alpha(j+k)]^m}. \quad (16)$$

Proof. From (5), we have

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = \frac{k^n}{(m-1)!(n-m-1)!}\int_0^{\infty} x^i [-\ln F(x)]^{m-1}\frac{f(x)}{F(x)} I(x)\, dx, \quad (17)$$

where

$$I(x) = \int_0^x y^j [-\ln F(y) + \ln F(x)]^{n-m-1}[F(y)]^{k-1} f(y)\, dy. \quad (18)$$

Taking $w = [-\ln F(y) + \ln F(x)]$ in (18), we find that

$$I(x) = \frac{1}{[\beta(1-e^{-\lambda})]^j}\int_0^{\infty} [-\ln\{1-(e^{-w}F(x))^{1/\alpha}\}]^j w^{n-m-1}[e^{-w}F(x)]^k\, dw. \quad (19)$$

On using (10) in (19), we get

$$I(x) = \sum_{p=0}^{\infty} \frac{z_p(j)[F(x)]^{\{(p/\alpha)+j+k\}}}{[\beta(1-e^{-\lambda})]^j}\int_0^{\infty} w^{n-m-1} e^{-\{(p/\alpha)+j+k\}w}\, dw. \quad (20)$$

$$= \frac{\alpha^{n-m}\Gamma(n-m)}{[\beta(1-e^{-\lambda})]^j}\sum_{p=0}^{\infty} \frac{z_p(j)[F(x)]^{\{(p/\alpha)+j+k\}}}{[p+\alpha(j+k)]^{n-m}}. \quad (21)$$

Substituting (21) in (17), we get

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = A\int_0^{\infty} x^i [-\ln F(x)]^{m-1}[F(x)]^{\{(p/\alpha)+j+k-1\}} f(x)\, dx, \quad (22)$$

where

$$A = \frac{k^n \alpha^{n-m}}{[\beta(1-e^{-\lambda})]^j (m-1)!}\sum_{p=0}^{\infty} \frac{z_p(j)}{[p+\alpha(j+k)]^{n-m}}.$$

Putting $t = [F(x)]^{1/\alpha}$ in (22) and simplifying the resulting expression after using (12), we find that

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = \frac{A\alpha^m}{[\beta(1-e^{-\lambda})]^i}\sum_{q=0}^{\infty} z_q(i)\int_0^1 t^{i+p+q+\alpha(j+k)-1}(-\ln t)^{m-1}\, dt$$

and hence the required expression given in (16).

Remark 3.1. At $j=0$ in (16), we have

$$E(Z_m^{(k)})^i = \frac{(\alpha k)^n}{[\beta(1-e^{-\lambda})]^i} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{z_p(0)\, z_q(i)}{(p+\alpha k)^{n-m}(i+p+q+\alpha k)^m} . \tag{23}$$

In view of Shawky and Bakoban (2008), by noting that

$$z_p(0)=1, \quad p=0 \quad \text{and} \quad z_p(0)=0\,, \quad p>0\,. \tag{24}$$

Making use of (24) in (23), we get

$$E(Z_m^{(k)})^i = \frac{(\alpha k)^m}{[\beta(1-e^{-\lambda})]^i} \sum_{q=0}^{\infty} \frac{z_q(i)}{(i+q+\alpha k)^m} ,$$

which is the exact expression for single moments from extended Erlang-truncated exponential distribution as obtained in (7).

Corollary 3.1. For $k=1$, in (16), the explicit expression for product moments of lower record values from extended Erlang-truncated exponential distribution has the form

$$E[X_{L(m)}^{\,i} X_{L(n)}^{\,j}] = \frac{\alpha^n}{[\beta(1-e^{-\lambda})]^{i+j}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{z_p(j)\, z_q(i)}{[p+\alpha(j+1)]^{n-m}[i+p+q+\alpha(j+1)]^m} .$$

Following Theorem contains the recurrence relations for product moments of $k-$th lower record values from the extended Erlang-truncated exponential distribution.

Before coming to the main result we shall prove the following Lemma.

Lemma 3.1. Fix a positive integer $k\ge 1$, for $1\le m\le n-2$ and $i, j = 0,1,\ldots$

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] - E[(Z_m^{(k)})^i (Z_{n-1}^{(k)})^j] = -\frac{jk^{n-1}}{(m-1)!(n-m-1)!} \int_\alpha^\beta \int_\alpha^x x^i y^{j-1}$$
$$\times [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^k \, dy\, dx\,. \tag{25}$$

Proof. From (5), we have

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] - E[(Z_m^{(k)})^i (Z_{n-1}^{(k)})^j] = \frac{k^n}{(m-1)!(n-m-1)!}$$
$$\times \int_\alpha^\beta \int_\alpha^x x^i y^j [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-2}$$
$$\times [F(y)]^{k-1} f(y) \left\{ [\ln F(x) - \ln F(y)] - \frac{(n-m-1)}{k} \right\} dy\, dx\,. \tag{26}$$

Let

$$h(x,y) = \frac{1}{k} [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^k \tag{27}$$

$$\frac{\partial}{\partial y} h(x,y) = [\ln F(x) - \ln F(y)]^{n-m-2} [F(y)]^{k-1} f(y)$$
$$\times \left\{ [\ln F(x) - \ln F(y)] - \frac{(n-m-1)}{k} \right\}. \tag{28}$$

Taking into account the value of (28) in (26), we find that

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] - E[(Z_m^{(k)})^i (Z_{n-1}^{(k)})^j] = \frac{k^n}{(m-1)!(n-m-1)!}$$
$$\times \int_\alpha^\beta x^i [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} \left\{ \int_\alpha^x y^j \frac{\partial}{\partial y} h(x,y)\, dy \right\} dx\,. \tag{29}$$

Now, in view of (27)

$$\int_\alpha^x y^j \frac{\partial}{\partial y} h(x,y)\, dy = -\frac{j}{k} \int_\alpha^x y^{j-1} [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^k \, dy\,. \tag{30}$$

On substituting (30) in (29) and simplifying, the required result is obtained.

Theorem 3.2. For the distribution given in (2) and $m \geq 1$, $m \geq k$ and $i, j = 0,1,\ldots$

$$E[(Z_m^{(k)})^i (Z_{m+1}^{(k)})^j] = E[(Z_m^{(k)})^{i+j}] + \frac{j}{\alpha\beta(1-e^{-\lambda})k}$$
$$\times \{E(Z_m^{(k)})^i (Z_{m+1}^{(k)})^{j-1} - E(\phi(Z_m^{(k)} Z_{m+1}^{(k)}))\} \tag{31}$$

and for $1 \leq m \leq n-2$, $i, j = 0,1,\ldots$

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = E[(Z_m^{(k)})^i (Z_{n-1}^{(k)})^j] + \frac{j}{\alpha\beta(1-e^{-\lambda})k}$$
$$\times \{E(Z_m^{(k)})^i (Z_n^{(k)})^{j-1} - E(\phi(Z_m^{(k)} Z_n^{(k)}))\}, \tag{32}$$

where

$$\phi(x, y)) = x^i y^{j-1} e^{\beta(1-e^{-\lambda})y}.$$

Proof. On using (3) in Lemma 3.1, we have

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] - E[(Z_m^{(k)})^i (Z_{n-1}^{(k)})^j] = \frac{jk^{n-1}}{(m-1)!(n-m-1)!\alpha\beta(1-e^{-\lambda})}$$
$$\times \int_0^\infty \int_0^x x^i y^{j-1} [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)\, dy\, dx$$
$$- \frac{jk^{n-1}}{(m-1)!(n-m-1)!\alpha\beta(1-e^{-\lambda})} \int_0^\infty \int_0^x x^i y^{j-1} e^{\beta(1-e^{-\lambda})y} [-\ln F(x)]^{m-1}$$
$$\times \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)\, dy\, dx,$$

and hence the result given in (32).

Now putting $n = m+1$ in (32) and noting that $E[(Z_m^{(k)})^i (Z_m^{(k)})^j] = E(Z_m^{(k)})^{i+j}$, yields (31).

Remark 3.2. At $i = 0$, Theorem 3.2 reduces to Theorem 2.2.

Corollary 3.2. The recurrence relation for product moments of lower record values from extended Erlang-truncated exponential distribution has the form

$$E(X_{L(m)}^i X_{L(n)}^j) = E(X_{L(m)}^i X_{L(n-1)}^j) + \frac{j}{\alpha\beta(1-e^{-\lambda})}$$
$$\times \{E(X_{L(m)}^i X_{L(n)}^{j-1}) - E[\phi(X_{L(m)} X_{L(n)})]\}.$$

IV. CHARACTERIZATION

This section contains the characterizations of extended Erlang-truncated exponential distribution by using, recurrence relations, conditional expectation and truncated moments.

Following Theorems contain characterizations of extended Erlang-truncated exponential distribution by recurrence relation for the single moments and conditional expectation of $k-$th lower record values.

Theorem 4.1. Fix a positive integer $k \geq 1$ and let j be a non-negative integer, a necessary and sufficient condition for a random variable X to be distributed with df given by (2) is that

$$E(Z_n^{(k)})^j = E(Z_{n-1}^{(k)})^j + \frac{j}{\alpha\beta(1-e^{-\lambda})k} [E(Z_n^{(k)})^{j-1} - E(\phi(Z_n^{(k)}))], \tag{33}$$

for $n = 1, 2, \ldots,$ $n \geq k$.

Proof. The necessary part is proved in (14). On the other hand if the recurrence relation (33) is satisfied, then on using (4), we have

$$\frac{k^n}{(n-1)!} \int_0^\infty x^j [-\ln F(x)]^{n-1} [F(x)]^{k-1} f(x)\, dx - \frac{k^{n-1}}{(n-2)!} \int_0^\infty x^j [-\ln F(x)]^{n-2} [F(x)]^{k-1} f(x)\, dx$$

$$= \frac{j}{\alpha\beta(1-e^{-\lambda})}\left[\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\,dx\right.$$
$$\left. -\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}e^{\beta(1-e^{-\lambda})x}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\,dx\right]. \tag{34}$$

$$\frac{k^n}{(n-1)!}\int_0^\infty x^j[-\ln F(x)]^{n-2}[F(x)]^{k-1} f(x)\left\{-\ln F(x)-\frac{n-1}{k}\right\}dx$$
$$= \frac{j}{\alpha\beta(1-e^{-\lambda})}\left[\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\,dx\right.$$
$$\left. -\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}e^{\beta(1-e^{-\lambda})x}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\,dx\right].$$

Let

$$h(x) = \frac{1}{k}[-\ln F(x)]^{n-1}[F(x)]^k . \tag{35}$$

$$h'(x) = [-\ln F(x)]^{n-2}[F(x)]^{k-1} f(x)\left\{-\ln F(x)-\frac{n-1}{k}\right\}. \tag{36}$$

Thus

$$\frac{k^n}{(n-1)!}\int_0^\infty x^j h'(x)dx = \frac{j}{\alpha\beta(1-e^{-\lambda})}\left[\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\,dx\right.$$
$$\left. -\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}e^{\beta(1-e^{-\lambda})x}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\,dx\right]. \tag{37}$$

Integrating the left hand side of (37) by parts and using the value of $h(x)$ from (35), we find that

$$-\frac{j\,k^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}[-\ln F(x)]^{n-1}[F(x)]^{k-1}\left\{F(x)-\frac{(e^{\beta(1-e^{-\lambda})x}-1)}{\alpha\beta(1-e^{-\lambda})} f(x)\right\}dx = 0 . \tag{38}$$

Now applying a generalization of the Müntz-Szász Theorem (see for example Hwang and Lin (1984)) to (38), we get

$$\frac{f(x)}{F(x)} = \frac{\alpha\beta(1-e^{-\lambda})}{(e^{\beta(1-e^{-\lambda})x}-1)} . \tag{39}$$

Integrating both the sides of (39) with respect to x between $(0, y)$, the sufficient part is proved.

Remark 4.1. If $k=1$ in (33), we obtain the following characterization of the of extended Erlang-truncated exponential distribution based on lower records

$$E(X_{L(n)}^{j}) = E(X_{L(n-1)}^{j}) + \frac{j}{\alpha\beta(1-e^{-\lambda})}\{E(X_{L(n)}^{j-1}) - E\phi(X_{L(n)})\}$$

for $n=1,2,\ldots$.

Theorem 4.2. Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $F(0)=0$ and $0 \le F(x) \le 1$ for all $x>0$, then

$$E[\xi(Z_n^{(k)})\,|\,(Z_l^{(k)})=x] = \xi(x)\left(\frac{k}{k+1}\right)^{n-l}, \quad l=m,\ m+1,\ m\ge k \tag{40}$$

if and only if

$$F(x) = (1-e^{-\beta(1-e^{-\lambda})x})^{\alpha}, \quad x \ge 0, \ \alpha, \beta, \lambda > 0,$$

where

$$\xi(x) = (1-e^{-\beta(1-e^{-\lambda})x})^{\alpha} .$$

Proof. From (6), we have

$$E[\xi(Z_n^{(k)}) \mid (Z_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!}\int_0^x (1-e^{-\beta(1-e^{-\lambda})y})^{\alpha}[\ln F(x) - \ln F(y)]^{n-m-1}$$

$$\times\left(\frac{F(y)}{F(x)}\right)^{k-1}\frac{f(y)}{F(x)}dy. \tag{41}$$

By setting $t = \frac{F(y)}{F(x)} = \frac{(1-e^{-\beta(1-e^{-\lambda})y})^{\alpha}}{(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha}}$ from (2) in (41), we have

$$E[\xi(Z_n^{(k)}) \mid (Z_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!}\xi(x)\int_0^1 [-\ln t]^{n-m-1} t^k dt. \tag{42}$$

(40) can be proved in view (12).
To prove sufficient part, we have

$$\frac{k^{n-m}}{(n-m-1)!}\int_0^x (1-e^{-\beta(1-e^{-\lambda})y})^{\alpha}[\ln F(x) - \ln F(y)]^{n-m-1}$$

$$\times[F(y)]^{k-1} f(y)dy = [F(x)]^k g_{n|m}(x), \tag{43}$$

where

$$g_{n|m}(x) = (1-e^{-\beta(1-e^{-\lambda})x})^{\alpha}\left(\frac{k}{k+1}\right)^{n-m}.$$

Differentiating (43) both sides with respect to x, we get

$$\frac{k^{n-m} f(x)}{F(x)(n-m-2)!}\int_0^x (1-e^{-\beta(1-e^{-\lambda})y})^{\alpha}[\ln F(x) - \ln F(y)]^{n-m-2}$$

$$\times[F(y)]^{k-1} f(y)dy = g'_{n|m}(x)[F(x)]^k + k\, g_{n|m}(x)[F(x)]^{k-1} f(x)$$

or

$$k\, g_{n|m+1}(x)[F(x)]^{k-1} f(x) = g'_{n|m}(x)[F(x)]^k + k\, g_{n|m}(x)[F(x)]^{k-1} f(x).$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{g'_{n|m}(x)}{k[g_{n|m+1}(x) - g_{n|m}(x)]}.$$

$$= \frac{(e^{\beta(1-e^{-\lambda})x} - 1)}{\alpha\beta(1-e^{-\lambda})}, \tag{44}$$

where

$$g'_{n|m}(x) = \alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha-1}\left(\frac{k}{k+1}\right)^{n-m},$$

$$g_{n|m+1}(x) - g_{n|m}(x) = \frac{1}{k}(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha}\left(\frac{k}{k+1}\right)^{n-m}.$$

Integrating both the sides of (44) with respect to x between $(0, y)$, the sufficiency part is proved.

Remark 4.2. If $k=1$, in (40), we obtain the following characterization of the extended Erlang-truncated exponential distribution based on lower record values.

$$E[\xi(X_{L_n}) \mid X_{L_l} = x] = \xi(x)(1/2)^{n-l}, \quad l = r, \quad r+1.$$

Following Theorem contains characterization of extended Erlang-truncated exponential distribution by truncated moments.

Theorem 4.3. Suppose an absolutely continuous (with respect to Lebesgue measure) random variable X has the df $F(x)$ and pdf $f(x)$ for $0 < x < \infty$, such that $f'(x)$ and $E(X \mid X \le x)$ exist for all x, $0 < x < \infty$, then

$$E(X \mid X \le x) = g(x)\eta(x), \tag{45}$$

where

$$\eta(x)=\frac{f(x)}{F(x)} \qquad \text{and}$$

$$g(x)=\frac{x(1-e^{-\beta(1-e^{-\lambda})x})}{\alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}}-\frac{\int_0^x(1-e^{-\beta(1-e^{-\lambda})u})^{\alpha}du}{\alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha-1}}$$

if and only if

$$f(x)=\alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha-1}, \quad x\geq 0, \quad \alpha,\ \beta,\ \lambda>0.$$

Proof. From (1), we have

$$E(X \mid X\leq x)=\frac{\alpha\beta(1-e^{-\lambda})}{F(x)}\int_0^x u\, e^{-\beta(1-e^{-\lambda})u}(1-e^{-\beta(1-e^{-\lambda})u})^{\alpha-1}du. \tag{46}$$

Integrating (46) by parts, taking '$e^{-\beta(1-e^{-\lambda})u}(1-e^{-\beta(1-e^{-\lambda})u})^{\alpha-1}$' as the part to be integrated and rest of the integrand for differentiation, we get

$$E(X \mid X\leq x)=\frac{1}{F(x)}\left\{x(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha}-\int_0^x(1-e^{-\beta(1-e^{-\lambda})u})^{\alpha}du\right\}. \tag{47}$$

Multiplying and dividing by $f(x)$ in (47), we have the result given in (45).

To prove sufficient part, we have from (45)

$$\frac{1}{F(x)}\int_0^x u\, f(u)\,du=\frac{g(x)f(x)}{F(x)}$$

or

$$\int_0^x u\, f(u)du=g(x)f(x). \tag{48}$$

Differentiating (48) on both sides with respect to x, we find that

$$x\, f(x)=g'(x)f(x)+g(x)f'(x).$$

Therefore,

$$\frac{f'(x)}{f(x)}=\frac{x-g'(x)}{g(x)}. \qquad \text{[Ahsanullah } et\ al. \text{ (2016)]}$$

$$=-\beta(1-e^{-\lambda})+\frac{(\alpha-1)\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}}{(1-e^{-\beta(1-e^{-\lambda})x})}, \tag{49}$$

where

$$g'(x)=x+g(x)\left(\beta(1-e^{-\lambda})-\frac{(\alpha-1)\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}}{(1-e^{-\beta(1-e^{-\lambda})x})}\right).$$

Integrating both the sides in (49) with respect to x, we get

$$f(x)=ce^{-\beta(1-e^{-\lambda})x}(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha-1}.$$

It is known that

$$\int_0^{\infty}f(x)dx=1.$$

Thus,

$$\frac{1}{c}=\int_0^{\infty}e^{-\beta(1-e^{-\lambda})x}(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha-1}dx=\frac{1}{\alpha\beta(1-e^{-\lambda})},$$

which proved that

$$f(x)=\alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}(1-e^{-\beta(1-e^{-\lambda})x})^{\alpha-1}, \quad x\geq 0, \quad \alpha,\ \beta,\ \lambda>0.$$

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REFERENCES

[1]. I.E. Okorie, A.C. Akpanta, J. Ohakwe and D.C.Chikezie, The extended Erlang-truncated exponential distribution: properties and application to rainfall data. *Heliyon*, **3,** 2017, 1-24.

[2]. P. Pawlas and D. Szynal, Relations for single and product moments of $k-$th record values from exponential and Gumbel distributions. *J. Appl. Statist. Sci.*, **7**, 1998, 53-62.

[3]. P. Pawlas and D. Syznal, Recurrence relations for single and product moments of $k-$th lower record values from the inverse distributions of Pareto's type and characterizations. *Discuss. Math. Probab. Stat.*, **20,** 2000, 223-231.

[4]. M. Bieniek and D. Syznal, Recurrence relations for distribution functions and moments of $k-$th record values. *J. Math. Sci.* (N.Y.), **111**, 2002, 3511-3518.

[5]. D. Kumar, $k-$th record values from Dagum distribution and characterization. *Discuss. Math. Probab. Stat.*, **36,** 2016, 25-41.

[6]. N. Balakrishnan and A.C. Cohen, *Order Statistics and Inference: Estimation Methods.* Academic Press, San Diego, 1991.

[7]. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*. Academic Press, New York, 2007.

[8]. A.I. Shawky and R.A. Bakoban, Characterization from exponentiated gamma distribution based on record values. *J. Stat. Theory Appl.*, **7**, 2008, 263-277.

[9]. J.S. Hwang and G.D. Lin, On a generalized moments problem II. *Proc. Amer. Math. Soc.*, **91**, 1984, 577-580.

[10]. M. Ahsanullah, M. Shakil and G.M.B. Kibria, Characterization of continuous distributions by truncated moment, *Journal of Modern Applied Statistical Methods*, **15,** 2016, 316-331.

B. Singh, "Moments of Extended Erlang-Truncated Exponential Distribution based on $k-$th Lower Record Values and Characterizations". International Journal of Mathematics and Statistics Invention (IJMSI), vol. 06, no. 03, 2018, pp. 65–74.