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Implicit Hybrid Block Six-Step Second Derivative Backward Differentiation Formula For The Solution Of Stiff Ordinary Differential Equations.

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***ABSTRACT**: In this paper, we present a higher order of implicit hybrid block second derivative backward differentiation formula for the solution of stiff initial value problems in ordinary differential equation. The developed scheme have shown to be A-stable and the graphical plots are displayed as figures while the performance of the schemes are very excellent compared to some numerical method of the same order.*
***KEYWORDS**: Hybrid block methods, Second Derivative, Collocation and interpolation, A-stability, Stiff equations*

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I INTRODUCTION

The development of numerical methods for approximating the solution of initial value problems of the form

$$\frac{dy}{dx} = f(x,y), \quad (a \le x \le b), \quad y(0) = y_0 \qquad (1)$$

Has attracted considerable attention in the recent decades and many individuals have shown interest in constructing efficient methods with good stability properties for the numerical integration of (1). Fatunla (1998) note that mathematical formulation of new models of physical situations in engineering and sciences often lead to systems of the form (1) and as such there was need to generate techniques to conveniently cope with these type of problem. Moreover, many numerical integration schemes to generate the numerical solution to problems of the form(1) have been proposed by several authors. The single-step include those developed by Fatunla(1986), Aashikpelokhai(1991), Ibijola(1997, 2000), Ibijola and Ogunrinde(2010), Ibijola and Sunday (2010,2011), Ibijola,Bamisile and Sunday (2011),to mention a few. Block methods for solving ordinary differential equations have initially have been proposed by Milne (1953).). Block methods have been considered by various authors among who are A-stable implicit one block methods with higher orders by Shampine and Watts (1972). The linear multistep methods(LMMs) generates discrete schemes which are used to solve first-order ordinary differential equations. Other researchers have introduced the continuous LMM usimg continuous collocation and interpolation to the approach leading to the development of the continuous LMMs of the form.

$$y(x) = \sum_{j=0}^{k} \alpha_j y_{n+j} + h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (2)$$

Where α_j and β_j are expressed as continuous functions of x and are at least differentiable once Awoyemi *et al* (2014). According to Okunuga and Ehigie (2009), the existing methods of deriving the linear multistep methods (LMMs) in discrete form include the interpolation approach, numerical integration, Taylor series expansion and through the determination of the order of linear multistep methods LMM. Continuous collocation and interpolation technique is now widely used for the derivation of LMMs, hybrid block methods. We develop a continuous hybrid second derivative block backward differentiation formula base on interpolation and collocation for the solution of stiff ordinary differential equations with constants step size.

II DEVELOPMET OF THE METHODS

The concept is to approximate the analytic solution $y(x)$ of (1) in the partition $\pi[a,b]$=$[a = x_0 < x_1 < \cdots < x_n = b]$ of the integration interval$[a,b]$ by a power series polynomial of the form.

$$y(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_p x^p = \sum_{j=0}^{p} a x^j \qquad (3)$$

Where $p = r + s - 1$
Which is twice-continuously differential function of $y(x)$.

$$y^{'(x)} = \sum_{j=0}^{r+s-1} j a_j x^{j-1} \qquad (4)$$

with the second derivative given by

$$y^{''(x)} = \sum_{j=0}^{r+s-1} j(1-j) a_j x^{j-2} \qquad (5)$$

where $x \in [a, b]$, the $a's$are real unknown parameters to be determined and $r + s$ is the sum of the number of collocation and interpolation points.Our aim is to utilize not only the interpolation point $\{x_i\}$but also several collocation point on the interpolation interval and to fit $y(x)$ for $y'(x)$ and $y''(x)$ and interpolate (3) at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ and collocate at (4) at $x = x_{n+5}, x_{n+\frac{11}{2}}, x_{n+6}$ where t and m are the point of interpolation and gives the steps $k = 6$ collocation and specifically for this scheme following system on non-linear equations to give the systems of equation.

$$y(x) = \sum_{j=0}^{10} a x^j \qquad (6)$$

$$y^{'(x)} = \sum_{j=0}^{10} j a_j x^{j-1} = f_{n+j} \qquad (7)$$

$$y^{''(x)} = \sum_{j=0}^{10} j(1-j) a_j x^{j-2} = g_{n+j} \qquad (8)$$

And these systems of equation written in matrix form $DC = I$ as
$D =$

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 & x_{n+1}^{10} \\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 & x_{n+2}^{10} \\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 & x_{n+3}^{10} \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 & x_{n+4}^{10} \\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 & x_{n+5}^{10} \\
0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 & 8x_{n+5}^7 & 9x_{n+5}^8 & 10x_{n+5}^9 \\
0 & 1 & 2x_{n+\frac{11}{2}} & 3x_{n+\frac{11}{2}}^2 & 4x_{n+\frac{11}{2}}^3 & 5x_{n+\frac{11}{2}}^3 & 6x_{n+\frac{11}{2}}^5 & 7x_{n+\frac{11}{2}}^6 & 8x_{n+\frac{11}{2}}^7 & 9x_{n+\frac{11}{2}}^8 & 10x_{n+\frac{11}{2}}^9 \\
0 & 1 & 2x_{n+6} & 3x_{n+6}^2 & 4x_{n+6}^3 & 5x_{n+6}^3 & 6x_{n+6}^5 & 7x_{n+6}^6 & 8x_{n+6}^7 & 9x_{n+6}^8 & 10x_{n+6}^9 \\
0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 15x_{n+5}^3 & 30x_{n+5}^3 & 42x_{n+5}^5 & 56x_{n+5}^7 & 72x_{n+5}^7 & 90x_{n+5}^6 \\
0 & 0 & 2 & 6x_{n+6} & 12x_{n+6}^2 & 15x_{n+6}^3 & 30x_{n+6}^3 & 42x_{n+6}^5 & 56x_{n+6}^7 & 392x_{n+6}^7 & 90x_{n+6}^6
\end{bmatrix} \qquad (9)$$

Similarly, we generate the continuous formulation of the new six step method as:
$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + \alpha_4(x)y_{n+4} + \alpha_5(x)y_{n+5} = h[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4} + \beta_5(x)f_{n+5} + \beta_{\frac{11}{2}}(x)f_{n+\frac{11}{2}} + \beta_6(x)f_{n+6}] + h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+1} + \gamma_2(x)g_{n+2} + \gamma_3(x)g_{n+3} + \gamma_4(x)g_{n+4} + \gamma_5(x)g_{n+5} + \gamma_6(x)g_{n+6}]$
(10)
Evaluating (10) at the following points at$x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+11/2}$ and x_{n+6} yields the following discrete methods which constitute the new three step block method.

$$y_n - \frac{1993213738125}{286998569032}y_{n+1} + \frac{1067332464375}{35874821129}y_{n+2} - \frac{28392952574375}{251123747903}y_{n+3} + \frac{149453022060000}{251123747903}y_{n+4} - \frac{1016307667706349}{2008989983224}y_{n+5} = \frac{15h}{502247495806}\left[3656048697273f_{n+5} - 24765907968000f_{n+\frac{11}{2}} + 6528463361500f_{n+6}\right] + \frac{75h^2}{251123747903}[392958026g_n + 1240595102637g_{n+5} - 112230609200g_{n+6}]$$

$$y_n + \frac{746837840375}{166494223784}y_{n+1} - \frac{731630635250}{20811777973}y_{n+2} + \frac{2791061817500}{20811777973}y_{n+3} - \frac{1992307001125}{2973111139}y_{n+4} + \frac{94180410540841}{166494223784}y_{n+5} = -\frac{15h}{41623555946}\left[273744238057f_{n+5} - 2189745766400f_{n+\frac{11}{2}} + 571809599200f_{n+6}\right] + \frac{225h^2}{20811777973}[196479013g_{n+1} - 37233930214g_{n+5} + 3262881012g_{n+6}]$$

$$y_n - \frac{155249227125}{4692969728}y_{n+1} - \frac{4232363125}{1759863648}y_{n+2} + \frac{199147232375}{586621216}y_{n+3} - \frac{1129310609625}{586621216}y_{n+4} + \frac{22809448731191}{14078909184}y_{n+5} = \frac{5h}{1173242432}\left[48703839707f_{n+5} - 502270156800f_{n+\frac{11}{2}} + 129590310900f_{n+6}\right] - \frac{25h^2}{586621216}[982359065g_{n+2} + 26203670167g_{n+5} - 2206554615g_{n+6}]$$

$$y_n - \frac{3603875625}{163850224}y_{n+1} + \frac{467625750}{1239313}y_{n+2} - \frac{47312838250}{194572141}y_{n+3} - \frac{679448104875}{194572141}y_{n+4} + \frac{10518859688619}{3113154256}y_{n+5} = -\frac{15h}{778288564}\left[17264541963f_{n+5} - 226757836800f_{n+\frac{11}{2}} + 57501934400f_{n+6}\right] + \frac{75h^2}{389144282}[1964790130g_{n+3} - 12144555243g_{n+5} + 971351960g_{n+6}]$$

$$y_n - \frac{25443306125}{1366585384}y_{n+1} + \frac{35290185875}{170823173}y_{n+2} - \frac{443911186625}{170823173}y_{n+3} - \frac{32747611000}{170823173}y_{n+4} + \frac{5912856014741}{1366585384}y_{n+5} = -\frac{15h}{341646346}\left[23226235457f_{n+5} - 227417497600f_{n+\frac{11}{2}} + 54685160300f_{n+6}\right] - \frac{225h^2}{170823173}[1964790130g_{n+4} + 4375423559g_{n+5} - 301370700g_{n+6}]$$

$$y_{n+\frac{11}{2}} + \frac{63565299}{64382242979840}y_n - \frac{848588125}{51505794383872}y_{n+1} + \frac{963792225}{6438224297984}y_{n+2} - \frac{7420860909}{6438224297984}y_{n+3} + \frac{178758015975}{1287644859596}y_{n+4} - \frac{260841860812071}{257528971919360}y_{n+5} = \frac{693h}{12876448595968}\left[5697556557f_{n+5} + 3619594240f_{n+\frac{11}{2}} - 249110400f_{n+6}\right] + \frac{2401245h^2}{6438224297984}[118789g_{n+5} + 5070g_{n+6}]$$

$$y_{n+6} - \frac{248}{196479013}y_n + \frac{3987}{196479013}y_{n+1} - \frac{34125}{196479013}y_{n+2} + \frac{253000}{196479013}y_{n+3} - \frac{2124000}{196479013}y_{n+4} - \frac{194559627}{196479013}y_{n+5} = -\frac{180h}{196479013}\left[242293f_{n+5} + \frac{614400}{196479013}f_{n+\frac{11}{2}} + 244533f_{n+5}\right] + \frac{1800h^2}{196479013}[633g_{n+5} - 1627g_{n+6}]$$

(11)

III ANALYSIS OF THE METHODS

The methods equation (10) is a specific members of the hybrid linear multistep methods (LMMs) which can be expressed as:

$\sum_{j=0}^{k}(\alpha_j y_{n+j} - h\beta_j f_{n+j} - h^2\beta_j f_{n+j}) = 0$ (12)

This can be written symbolically as:

$$\rho(E)y_n - h^2\sigma(E) = 0 \quad (13)$$

E is the shift operator defined as $E^i y_n = y_{n+j}$ and $\rho(E)$and $\sigma(E)$ are respectively the first and second characteristics polynomial of the LMM defined as:

$$\rho(E) = \sum_{j=0}^{k} \alpha_j E^j, \qquad \sigma(E) = \sum_{j=0}^{k} \beta_j E^j \quad (14)$$

Following (Fatunla, 1991) and(Lambert,1973) we defined the local truncation error associated with (12) to be linear difference operator

$$[y(t);h] = \sum_{j=0}^{k} \alpha_j y_{n+j} - h\beta_k f_{n+k} - h^2\gamma_k g_{n+k} \quad . \quad (15)$$

Assuming that $y(t)$ is sufficiently differentiable, we can we expand the terms in (15) as a Taylor series and comparing the coefficients of h gives

$$L[y(t);h] = c_0 y(t) + c_1 h y^{'(t)} + c_2 h^2 y^{''(t)} + \cdots + c_p h^p y^p(t) + \cdots \quad (16)$$

Where the constant coefficients $C_p, p = 0, 1,2 \ldots., j = 1,2, \ldots k$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j,$$

.

.

.

$$C_q = [\frac{1}{q!}\sum_{j=0 j^q}^{k} \alpha_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\beta_j] \quad (17)$$

Where

$C_0 = C_1 = C_2 \ldots C_p = 0$ and $C_{p+1} \neq 0$. Therefore, C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{(p+1)}(t_n)$ is the principal local truncation error at appoint t_n.

Definition 1 The $k-step$ implicit hybrid block linear method (12) and the associated linear difference operator (17) are said to be to have order p if $C_0 = C_1 = C_2 \ldots C_p = 0$ and $C_{p+1} \neq 0$.

Definition 2 The term C_{p+1} is called the error constant and implies that the local truncation error for the implicit block hybrid formula is given by$C_{p+1}h^{p+1}y^{(p+1)}$.

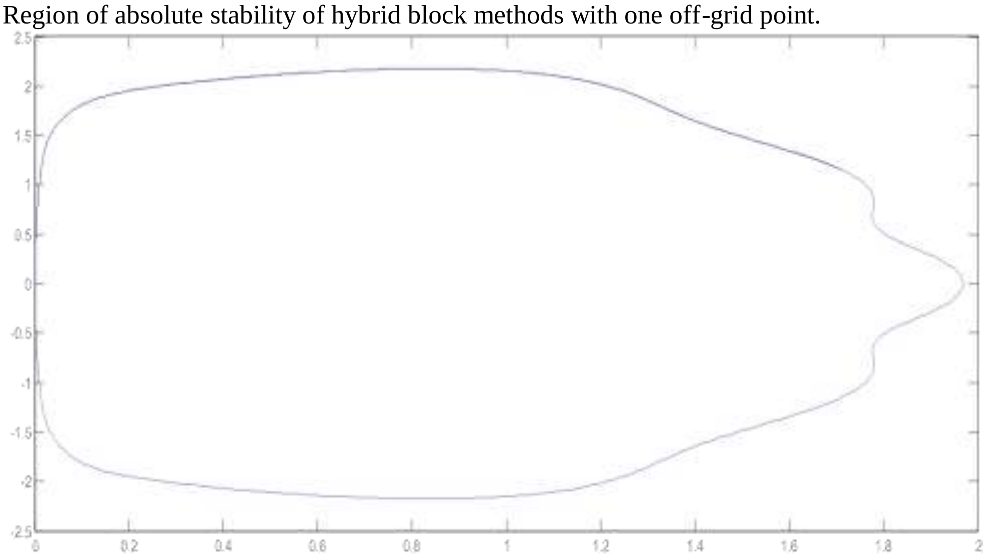
The method K = 6 is of order 10 as a block and has error constant

$$C_{11} = [\frac{2115223185675}{176791118523712}, -\frac{1688803284305}{153840662776416}, -\frac{215910422275}{8672608057344}, -\frac{342499085}{8561174204}, -\frac{146312003735}{2525449789632}, -\frac{2027487}{24238020886528}, \frac{895}{7119474824}]^T$$

Definition 3 The continuous implicit $six-step$ hybrid method (12) is said to be consistent if it satisfies the conditions order $p \geq 1$

Definition 4 A numerical method is said to be A-stable if the whole of the left-half plane $\{Z: Re(Z) \leq 0\}$ is contained in the region. $\{Z: Re(Z) \leq 1\}$Where$R(Z)$ is called the stability polynomial of the method (Lambert,1973)

Figure 1: Region of absolute stability of hybrid block methods with one off-grid point.



IV NUMERIAL EXPERIMENT

In this study, the concern is the application of developed hybrid block scheme on some non-linear systems initial value problems. The errors obtained from computed and exact values are shown in the table below.

Example 1: Stiff linear system

$$y_1' = 1.01y_1 - y_2 \qquad y_1(0) = 0$$
$$y_2' = 2y_1 - 100.005y_2 + 99.995y_3 \qquad y_2(0) = 0$$
$$y_3' = 2y_1 + 99.995y_2 - 100.005y_3 \qquad y_3(0) = 0$$

The exact solutions are:

$$y_1(x) = \mathrm{e}^{-0.01x}(\cos 2\mathrm{x} - \sin 2\mathrm{x})$$
$$y_2(x) = \mathrm{e}^{-0.01x}[(\cos 2\mathrm{x} + \sin 2\mathrm{x}) + \mathrm{e}^{-200x}$$
$$y_3(x) = \mathrm{e}^{-0.01x}(\cos 2\mathrm{x} + \sin 2\mathrm{x})$$

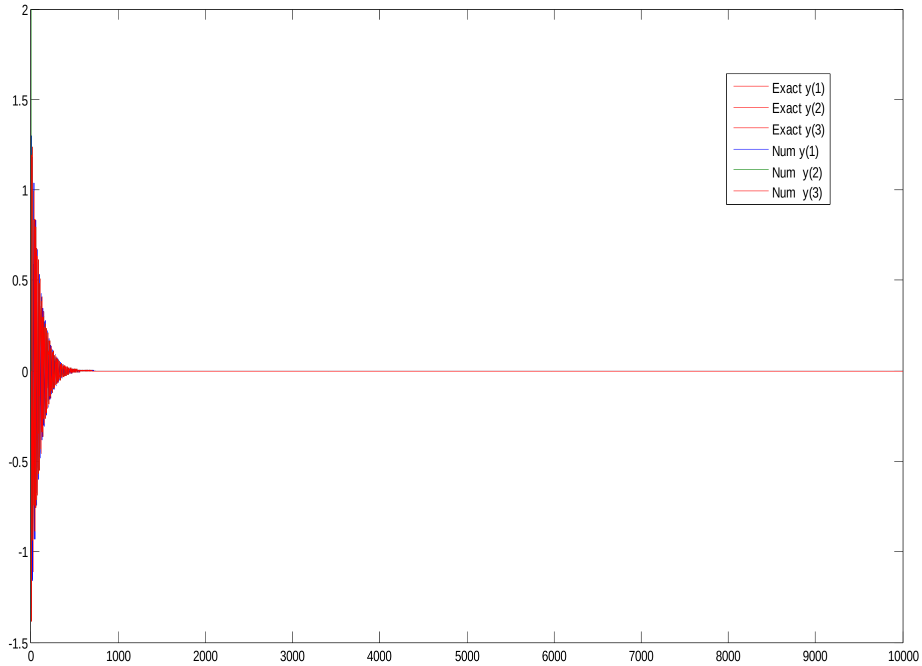


Figure 2: Solution curve for example 1 using the proposed hybrid block method

Example 2:

$$y_1' = -2y_1 + y_2 + 2sinx$$
$$y_2' = 998y_1 - 999y_2(cosx - sinx)$$

The exact solutions are:

$$y_1(x) = 2\mathrm{e}^{-x} + sinx \qquad y_1(0) = 2$$
$$y_2(x) = 2\mathrm{e}^{-x} + cosx \qquad y_2(0) = 3$$

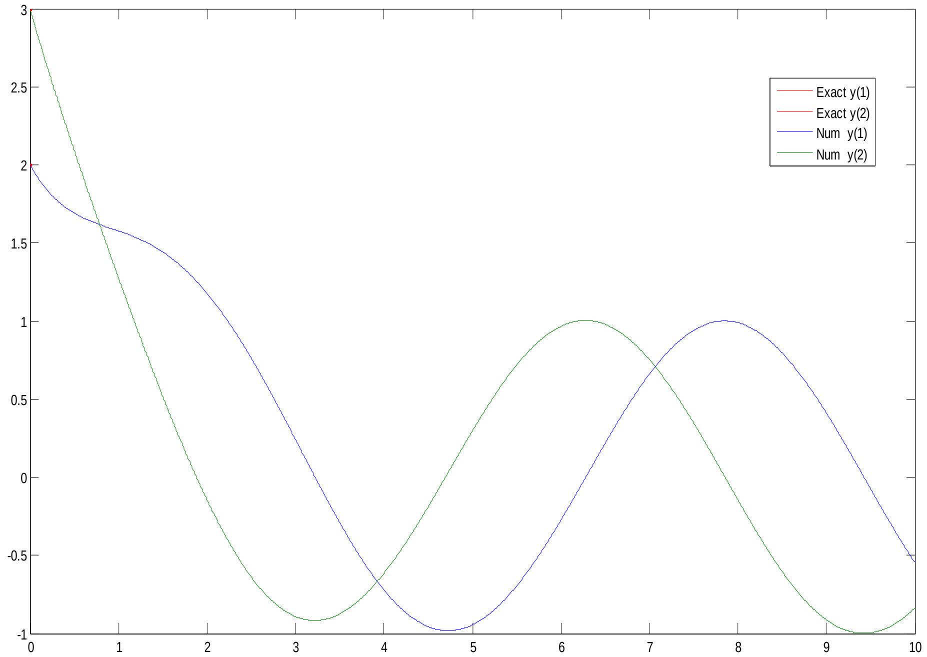


Figure 3: Solution curve for example 2 using the proposed hybrid block method

Example3:

$$y_1' = -100y_1 + 9.90y_2 \qquad y_1(0) = 1$$
$$y_2' = 0.1y_1 - y_2 \qquad y_1(0) = 10$$

The exact solutions are:

$$y_1(x) = \mathrm{e}^{-0.99x}$$
$$y_2(x) = 10\mathrm{e}^{-0.99x}$$

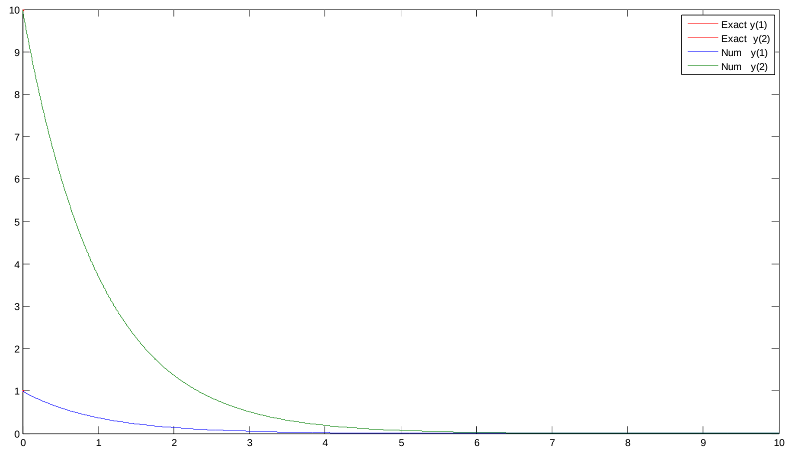


Figure 4: Solution curve for example 3 using the proposed hybrid block method

V CONCLUSION

In this research, the methods developed consist of some members in hybrid methods and have smaller error constants compare to the existing block methods of the same order. The result presented in this research shows that the graphical solution of the developed method give better accuracy and region of absolute stability plotted in figures 1 have shown to be A-stable. The hybrid block methods of step six are very promising for both stiff and non-stiff.

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