International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 5 Issue 7 || August. 2017 || PP-01-10

Birth and Death Processeswith Finished Number of States

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ABSTRACT:*A realistic description of the development of a population must obviously take into account both the births and deaths of the individuals composing it, this description can be illustrated by a simple model that is obtained by combining* ***the Birth Processes model*** *and* ***the Death Processes model****, this model is called* ***the Birth and Death Processes****, which represents an important class of Markov Processes applied in the study of waiting phenomena.The model of the Birth and Death Processes (See p.3 in* ***[1]****, p.4 and 5 in* ***[6]****) implicate the existence of* ***the famous Chapman Kolmogorov's equation****,which has been unresolved until now, considering the interest of this problem (See p.4 in* ***[1]****), whose aim to find the law of* ***the Birth and Death Processes****, we have solved in this article this equation, and thus find the law of this process according to three cases.*
Thanks to this law, many problems can be solved,usedin particularinbiology, demography, physics, sociology, statistic..., to account for the evolution of the size of a population.

KEYWORDS: *Problem modeling, Reduction of the matrix, Recurrent Sequences of Order 2andIdentification of the law.*

Date of Submission: 18-07-2017 Date of acceptance: 10-08-2017

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1. INTRODUCTION

A birth-death process is a **continuous-time Markovchain** that counts the number of particles in a system over time. In the general process with **n** current particles, a new particle is born with instantaneous **rate** λ_n and a particle dies with instantaneous **rate** μ_n.

Birth-death processes (**BDP**) have a rich history in probabilistic modeling, including applications in ecology, genetics, statistic and evolution.These probabilities exhibit their usefulness in many modeling applications since the probabilities do not depend on the possibly unobserved path taken by the process from **m** to **n** and hence make possible analyses of discretely sampled or partially observed processes.

In this article, and as part of the propagation of the use of the **BDPs law**, we will present the solutions of the first three cases of **Chapman Kolmogorov's equation**, knowing that our study will be limited to **n-states**. Hence, in the first section we will present the model of the **BDPs** and well define the **Chapman Kolmogorov's equation**, in the second section we will explain the approach followed to solve this equation, and in the third section we will find the resolution of Equation in **three cases**.

2. Presentation of the model

To model this process, our study will be limited on the states between **1** and **n**, so we identify the states of this process with: $\mathbf{P_i(t)}$ (Where**i=1,…,n**)

Knowing that:$\mathbf{P_j(t) = P(X_t = j)}$; (Where $\sum_{i=1}^{n} \mathbf{P(X_t = i) = 1}$)

(Where$\mathbf{X_t}$is **adiscrete** and **homogeneousstochasticprocess** $(\mathbf{t} \in \mathbb{R}^+)$)

Let**P(t)**the column matrix of type**(n, 1)**such as:$\mathbf{P(t)^t = (P_1(t), P_2(t), \dots, P_n(t)}$ **;** $(\mathbf{t} \in \mathbb{R}^+)$

Let:$\mathbf{P_{ij}(\Delta t) = P(X_{t+\Delta t} = j / X_t = i)}$(1) **(The transition probability** of the state**i** in the state **j)**

- Definition:

- **The Process of Birth and Death** is defined as follows (See $[\mathbf{1}]$):

$$① \mathbf{P_{ij}}(\Delta \mathbf{t}) = \begin{cases} \lambda_i \Delta t + o(\Delta t) & ; \quad \mathbf{si}:\ j = i+1 \\ \mu_i \Delta t + o(\Delta t) & ; \quad \mathbf{si}:\ j = i-1 \\ 1 - (\lambda_i + \mu_i)\Delta t + o(\Delta t) & ; \quad \mathbf{si}:\ j = i \\ o(\Delta t) & ; \quad \mathbf{si}:\ |j - i| \geq 2 \end{cases} (2)$$

- Proposition:

- Let $\mathbf{P_i(t)}$ **a process of birth and death** $(\mathbf{i = 1, \dots, n})$.
With $\lambda_{\mathbf{k}}(\mathbf{k = 1, \dots, n})$ is **the birth rate** and $\mu_{\mathbf{k}}(\mathbf{k = 1, \dots, n})$ is **the death rate.**
- Thus we have the following **system of linear differential equations**(3)(See [1]):

$$\begin{cases} P_1'(t) = -(\lambda_1 + \mu_1)P_1(t) + \mu_2 P_2(t) \\ \dots \\ P_j'(t) = \lambda_{j-1}P_{j-1}(t) - (\lambda_j + \mu_j)P_j(t) + \mu_{j+1}P_{j+1}(t)\ ;\ j = 2, \dots, n-1 \\ \dots \\ P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) \end{cases}$$

- Proof**:**

According to **the Bays formula**, we have: $\mathbf{P(X_{t+\Delta t} = j) = \sum_{i=1}^{n} P(X_{t+\Delta t} = j\ , X_t = i)}$(4)

So, $\mathbf{P(X_{t+\Delta t} = j) = \sum_{i=1}^{n} P(X_{t+\Delta t} = j\ /X_t = i)\ .\ P(X_t = i)}$

Thus we obtain the relation: ② $\mathbf{P_j(t + \Delta t) = \sum_{i=1}^{n} P_i(t)\ .\ P_{ij}(\Delta t)}$(5)

By replacing ① in ②, we will have:

$$P_j(t + \Delta t) = P_{j-1}(t).P_{j-1j}(\Delta t) + P_j(t).P_{jj}(\Delta t) + P_{j+1}(t).P_{j+1j}(\Delta t) + o(\Delta t)$$
$$P_j(t + \Delta t) = \lambda_{j-1}\Delta t.P_{j-1}(t) + \left(1 - (\lambda_j + \mu_j)\Delta t\right)P_j(t) + \mu_{j+1}\Delta t\, P_{j+1}(t) + o(\Delta t)$$
$$P_j(t + \Delta t) - P_j(t) = (\lambda_{j-1}P_{j-1}(t) - (\lambda_j + \mu_j)P_j(t) + \mu_{j+1}P_{j+1}(t))\Delta t + o(\Delta t)$$
$$\frac{P_j(t + \Delta t) - P_j(t)}{\Delta t} = \lambda_{j-1}P_{j-1}(t) - (\lambda_j + \mu_j)P_j(t) + \mu_{j+1}P_{j+1}(t) + \varepsilon(\Delta t)$$

With $\boldsymbol{\varepsilon}(\mathbf{\Delta t}) = \frac{o(\Delta t)}{\Delta t}$ (Such that: $\mathbf{\lim_{\Delta t \to 0} \varepsilon(\Delta t) = 0}$)

$$(\Delta t \to 0) \Rightarrow P_j'(t) = \lambda_{j-1}P_{j-1}(t) - (\lambda_j + \mu_j)P_j(t) + \mu_{j+1}P_{j+1}(t) \quad (6)$$

Therefore, we obtain: $\mathbf{P'(t) = A.P(t)}$(7)

With $\mathbf{A} \in \mathbf{M_N}(\mathbb{R})$ and $\mathbf{P}$ a column matrix of type $(\mathbf{n},\ \mathbf{1})$**:**

$$A = \begin{pmatrix} -(\lambda_1 + \mu_1) & \mu_2 & 0 \\ \lambda_1 & \ddots & \mu_n \\ 0 & \lambda_{n-1} & -(\lambda_n + \mu_n) \end{pmatrix}$$

This represents **the famous Chapman Kolmogorov equation** (See **[1]** and **[6]**).

3. **Resolution of equation:** $\mathbf{P'(t) = A.P(t)}$(7)

3.1 Procedure followed in solving equation:

→ Goal**:** Solve the equation: $\mathbf{P'(t) = A.P(t)}$(7)

(The matrix **A** is **non-singular** and **tridiagonal** (described above)**)**

That is, solve the following **system of linear differential equations**(3)(See [1]):

$$\begin{cases} P_1'(t) = -(\lambda_1 + \mu_1)P_1(t) + \mu_2 P_2(t) \\ \dots \\ P_j'(t) = \lambda_{j-1}P_{j-1}(t) - (\lambda_j + \mu_j)P_j(t) + \mu_{j+1}P_{j+1}(t)\ ;\ j = 2, \dots, n-1 \\ \dots \\ P_n'(t) = \lambda_{n-1}P_{n-1}(t) - \mu_n P_n(t) \end{cases}$$

So to solve this equation, we try **to diagonalize** the matrix **A**(See[3]) and[4]).
Thus, **firstly**, we will prove that the matrix **A** is **diagonalizable**.
Secondly, **the eigenvalues** of the matrix **A** and **the eigenvectors associated** will be searched.
Third, the matrix **S**, whose columns are **the eigenvectors associated** with **the eigenvalues** of the matrix **A**, will be determined.

Therefore, we have: $\mathbf{A = SDS^{-1}}$(8)

(**D** is **a diagonal matrix** with **the proper values** of the matrix **A** on her **principal diagonal**)

Next we will put **the following change of variable**: $\mathbf{Q(t) = S^{-1}P(t)}$(9)

So, $\mathbf{Q}'(\mathbf{t}) = \mathbf{S^{-1}P'(t)} = \mathbf{S^{-1}AP(t)} = \mathbf{S^{-1}ASQ(t)} = \mathbf{DQ(t)}$ **(**Because:$\mathbf{P(t)} = \mathbf{SQ(t)}$(10)**)**
Thus we solve **first** the equation:$\mathbf{Q}'(\mathbf{t}) = \mathbf{DQ(t)}$(11)

Then we conclude**P(t)** according to the relation**:**$\mathbf{P(t)} = \mathbf{SQ(t)}$(10)

3.2 Demonstration of the diagonalization of the matrix **A:**

- Lemma①:

- Let$\mathbf{A} \in \mathbf{M_n}(\mathbb{R})$a **tridiagonal symmetric real** matrix**,** such that:$\mathbf{b_i} \neq \mathbf{0}$for all **i** between **1** and **n-1**.

$$\mathbf{A} = \begin{pmatrix} \mathbf{a_1} & \mathbf{b_1} & \mathbf{0} \\ \mathbf{b_1} & \ddots & \mathbf{b_{n-1}} \\ \mathbf{0} & \mathbf{b_{n-1}} & \mathbf{a_n} \end{pmatrix}$$

-The associatedproper spaceatany **eigenvalue** of the matrix **A,** is of **dimension 1,** so**theeigenvalues** of the matrix**A** are **simples** (See **[5]**).

- Proof**:**

Since the matrix **A** is **real symmetric**, all its **propervalues** are **real**.
For any **propervalue** λ of the matrix**A,** we notice:

$$\mathbf{A}_\lambda = \mathbf{A} - \lambda\mathbf{I_n} = \begin{pmatrix} \mathbf{a_1} - \lambda & \mathbf{b_1} & \mathbf{0} \\ \mathbf{b_1} & \ddots & \mathbf{b_{n-1}} \\ \mathbf{0} & \mathbf{b_{n-1}} & \mathbf{a_n} - \lambda \end{pmatrix}$$

And$\mathbf{B}_\lambda$is the matrix **extracted** from the matrix$\mathbf{A}_\lambda$By deleting **the first line** and **the last column**, let:

$$\mathbf{B}_\lambda = \begin{pmatrix} \mathbf{b_1} & \mathbf{a_2} - \lambda & \mathbf{0} \\ \vdots & \ddots & \mathbf{a_{n-1}} - \lambda \\ \mathbf{0} & \cdots & \mathbf{b_{n-1}} \end{pmatrix}$$

Then, we have:$\mathbf{det}(\mathbf{B}_\lambda) = \Pi_{\mathbf{i=1}}^{\mathbf{n-1}}\mathbf{b_i} \neq \mathbf{0}$because: $\mathbf{b_i} \neq \mathbf{0}$for $\mathbf{i} = \mathbf{1}, \dots, \mathbf{n-1}$
So, $\mathbf{rang}(\mathbf{A}_\lambda) \geq \mathbf{n-1}$
Thus, we have:$\forall\, \lambda \in \mathbb{R}$ $\mathbf{dim}\big(\mathbf{Ker}(\mathbf{A} - \lambda\mathbf{I_n})\big) \leq \mathbf{1}$**(**Because: $\mathbf{rang}(\mathbf{A}_\lambda) + \mathbf{dim}\big(\mathbf{ker}(\mathbf{A}_\lambda)\big) = \mathbf{n}$**)**
Therefore:$\mathbf{dim}\big(\mathbf{Ker}(\mathbf{A} - \lambda\mathbf{I_n})\big) = \mathbf{1}$ (Because: $\mathbf{dim}\big(\mathbf{Ker}(\mathbf{A} - \lambda\mathbf{I_n})\big) \neq 0$)
Knowing that the matrix **A** is **diagonalizable**, noting $\lambda_1, \dots, \lambda_p$les**p real proper valuestwo to two distinct** from the matrix**A**, such that:$\mathbf{p} \leq \mathbf{n}$,
We have the following relation(**E**is a **vector space**):

$$\mathbf{E} = \oplus_{\mathbf{k=1}}^{\mathbf{p}}\mathbf{Ker}(\mathbf{A} - \lambda_\mathbf{k}\mathbf{I_n}) \quad ; \quad \mathbf{k=1, \dots, p} \quad \big(\text{With}\,\mathbf{dim}\big(\mathbf{Ker}(\mathbf{A} - \lambda_\mathbf{k}\mathbf{I_n})\big) = \mathbf{1}\big)$$

This requires:**p=n**, thus the matrix**A**has **nsimple real eigenvalues two to two distinct**.

- Lemma②:

- Let$\mathbf{n} \geq \mathbf{3},$ weconsider the matrix$\mathbf{A} \in \mathbf{M_n}(\mathbb{R})$as following:

$$\mathbf{A} = \begin{pmatrix} \mathbf{a_1} & \mathbf{c_1} & \mathbf{0} \\ \mathbf{b_2} & \ddots & \mathbf{c_{n-1}} \\ \mathbf{0} & \mathbf{b_n} & \mathbf{a_n} \end{pmatrix}$$

Such that:$\mathbf{b_k} > 0$ and $\mathbf{c_{k'}} > 0$for**k=2, …, n** and$\mathbf{k}' = \mathbf{1}, \dots, \mathbf{n-1}$
- The matrix **A** has **the same characteristic polynomial** as the matrix **T**,such that:

$$\mathbf{T} = \begin{pmatrix} \mathbf{a_1} & \sqrt{\mathbf{b_2c_1}} & \mathbf{0} \\ \sqrt{\mathbf{b_2c_1}} & \ddots & \sqrt{\mathbf{b_nc_{n-1}}} \\ \mathbf{0} & \sqrt{\mathbf{b_nc_{n-1}}} & \mathbf{a_n} \end{pmatrix}$$

(Because the matrix **A** is **similar** to the matrix **T**

- Proof:

By noting that for any **diagonal** matrix$\mathbf{D} = \mathbf{diag}(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$and any matrix$\mathbf{A} \in \mathbf{M_n}(\mathbb{R})$**o**f columns$\mathbf{C_1}, \dots, \mathbf{C_n}$and lines$\mathbf{L_1}, \dots, \mathbf{L_n}$, we have:$\mathbf{DA} = \begin{pmatrix} \boldsymbol{\alpha}_1\mathbf{L_1} \\ \dots \\ \boldsymbol{\alpha}_n\mathbf{L_n} \end{pmatrix}and\mathbf{AD} = (\boldsymbol{\alpha}_1\mathbf{C_1}, \dots, \boldsymbol{\alpha}_n\mathbf{C_n})$
It can be deduced that in the case where all $\boldsymbol{\alpha}_\mathbf{k} \neq \mathbf{0}$ $(\mathbf{k} = \mathbf{1}, \dots, \mathbf{n})$, we have:$\mathbf{D^{-1}AD} = \left(\frac{\alpha_j}{\alpha_i}\mathbf{a_{ij}}\right)_{\mathbf{1 \leq i,j \leq n}}$

Since the matrix **A** is **tridiagonal**, then we have(12):$\mathbf{D^{-1}AD} = \begin{pmatrix} \mathbf{a_1} & \frac{\alpha_2}{\alpha_1}\mathbf{c_1} & \mathbf{0} \\ \frac{\alpha_1}{\alpha_2}\mathbf{b_2} & \ddots & \frac{\alpha_n}{\alpha_{n-1}}\mathbf{c_{n-1}} \\ \mathbf{0} & \frac{\alpha_{n-1}}{\alpha_n}\mathbf{b_n} & \mathbf{a_n} \end{pmatrix}$

And defining the sequence$(\boldsymbol{\alpha_k})_{1\leq k\leq n}$by(13):$\begin{cases} \boldsymbol{\alpha_1 = 1} \\ \boldsymbol{\alpha_k = \alpha_{k-1}}\sqrt{\frac{b_k}{c_{k-1}}} \quad ; \quad \mathbf{k = 2, \dots, n} \end{cases}$

So (14), $\frac{\alpha_{k-1}}{\alpha_k}\mathbf{b_k} = \sqrt{\frac{c_{k-1}}{b_k}}\mathbf{b_k} = \sqrt{\mathbf{b_k c_{k-1}}} = \sqrt{\frac{b_k}{c_{k-1}}}\mathbf{c_{k-1}} = \frac{\alpha_k}{\alpha_{k-1}}\mathbf{c_{k-1}} \quad ; \quad \mathbf{k = 2, \dots, n}$

Ultimately, the matrix **A** is **similar** to the matrix$\mathbf{T}$($\mathbf{D^{-1}AD = T}$), such that:

$$\mathbf{T} = \begin{pmatrix} \mathbf{a_1} & \sqrt{\mathbf{b_2 c_1}} & \mathbf{0} \\ \sqrt{\mathbf{b_2 c_1}} & \ddots & \sqrt{\mathbf{b_n c_{n-1}}} \\ \mathbf{0} & \sqrt{\mathbf{b_n c_{n-1}}} & \mathbf{a_n} \end{pmatrix}$$

- Lemma③:

- The matrix **T** is **diagonalizable** with **simple eigenvalues,**and the **same** is **true** for the matrix **A**.

- Proof:

The matrix **T** is **real symmetric**, then it is **diagonalizable**.
And since it is **tridiagonal** with:$\sqrt{\mathbf{b_k c_{k-1}}} \neq \mathbf{0}(\mathbf{k = 2, \dots, n})$
Therefore according to the lemma①, the matrix **T**has**n simpleeigenvalues.**
According to the Lemma②, the matrix **A**is **similar** to the matrix **T**, thus it is d**iagonalizable** and has **n simple eigenvalues (Thesame** as the matrix **T).**

3.3 Resolution of the Chapman Kolmogorov equation:

We will solve the equation of **Chapman Kolmogorov** according to **three cases:**

3.3.1 $\forall\, i = 1, \dots, n \quad ; \quad \mu_i = \lambda_i = \beta$

So, $\quad \forall\, \mathbf{i = 1, \dots, n} \quad ; \quad -(\boldsymbol{\lambda_i + \mu_i}) = -(\boldsymbol{\beta + \beta}) = \boldsymbol{-2\beta}$

Thus the matrix **A** becomes:$\mathbf{A} = \begin{pmatrix} -2\boldsymbol{\beta} & \boldsymbol{\beta} & \mathbf{0} \\ \boldsymbol{\beta} & \ddots & \boldsymbol{\beta} \\ \mathbf{0} & \boldsymbol{\beta} & -2\boldsymbol{\beta} \end{pmatrix}$

To do this,we look for the real$\boldsymbol{\alpha}$such that there is a vector $\mathbf{X} = (\mathbf{x_i})_{1\leq i\leq n} \neq 0$ of$\mathbb{R}^{\mathbf{n}}$Such that:$\mathbf{AX = \boldsymbol{\alpha}X}$

Then, we have(15): $\begin{cases} \boldsymbol{\beta}\mathbf{x_2} - (\boldsymbol{\alpha + 2\beta})\mathbf{x_1} = \mathbf{0} \\ \boldsymbol{\beta}\mathbf{x_{k+1}} - (\boldsymbol{2\beta + \alpha})\mathbf{x_k} + \boldsymbol{\beta}\mathbf{x_{k-1}} = \mathbf{0} \quad ; \quad \mathbf{k = 2, \dots, n-1} \\ -(\boldsymbol{2\beta + \alpha})\mathbf{x_n} + \boldsymbol{\beta}\mathbf{x_{n-1}} = \mathbf{0} \end{cases}$

Thus, by positing: $\mathbf{x_0 = x_{n+1} = 0}$
We obtain for the finite sequence$(\mathbf{x_k})$,the following **recurring sequence of order 2**(16):

$$\forall\, \mathbf{k = 1, \dots, n}\, \boldsymbol{\beta}\mathbf{x_{k+1}} - (\boldsymbol{2\beta + \alpha})\mathbf{x_k} + \boldsymbol{\beta}\mathbf{x_{k-1}} = \mathbf{0} \text{(See [7])}$$

Her**characteristic equation** is given by: $\boldsymbol{\beta}\mathbf{r^2} - (\boldsymbol{2\beta + \alpha})\mathbf{r} + \boldsymbol{\beta} = \mathbf{0}$(17)
Of**discriminant**: $\Delta = (\boldsymbol{2\beta + \alpha})^2 - \boldsymbol{4\beta^2}$(18)

We will discuss the solutions according to the sign of □ and the values of the initials conditions:

<u>1stcase:</u> $\lambda \in]-\infty; -\boldsymbol{4\beta}[\cup]\mathbf{0}; +\infty[\Rightarrow \Delta > 0$
Therefore **the characteristic equation** admits two **conjugate real solutions**$\mathbf{r}_-$ and $\mathbf{r}_+$given by:

$$\mathbf{r}_\pm = \frac{\boldsymbol{\alpha + 2\beta}}{\boldsymbol{2\beta}} \pm \sqrt{\left(\frac{\boldsymbol{\alpha + 2\beta}}{\boldsymbol{2\beta}}\right)^2 - \mathbf{1}} \text{(19)}$$

Therefore, $(\mathbf{x_k})_{1\leq k,i\leq n}$is given by:$\mathbf{x_k = \gamma_- r_-^k + \gamma_+ r_+^k}$(20)
Where the coefficients $\boldsymbol{\gamma_-}$and$\boldsymbol{\gamma_+}$are provided by the following conditions:$\mathbf{x_0 = x_{n+1} = 0}$
Thus we obtain:$\begin{cases} \boldsymbol{\gamma_- + \gamma_+ = 0} \\ \boldsymbol{\gamma_-(r_-^{n+1} - r_+^{n+1}) = 0} \end{cases}$
Therefore, we have:$\boldsymbol{\gamma_- = \gamma_+ = 0}$so,$\mathbf{X = (0)}$ **Which is excluded.**

→ **Therefore, this case is empty.**

$\underline{\mathbf{2^{nd}case}: \boldsymbol{\alpha = -2\beta + 2\beta}}$ $\Rightarrow \Delta = \mathbf{0}$
Then **the characteristic equation** admits a **double real solution, (**Nominated$\mathbf{r_0}$), such that(21):

$$\mathbf{x_k = (\gamma_- + k\gamma_+) r_0^k} \; ; \quad \mathbf{k=1, \dots, n}$$

The condition:$\mathbf{x_0 = x_{n+1} = 0}$give $\boldsymbol{\gamma_- = \gamma_+ = 0}$
In the end: $\mathbf{X = (0)}$ **which is excluded**.

→ **This second case is also empty.**

$\underline{\mathbf{3^{rd}case}: \boldsymbol{\alpha \in]-4\beta; 0[}}$ $\Rightarrow \Delta < \mathbf{0}$
Thus, we can write $\boldsymbol{\alpha}$under the form:$\boldsymbol{\alpha = 2\beta\,(\cos\theta - 1)}$ with$\boldsymbol{\theta} \in]\mathbf{0}, \boldsymbol{\pi}[\cup]\boldsymbol{\pi}, \mathbf{2}\boldsymbol{\pi}[$
Thus **the characteristic equation** becomes(22):

$$\mathbf{r^2 - (2\cos\theta) r + 1 = 0} \text{ (Because:} \boldsymbol{\beta \neq 0}\text{)}$$

Who has two **conjugate complex solutions**$\boldsymbol{\omega}$and$\boldsymbol{\overline{\omega}}$given by:$\boldsymbol{\omega = e^{i\theta}}and\boldsymbol{\overline{\omega} = e^{-i\theta}}$
Which give(23):$\mathbf{x_k = \gamma_- \rho^k \cos k\theta + \gamma_+ \rho^k \sin k\theta}$; **k=1, …, n**
With: $\boldsymbol{\rho = |\omega| = 1}and\boldsymbol{\theta = \arg(\omega)}$
Thus, **the recurring sequence of order2** becomes: $\mathbf{x_k = \gamma_- \cos k\theta + \gamma_+ \sin k\theta}$(24)
Using the initials conditions:$\mathbf{x_0 = x_{n+1} = 0}$
We obtain for $\mathbf{x_0 = 0}$: $\boldsymbol{\gamma_- = 0}$
So, $\mathbf{x_k = \gamma_+ \sin k\theta}$
The condition:$\mathbf{x_{n+1} = 0}$gives$\boldsymbol{\gamma_+ = 0}$or $\mathbf{\sin(n+1)\theta = 0}$
If$\boldsymbol{\gamma_+ = 0}$than$\mathbf{X = (0)}$ → **therefore, it is excluded**.
If$\mathbf{\sin(n+1)\theta = 0}$than$\boldsymbol{\theta = \frac{k\pi}{n+1}}$
Therefore, **the eigenvalues** of the matrix **A** are of the form(25):

$$\forall\, \boldsymbol{\alpha_k \in]-4\beta, 0[\ \alpha_k = 2\beta\left(\cos\left(\frac{k\pi}{n+1}\right) - 1\right)} \; ; \quad \mathbf{k=1, \dots, n}$$

And, **the associatedeigenvectors** are of the form(26):$\mathbf{(x_k)_j = \gamma_+ \sin\left(j\frac{k\pi}{n+1}\right); 1 \leq k, j \leq n}$

We return to our equation:$\mathbf{P'(t) = A.P(t)}$(7)
We have **S** a matrix, whose columns are **theeigenvectors associated** to **the eigenvalues**of the matrix **A**, such that it is presented in the following form(27):

$$\mathbf{S} = \boldsymbol{\gamma_+}\begin{pmatrix} \sin\left(\frac{\pi}{n+1}\right) & \cdots & \sin\left(\frac{n\pi}{n+1}\right) \\ \vdots & \ddots & \vdots \\ \sin\left(\frac{n\pi}{n+1}\right) & \cdots & \sin\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix}$$

So, as explained in paragraph (**3→3.1**) above, we**first** solve the following linear differential equation:

$$\mathbf{Q'(t) = DQ(t)}\text{(11)}$$

$$\text{We have:}\mathbf{Q'(t)} = \begin{pmatrix} 2\beta\left(\cos\left(\frac{\pi}{n+1}\right) - 1\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 2\beta\left(\cos\left(\frac{\pi}{n+1}\right) - 1\right) \end{pmatrix}\begin{pmatrix} Q_1(t) \\ \vdots \\ Q_n(t) \end{pmatrix}$$

$$\text{So(28),}\begin{pmatrix} Q_1'(t) \\ \dots \\ Q_n'(t) \end{pmatrix} = \begin{pmatrix} 2\beta\left(\cos\left(\frac{\pi}{n+1}\right) - 1\right) Q_1(t) \\ \vdots \\ 2\beta\left(\cos\left(\frac{n\pi}{n+1}\right) - 1\right) Q_n(t) \end{pmatrix}$$

Thus,we obtain (29):$\mathbf{Q_k(t) = \delta_k e^{-2\beta\left(\cos\left(\frac{k\pi}{n+1}\right) - 1\right)t}}$; $\mathbf{k = 1, \dots, n}$
With$\boldsymbol{\delta_k}$is a constant to be determined, if we have an initial condition.

Therefore, and according to the following relation:$\mathbf{P(t) = SQ(t)}$(10)

We have(30):$\mathbf{P(t)} = \boldsymbol{\gamma}_+ \begin{pmatrix} \sin\left(\frac{\pi}{n+1}\right) & \cdots & \sin\left(\frac{n\pi}{n+1}\right) \\ \vdots & \ddots & \vdots \\ \sin\left(\frac{n\pi}{n+1}\right) & \cdots & \sin\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix} \begin{pmatrix} \delta_1 e^{-2\beta\left(\cos\left(\frac{\pi}{n+1}\right)-1\right)t} \\ \vdots \\ \delta_n e^{-2\beta\left(\cos\left(\frac{n\pi}{n+1}\right)-1\right)t} \end{pmatrix}$

Finally,$\mathbf{P_j(t)}$is in the following form(31):

$$\mathbf{P_j(t)} = \boldsymbol{\gamma}_+ \sum_{k=1}^{n} \boldsymbol{\delta}_k \sin\left(\frac{jk\pi}{n+1}\right) e^{-2\beta\left(\cos\left(\frac{k\pi}{n+1}\right)-1\right)t} \; ; \quad \mathbf{j = 1, \dots, n}$$

Noting that the constant$\boldsymbol{\gamma}_+$is obtained if we have an initial condition (Example:$\mathbf{(X_k)_0 = cst}$; $\mathbf{k = 1, \dots, n}$**)**
Noting that the constant$\boldsymbol{\varphi}_k$is obtained if we have an initial condition (Example: $\boldsymbol{\delta}_k(\mathbf{0}) = \mathbf{cst}$; $\mathbf{k = 1, \dots, n}$)

3.3.2 $\forall\ i = 1, \dots, n \quad ; \quad \lambda_i = \lambda \ \text{ et } \ \mu_i = \mu \ \ (\lambda \neq \mu)$

Thus, we obtain the matrix **A** in the form:$\mathbf{A} = \begin{pmatrix} -(\lambda + \mu) & \mu & 0 \\ \lambda & \ddots & \mu \\ 0 & \lambda & -(\lambda + \mu) \end{pmatrix}$

First we will look for the **eigenvalues**and **theassociatedeigenvector**of the matrix**A**.
Let$\boldsymbol{\alpha}$an**eigenvalue**of the matrix**A**and$\mathbf{x} \in \mathbb{R}^n - \{\mathbf{0}\}$an**associatedeigenvector,** then(32):

$$\begin{cases} -(\lambda + \mu)x_1 + \mu x_2 = \alpha x_1 \\ \dots \\ \lambda x_{k-1} - (\lambda + \mu)x_k + \mu x_{k+1} = \alpha x_k; \;\; k = 2, \dots, n-1 \\ \dots \\ \lambda x_{n-1} - (\lambda + \mu)x_n = \alpha x_n \end{cases}$$

Thus,
$$\begin{cases} \mu x_2 - \big(\alpha + (\lambda + \mu)\big)x_1 = 0 \\ \dots \\ \mu x_{k+1} - \big((\lambda + \mu) + \alpha\big)x_k + \lambda x_{k-1} = 0 \;;\;\; k = 2, \dots, n-1 \\ \dots \\ -\big(\alpha + (\lambda + \mu)\big)x_n + \lambda x_{n-1} = 0 \end{cases}$$

Thus,by putting$\mathbf{x_0 = x_{n+1} = 0}$,we obtain for the finite sequence$\mathbf{(x_k)_{1 \leq j \leq n}}$the following recurrence relation of order 2(33):$\mu \mathbf{x_{k+1}} - \big((\lambda + \mu) + \alpha\big)\mathbf{x_k} + \lambda \mathbf{x_{k-1}} = \mathbf{0} \quad ; \quad \mathbf{k = 1, \dots, n}$(See **[7]**)

Her**characteristic equation** is given by:$\mu \mathbf{r^2} - \big((\lambda + \mu) + \alpha\big)\mathbf{r} + \lambda = \mathbf{0}$(34)
Of **discriminant**:$\Delta = \big((\lambda + \mu) + \alpha\big)^2 - 4\lambda\mu$(35)

We will discuss the solutions according to the sign of and the values of the initials conditions:

<u>**1st case :**</u> $\alpha \in \left]-\infty ; -(\lambda + \mu) - 2\sqrt{\lambda\mu}\right[\cup \left]-(\lambda + \mu) + 2\sqrt{\lambda\mu} ; +\infty\right[\Rightarrow \Delta > 0$
<u>**2ème case**: $\alpha = -(\lambda + \mu) + 2\sqrt{\lambda\mu}$</u> $\Rightarrow \Delta = 0$
<u>**3rd case**: $\alpha \in \left]-(\lambda + \mu) - 2\sqrt{\lambda\mu} ; -(\lambda + \mu) + 2\sqrt{\lambda\mu}\right[$</u>$\Rightarrow \Delta < 0$

Thus, as we see in the first case where the coefficients $\lambda_k$and$\mu_k$are equal to the same constant,if we choose the following conditions:$\mathbf{x_0 = x_{n+1} = 0}$
The solutions exist for: $\Delta < 0$
By putting: $\alpha = -(\lambda + \mu) + 2\sqrt{\lambda\mu}\cos\theta \quad ; \quad \theta \in]0, \pi[\cup]\pi, 2\pi[$

Thecharacteristic equation becomes:$\mu r^2 - 2\sqrt{\lambda\mu}\cos\theta\, r + \lambda = 0$(36)
Thus, as$\lambda \neq 0$, we have:$\frac{\mu}{\lambda} r^2 + 2\sqrt{\frac{\mu}{\lambda}}\cos\theta\, r + 1 = 0$

We have the following relation(37): $\frac{\mu}{\Lambda}\mathbf{r}^2 + 2\sqrt{\frac{\mu}{\Lambda}}\cos\theta\,\mathbf{r} + 1 = \left(\sqrt{\frac{\mu}{\Lambda}}\mathbf{r} - e^{i\theta}\right)\left(\sqrt{\frac{\mu}{\Lambda}}\mathbf{r} - e^{-i\theta}\right)$

The characteristic equation admits two complex conjugate solutions $\boldsymbol{\omega}$ and $\overline{\boldsymbol{\omega}}$ defined by:

$$\boldsymbol{\omega} = \sqrt{\frac{\Lambda}{\mu}}e^{i\theta} \text{ et } \overline{\boldsymbol{\omega}} = \sqrt{\frac{\Lambda}{\mu}}e^{-i\theta}$$

Which give(23): $\mathbf{x_k} = \gamma_-\rho^k\cos k\theta + \gamma_+\rho^k\sin k\theta \quad ; \quad$ **k=1, …, n**

With: $\rho = |\omega| = \sqrt{\frac{\Lambda}{\mu}}$ and $\theta = \arg(\omega)$

Thus, **the recurring sequence of order 2** becomes(38):

$$\mathbf{x_k} = \gamma_-\left(\sqrt{\frac{\Lambda}{\mu}}\right)^k\cos k\theta + \gamma_+\left(\sqrt{\frac{\Lambda}{\mu}}\right)^k\sin k\theta \quad ; \quad k=1, \dots, n$$

By using the initial conditions: $\mathbf{x_0} = \mathbf{x_{n+1}} = 0$

We obtain for $\mathbf{x_0} = 0$: $\gamma_- = 0$

So(39), $\mathbf{x_k} = \gamma_+\left(\sqrt{\frac{\Lambda}{\mu}}\right)^k\sin k\theta; \quad$ **k=1, …, n**

The condition: $\mathbf{x_{n+1}} = 0$ gives $\gamma_+ = 0$ or $\sin(n+1)\theta = 0$

If $\gamma_+ = 0$ than $\mathbf{X} = (0) \quad \rightarrow$ **Therefore, it is excluded.**

If $\sin(n+1)\theta = 0$ than $\theta = \frac{k\pi}{n+1}$

Therefore, **the eigenvalues** of the matrix **A**(Nominated α_k, $k = 1, \dots, n$) are of the form(40):

$$\forall\alpha_k \in \left]-(\Lambda+\mu) - 2\sqrt{\Lambda\mu}; -(\Lambda+\mu) + 2\sqrt{\Lambda\mu}\right[\alpha_k = -(\Lambda+\mu) + 2\sqrt{\Lambda\mu}\cos\left(\frac{k\pi}{n+1}\right)$$

And **the associated eigenvectors** of the matrix **A** are of the form(41):

$$(\mathbf{x_k})_j = \gamma_+\left(\sqrt{\frac{\Lambda}{\mu}}\right)^k\sin\left(j\frac{k\pi}{n+1}\right) \quad ; \quad 1 \le k, j \le n$$

We return to our equation: $\quad \mathbf{P}'(\mathbf{t}) = \mathbf{A}.\mathbf{P}(\mathbf{t})$(7)

We have **S** a matrix, whose columns are **the eigenvectors** associated to **the eigenvalues** of the matrix **A**, such that it is presented in the following form(42):

$$\mathbf{S} = \gamma_+\begin{pmatrix} \sqrt{\frac{\Lambda}{\mu}}\sin\left(\frac{\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\Lambda}{\mu}}\right)^n\sin\left(\frac{n\pi}{n+1}\right) \\ \vdots & \ddots & \vdots \\ \sqrt{\frac{\Lambda}{\mu}}\sin\left(\frac{n\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\Lambda}{\mu}}\right)^n\sin\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix}$$

Therefore, as explained in paragraph (**3→3.1**) above, we first solve **the following differential equation**:

$$\mathbf{Q}'(\mathbf{t}) = \mathbf{D}\mathbf{Q}(\mathbf{t})(11)$$

Then, we have:

$$\mathbf{Q}'(\mathbf{t}) = \begin{pmatrix} -(\Lambda+\mu) + 2\sqrt{\Lambda\mu}\cos\left(\frac{\pi}{n+1}\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -(\Lambda+\mu) + 2\sqrt{\Lambda\mu}\cos\left(\frac{n\pi}{n+1}\right) \end{pmatrix}\begin{pmatrix} \mathbf{Q_1}(\mathbf{t}) \\ \vdots \\ \mathbf{Q_n}(\mathbf{t}) \end{pmatrix}$$

So(43), $\begin{pmatrix} \mathbf{Q}_1'(\mathbf{t}) \\ \dots \\ \mathbf{Q}_n'(\mathbf{t}) \end{pmatrix} = \begin{pmatrix} \left(-(\Lambda+\mu) + 2\sqrt{\Lambda\mu}\cos\left(\frac{\pi}{n+1}\right)\right)\mathbf{Q_1}(\mathbf{t}) \\ \vdots \\ \left(-(\Lambda+\mu) + 2\sqrt{\Lambda\mu}\cos\left(\frac{n\pi}{n+1}\right)\right)\mathbf{Q_n}(\mathbf{t}) \end{pmatrix}$

Thus(44), $\quad \mathbf{Q_k}(\mathbf{t}) = \delta_k e^{\left(-(\Lambda+\mu)+2\sqrt{\Lambda\mu}\cos\left(\frac{k\pi}{n+1}\right)\right)t} \quad ; \quad k = 1, \dots, n$

With δ_k is a constant to be determined, if we have an initial condition.

Therefore, and according to the following relation: $\mathbf{P(t) = SQ(t)}$(10)

We have(45)**:**

$$\mathbf{P(t)} = \boldsymbol{\gamma}_+ \begin{pmatrix} \sqrt{\frac{\Lambda}{\mu}}\sin\left(\frac{\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\Lambda}{\mu}}\right)^n \sin\left(\frac{n\pi}{n+1}\right) \\ \vdots & \ddots & \vdots \\ \sqrt{\frac{\Lambda}{\mu}}\sin\left(\frac{n\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\Lambda}{\mu}}\right)^n \sin\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix} \begin{pmatrix} \delta_1 e^{\left(-(\Lambda+\mu)+2\sqrt{\Lambda\mu}\cos\left(\frac{\pi}{n+1}\right)\right)t} \\ \vdots \\ \delta_n e^{\left(-(\Lambda+\mu)+2\sqrt{\Lambda\mu}\cos\left(\frac{n\pi}{n+1}\right)\right)t} \end{pmatrix}$$

Finally, $\mathbf{P_j(t)}$is in the following form(46):

$$\mathbf{P_j(t)} = \boldsymbol{\gamma}_+ \sum_{k=1}^{n} \delta_k \left(\sqrt{\frac{\Lambda}{\mu}}\right)^k \sin\left(\frac{jk\pi}{n+1}\right) e^{\left(-(\Lambda+\mu)+2\sqrt{\Lambda\mu}\cos\left(\frac{k\pi}{n+1}\right)\right)t} \;;\quad \mathbf{j = 1, \dots, n}$$

Noting that the constant $\boldsymbol{\gamma}_+$is obtained if we have an initial condition (Example:$\mathbf{(X_k)_0 = cst}$; $\mathbf{k = 1, \dots, n}$**)**
Noting that the constant$\boldsymbol{\delta_k}$is obtained if we have an initial condition (Example: $\boldsymbol{\delta_k(0) = cst}$; $\mathbf{k = 1, \dots, n}$)

3.3.3 $\forall\, i = 1, \dots, n \;;\; \mu_i = \Lambda_i = \beta_i$

Thus we obtain the matrix **A** of the following form:$\mathbf{A} = \begin{pmatrix} -2\beta_1 & \beta_1 & 0 \\ \beta_1 & \ddots & \beta_{n-1} \\ 0 & \beta_{n-1} & -2\beta_n \end{pmatrix}$

We'll look for **the eigenvalues** and **the associatedeigenvectors** of the matrix **A.**
Let $\boldsymbol{\alpha}$be **an eigenvalue** of the matrix**A** and$\mathbf{x} \in \mathbb{R}^n - \{0, \dots, 0\}$ **an associated eigenvector, s**uch that:$\mathbf{AX = \alpha X}$

Then, we have(47):
$$\begin{cases} -2\beta_1 x_1 + \beta_1 x_2 = \alpha x_1 \\ \dots \\ \beta_{k-1}x_{k-1} - 2\beta_k x_k + \beta_{k+1}x_{k+1} = \alpha x_k; \quad k = 2, \dots, n-1 \\ \dots \\ \beta_{n-1}x_{n-1} - 2\beta_n x_n = \alpha x_n \end{cases}$$

So (48),$\boldsymbol{\beta_{k+1}x_{k+1} - (2\beta_k + \alpha)x_k + \beta_{k-1}x_{k-1} = 0} \;;\; \mathbf{k = 1, \dots, n}$(See **[7]**)

We put:$\mathbf{x_k = q^k}$

So we get:$\boldsymbol{\beta_{k+1}q^{k+1} - (2\beta_k + \alpha)q^k + \beta_{k-1}q^{k-1} = 0}$

Thus for **k=n** we obtain:$\boldsymbol{\beta_{n+1}q^{n+1} - (2\beta_n + \alpha)q^n + \beta_{n-1}q^{n-1} = 0}$

Dividing the last equation by: $\mathbf{q^{k-n}}$

The relation becomes:$\boldsymbol{\beta_{n+1}q^{k+1} - (2\beta_n + \alpha)q^k + \beta_{n-1}q^{k-1} = 0}$(49)

By dividing the equation by: $\mathbf{q^{k-1}}$

The **characteristic equation**becomes: $\boldsymbol{\beta_{n+1}q^2 - (2\beta_n + \alpha)q + \beta_{n-1} = 0}$(50)

Of **discriminant**: $\Delta = \boldsymbol{(2\beta_n + \alpha)^2 - 4\beta_{n-1}\beta_{n+1}}$(51)

We will discuss the solutions according to the sign of □ and the values of the initials conditions:

<u>1st case</u> : $\boldsymbol{\alpha} \in \left]-\infty\,; -2\left(\beta_n + \sqrt{\beta_{n-1}\beta_{n+1}}\right)\right[\cup\left]-2\left(\beta_n - \sqrt{\beta_{n-1}\beta_{n+1}}\right); +\infty\right[\Rightarrow \Delta > 0$
<u>2nd case</u>: $\boldsymbol{\alpha = -2\left(\beta_n \pm \sqrt{\beta_{n-1}\beta_{n+1}}\right)} \Rightarrow \Delta = \mathbf{0}$

3^{rd}case: $\alpha \in]-2(\beta_n + \sqrt{\beta_{n-1}\beta_{n+1}}); -2(\beta_n - \sqrt{\beta_{n-1}\beta_{n+1}})[\Rightarrow \Delta < 0$

Thus, as we see in the first case where the coefficients λ_k and μ_k are equal to the same constant,if we choose the following conditions:$x_0 = x_{n+1} = 0$
The solutions exist for: $\Delta < 0$
Thus we put:$\alpha = -2\,\beta_n + 2\sqrt{\beta_{n-1}\beta_{n+1}}\cos\theta$; $\theta \in]0,\pi[\cup]\pi,2\pi[$
Hence **the characteristic equation** becomes:$\beta_{n+1}q^2 - 2q\sqrt{\beta_{n-1}\beta_{n+1}}\cos\theta + \beta_{n-1} = 0$(52)
Therefore, as $\beta_{n-1} \neq 0$ $(k = 1, \dots, n)$, we have:$\frac{\beta_{n+1}}{\beta_{n-1}}q^2 + 2q\sqrt{\frac{\beta_{n+1}}{\beta_{n-1}}}\cos\theta + 1 = 0$
Thus we have the following relation(53):

$$\frac{\beta_{n+1}}{\beta_{n-1}}q^2 + 2\sqrt{\frac{\beta_{n+1}}{\beta_{n-1}}}\cos\theta\, q + 1 = \left(\sqrt{\frac{\beta_{n+1}}{\beta_{n-1}}}q - e^{i\theta}\right)\left(\sqrt{\frac{\beta_{n+1}}{\beta_{n-1}}}q - e^{-i\theta}\right)$$

Which give(23): $x_k = \rho^k(\gamma_- \cos k\theta + \gamma_+ \sin k\theta)$; k=1, …, n
With: $\rho = |\omega| = \sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}$ and $\theta = \arg(\omega)$
Thus **the sequence of recurrence of order 2** admits as solution(54):

$$x_k = \left(\sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\right)^k (\gamma_- \cos k\theta + \gamma_+ \sin k\theta) \quad ; \quad k=1, \dots, n$$

Using the initials conditions:$x_0 = x_{n+1} = 0$
We obtain for $x_0 = 0$: $\gamma_- = 0$
So(55),$x_k = \gamma_+\left(\sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\right)^k \sin k\theta$;k=1, …,n
The condition $x_{n+1} = 0$ gives $\gamma_+ = 0$ or $\sin(n+1)\theta = 0$
If $\gamma_+ = 0$ than $X = (0)$ → **Therefore, it is excluded.**
If $\sin(n+1)\theta = 0$ than $\theta = \frac{k\pi}{n+1}$
Thus, **the eigenvalues** of the matrix **A**(Called α_k; $k = 1, \dots, n$) are of the form(56):

$$\forall\, \alpha_k \in]-2(\beta_n + \sqrt{\beta_{n-1}\beta_{n+1}}); -2(\beta_n - \sqrt{\beta_{n-1}\beta_{n+1}})[\ \ \alpha_k = -2\left(\beta_n - \sqrt{\beta_{n-1}\beta_{n+1}}\cos\left(\frac{k\pi}{n+1}\right)\right)$$

And **the eigenvectorsassociated** with **the eigenvalues** of the matrix **A** are (57):

$$(x_k)_j = \gamma_+\left(\sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\right)^k \sin\left(j\frac{k\pi}{n+1}\right) \quad ; \quad 1 \le j, k \le n$$

We return to our equation: $P'(t) = A.P(t)$(7)
We have **S** a matrix whose columns are **the eigenvectors associated** with **the eigenvalues** of the matrix **A**, such that it is presented in the following form(58):

$$S = \gamma_+\begin{pmatrix} \sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\sin\left(\frac{\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\right)^n \sin\left(\frac{n\pi}{n+1}\right) \\ \vdots & \ddots & \vdots \\ \sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\sin\left(\frac{n\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\right)^n \sin\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix}$$

We **first** solve the following **differential equation**:$Q'(t) = DQ(t)$(11)

Then, we have:$Q'(t) = -2\begin{pmatrix} \beta_n - \sqrt{\beta_{n-1}\beta_{n+1}}\cos\left(\frac{\pi}{n+1}\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_n - \sqrt{\beta_{n-1}\beta_{n+1}}\cos\left(\frac{n\pi}{n+1}\right) \end{pmatrix}\begin{pmatrix} Q_1(t) \\ \vdots \\ Q_n(t) \end{pmatrix}$

So(59), $\begin{pmatrix} \mathbf{Q_1'(t)} \\ \dots \\ \mathbf{Q_n'(t)} \end{pmatrix} = -\mathbf{2}\begin{pmatrix} \left(\boldsymbol{\beta_n} - \sqrt{\boldsymbol{\beta_{n-1}\beta_{n+1}}}\,\mathbf{cos}\left(\frac{\pi}{n+1}\right)\right)\mathbf{Q_1(t)} \\ \vdots \\ \left(\boldsymbol{\beta_n} - \sqrt{\boldsymbol{\beta_{n-1}\beta_{n+1}}}\,\mathbf{cos}\left(\frac{n\pi}{n+1}\right)\right)\mathbf{Q_n(t)} \end{pmatrix}$

Thus(60), $\mathbf{Q_k(t)} = \boldsymbol{\delta_k} \mathbf{e}^{\left(\beta_n - \sqrt{\beta_{n-1}\beta_{n+1}}\cos\left(\frac{k\pi}{n+1}\right)\right)t}$; $\mathbf{k = 1, \dots, n}$

With $\boldsymbol{\delta_k}$ a constant to be determined, if one has an initial condition.

Therefore, and according to the following relation: $\mathbf{P(t) = SQ(t)}$(10)

We have

(61):$\mathbf{P(t)} = \boldsymbol{\gamma_+} \begin{pmatrix} \sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\,\mathbf{sin}\left(\frac{\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\right)^{n} \mathbf{sin}\left(\frac{n\pi}{n+1}\right) \\ \vdots & \ddots & \vdots \\ \sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\,\mathbf{sin}\left(\frac{n\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\right)^{n} \mathbf{sin}\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix} \begin{pmatrix} \boldsymbol{\delta_1} \mathbf{e}^{\left(\beta_n - \sqrt{\beta_{n-1}\beta_{n+1}}\cos\left(\frac{\pi}{n+1}\right)\right)t} \\ \vdots \\ \boldsymbol{\delta_n} \mathbf{e}^{\left(\beta_n - \sqrt{\beta_{n-1}\beta_{n+1}}\cos\left(\frac{n\pi}{n+1}\right)\right)t} \end{pmatrix}$

Finally, $\mathbf{P_j(t)}$ is in the following form(62):

$$\mathbf{P_j(t)} = \boldsymbol{\gamma_+} \sum_{k=1}^{n} \boldsymbol{\delta_k} \left(\sqrt{\frac{\beta_{n-1}}{\beta_{n+1}}}\right)^{k} \mathbf{sin}\left(\frac{jk\pi}{n+1}\right) \mathbf{e}^{\left(\beta_n - \sqrt{\beta_{n-1}\beta_{n+1}}\cos\left(\frac{n\pi}{n+1}\right)\right)t}; \quad \mathbf{j = 1, \dots, n}$$

Note that the constant $\boldsymbol{\gamma_+}$is obtained if an initial condition exists (Example:$\mathbf{(X_k)_0 = cst}$; $\mathbf{k = 1, \dots, n}$)
Note also that the constant $\boldsymbol{\delta_k}$is obtained if an initial condition exists (Example: $\boldsymbol{\delta_k}\mathbf{(0) = cst}$; $\mathbf{k = 1, \dots, n}$)

4. Conclusion

Many important stochastic counting models can be written as general **Birth-Death Processes** (**BDPs**). **BDPs** are continuous-time Markov chains on the non-negative integers in which only jumps to adjacent states are allowed. **BDPs** can be used to easily parameterize a rich variety of probability distributions onthe non-negative integers, and straightforward conditions guarantee that these distributions are proper.**BDPs** also provide a mechanistic interpretation – birth and death of actual particles or organisms – that has proven useful in ecology, physics, and chemistry.

Finally, thanks to this law, several problems can be solved, used more particularly in biology, demography, physics, sociology, statistics ... etc. And also to account for the changing size of any type of population and the problems related to waiting phenomena.

Acknowledgements

I would like to express my very great appreciation to **Dr A.Arbai** for his valuable and constructive suggestions and his useful critiques during the planning and development of this research work. His willingness to give his time so generously has been very much appreciated.

I would also like to express my deep gratitude to **Dr M.Benslimane**, my research supervisors, for his patient guidance and his enthusiastic encouragement of this research work.

References

[1]. Forrest W. Crawford_ and Marc A. Suchardy, Birth-death processes, July 28, 2014, cite as: arXiv: 1301.1305v2 [stat.ME] 25 Jul, 2014.
[2]. Forrest W. Crawford and Marc A. Suchard, Transition probabilities for general birth-death processes with applications in ecology,genetics, and evolution,Typeset on September 26, 2011.
[3]. Patrice Wira, Recall on the Matrix,Faculté des Sciences et Techniques (2000-2001), Universityof Haute Alsace.
[4]. Martin Otto, Linear Algebra II, Summer Term 2011.
[5]. E. Kowalski, kowalski@math.ethz.ch, Linear Algebra,ETH Zurich-Fall2015 andSpring 2016, Version of September 15, 2016.
[6]. AntoninaMitrofanova, NYU, Continuous times Markov Chains, Poisson Process, Birth and Death process,December 18, 2007, department of Computer Science.
[7]. Recurrences, MATH 579 Spring 2012 Supplement.