International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 5 Issue 6 || June. 2017 || PP-26-28

The Schur Complement of Gyroscopically Stabilized Quadratic Pencils

Aleksandra Kostić[1]
[1](*Faculty of Mechanical Engineering/ University of Sarajevo, Bosnian and Herzegovina*)
Corresponding Author: Aleksandra Kostić

ABSTRACT: *In this paper we consider a taking advantage of the Schur complement of gyroscopically stabilized quadratic pencils. All eigenvalues of the gyroscopically stabilized quadratic problems are real and lying in four disjoint intervals. Since the bounds of these intervals are not exactly known we cannot apply variational characterization for this kind of problems in full. By using the advantage of the Schur complement of gyroscopically stabilized quadratic pencil, we aim to construct functionals that will allow the application of variational characterization for this kind of problems in full.*

KEYWORDS: *Eigenvalue, gyroscopically pencil, minimax principle, Schur complement*

Date of Submission: 11-07-2017 Date of acceptance: 20-07-2017

I. INTRODUCTION

Eigenvalue problems occur when the system vibrates. The eigenvalue problems are divided into: linear and nonlinear. The quadratic eigenvalue problems (QEP) are a special case of the nonlinear eigenvalue problems. A gyroscopically stabilized quadratic eigenvalue problems (GSS) are a special case of QEP. More information on the problem of the eigenvalues is found in [1].

GSS has 2n real eigenvalues. The important features of gyroscopically stabilized quadratic systems are considered in [2,3] Voss in [4] presents an important tool for finding eigenvalues is variational characterization of the nonlinear eigenvalue problems.

In this paper we bring the taking of the Schur complement of gyroscopically stabilized quadratic pencils. We present the properties of the Schur complement as well. Our aim is to construct functionals that will enable us to apply variational characterization for this kind of problems in full by taking advantage of the Schur complement of gyroscopically stabilized quadratic pencil.

The remainder of the paper is organized in the following manner:
Section 2 of this paper presents the problem definition and the basic results about gyroscopically stabilized quadratic systems. In Section 3 we consider the taking of the Schur complement of gyroscopically stabilized quadratic pencils. We give conclusion and indications for further research in Section 4.

II. PROBLEM DEFINITION AND BASIC RESULTS

In this section we bring definition and basic results for this type of problems.
A quadratic matrix pencil:

$$\mathbf{Q}(\lambda) := \lambda^2\mathbf{I} + \lambda\mathbf{B} + \mathbf{C}, \quad \mathbf{B} = \mathbf{B}^{\boldsymbol{H}}, \det\mathbf{B} \neq 0, \mathbf{C} = \mathbf{C}^{\boldsymbol{H}} > 0, \mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}^{H} & \mathbf{C}_{22} \end{pmatrix} \tag{1}$$

is gyroscopically stabilized if for some *k>0* it holds that:

$$|\mathbf{B}| > k\mathbf{I} + k^{-1}\mathbf{C} \tag{2}$$

where $|\mathbf{B}|$ denotes the positive squere root of $\mathbf{B}^2$. Without loss of generality we can assume that $\mathbf{B} = \begin{pmatrix} \mathbf{B_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_2} \end{pmatrix}$, where $\mathbf{B_1}$ and $\mathbf{B_2}$ are diagonal, $\mathbf{B_1} > 0$ and $\mathbf{B_2} < 0$.

$\mathbf{Q}(\lambda)\mathbf{x}=\mathbf{0}$ has *2n* real eigenvalues, see [1].Although it is well known that eigenvalues of gyroscopically stabilized quadratic systems lie in four disjoint intervals $\Delta_j := [\alpha_j, \beta_j], \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 0 < \alpha_3 < \beta_3 < \alpha_4 < \beta_4$, in practice the bounds of these intervals are not known. This makes variational characterization for this type of problems only partially feasible. Let us briefly summarize the variational characterization for this type of problems. With this aim we consider the equation:

$$f(\lambda\,;\,\mathbf{x}) := \mathbf{x}^{\mathbf{H}}\mathbf{Q}(\lambda)\mathbf{x} = 0 \tag{3}$$

Then for $\mathbf{x} \neq \mathbf{0}$ the two complex roots of (3) are with:

$$p_{+}(\mathbf{x}) := -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}} + \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}} \tag{4}$$

$$p_{-}(\mathbf{x}) := -\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}} - \sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}} \tag{5}$$

Between these complex roots (3) and (4) there are all eigenvalues of the corresponding eigenvalue problem. Eigenvalues obtained by the functional (4) are called the eigenvalues of a positive type. Eigenvalues obtained by the functional (5) are called the eigenvalues of a negative type. For the eigenvector $\mathbf{x} \neq \mathbf{0}$ there is $\mathbf{Q}(\lambda)\mathbf{x}=\mathbf{0}$ and therefore $f(\lambda\,;\,\mathbf{x}) := \mathbf{x}^{\mathbf{H}}\mathbf{Q}(\lambda)\mathbf{x} = 0$.

Let:

$$p_{-}^{+}(\mathbf{x}) := \begin{cases} p_{-}(\mathbf{x}) \text{ if } p_{-}(\mathbf{x}) > 0 \\ \infty \text{ else} \end{cases} \quad p_{+}^{+}(\mathbf{x}) := \begin{cases} p_{+}(\mathbf{x}) \text{ if } p_{+}(\mathbf{x}) > 0 \\ 0 \text{ else} \end{cases} \tag{6}$$

$$p_{-}^{-}(\mathbf{x}) := \begin{cases} p_{-}(\mathbf{x}) \text{ if } p_{-}(\mathbf{x}) < 0 \\ \infty \text{ else} \end{cases} \quad p_{+}^{-}(\mathbf{x}) := \begin{cases} p_{+}(\mathbf{x}) \text{ if } p_{+}(\mathbf{x}) < 0 \\ 0 \text{ else} \end{cases} \tag{7}$$

If $\max\{p_{-}^{+}(\mathbf{x})\} < \min\{p_{+}^{+}(\mathbf{x})\}$ then all eigenvalues in Δ_3 and Δ_4 are minmax and maxmin values of $p_{-}^{+}$and p_{+}^{+}, respectively. If $\max\{p_{-}^{-}(\mathbf{x})\} < \min\{p_{+}^{-}(\mathbf{x})\}$ then all eigenvalues in Δ_1 and Δ_2 are minmax and maxmin values of $p_{-}^{-}$and p_{+}^{-}, respectively.

From these variational characterizations of eigenvalues one obtains a Sylvester Theorem for gyroscopically stabilized quadratic eigenvalue problems in an obvious way.

III. TAKING ADVANTAGE OF THE SCHUR COMPLEMENT

The idea of applying the Schur complement of gyroscopically stabilized quadratic pencils appears for the first time in [1]. In this paper we are developing this idea with its positive and negative sides.

Consider the quadratic pencils:

$$\mathbf{Q}_1(\lambda) := \lambda^2\mathbf{I} + \lambda\mathbf{B}_1 + \mathbf{C}_{11} \tag{8}$$

$$\mathbf{Q}_2(\lambda) := \lambda^2\mathbf{I} - \lambda\mathbf{B}_2 + \mathbf{C}_{12} \tag{9}$$

Due to assumption (2) $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are both hyperbolic.

For the Schur complement $\mathbf{Q}$ with respect to $\mathbf{y}$ and for $\lambda>0$**:**

$$\mathbf{T}(\lambda) := \mathbf{Q}_2(\lambda) - \mathbf{C}_{12}^H\left(\mathbf{Q}_1(\lambda)\right)^{-1}\mathbf{C}_{12} \tag{10}$$

It is obvious that $\mathbf{T}(\lambda)$ for $\lambda>0$ is well defined because $\mathbf{Q}_1(\lambda) > 0$ for every $\lambda>0$. It means that for such λ $\left(\mathbf{Q}_1(\lambda)\right)^{-1}$exists. It holds that $(\lambda, \mathbf{y})$ is an eigenpair of the rational eigenvalue problem $\mathbf{T}(\lambda)\mathbf{y} = \mathbf{0}$ if and only if $(\lambda, \mathbf{x}), \mathbf{x} = (\mathbf{z}^H, \mathbf{y}^H)^H, \mathbf{z} = \left(\mathbf{Q}_1(\lambda)\right)^{-1}\mathbf{C}_{12}\mathbf{y}$ is an eigenpair of $\mathbf{Q}(\lambda)\mathbf{y} = \mathbf{0}$. Last statement is proved in [1].

Analogously to $f(\lambda\,;\,\mathbf{x})$ we define the function:

$$q(\lambda\,;\,\mathbf{y}) := \mathbf{y}^H\,\mathbf{T}(\lambda)\mathbf{y} = \mathbf{y}^H(\lambda^2\mathbf{I} - \lambda\mathbf{B}_2 + \mathbf{C}_{22}\,)\mathbf{y} - \mathbf{y}^H\mathbf{C}_{12}^H\left(\mathbf{Q}_1(\lambda)\right)^{-1}\mathbf{C}_{12}\mathbf{y} \tag{11}$$

Clearly, the eigenvalues of $\mathbf{T}(\cdot)$ and $\mathbf{Q}(\cdot)$ are the roots of the function $q(\lambda\,;\,\mathbf{y})$ where $\mathbf{y}$ denotes an eigenvector corresponding to λ. Eigenvalue λ from $\mathbf{T}(\lambda)\mathbf{y} = \mathbf{0}$ will be called positive eigenvalue of the negative type if it stands that $q(\lambda\,;\,\mathbf{y}) = 0$ and $q'(\lambda\,;\,\mathbf{y}) < 0$. Eigenvalue λ from $\mathbf{T}(\lambda)\mathbf{y} = \mathbf{0}$ will be called positive eigenvalue of the positive type if it stands that $q(\lambda\,;\,\mathbf{y}) = 0$ and $q'(\lambda\,;\,\mathbf{y}) > 0$.

Theorem1. The function $q(\lambda\,;\,\mathbf{y})$ has the following propperties:

1) $q(0\,;\,\mathbf{y}) > 0$ for every $\mathbf{y} \neq \mathbf{0}$.
2) For every $\mathbf{y} \neq \mathbf{0}$ and $\lambda>0$ the function $q(\lambda\,;\,\mathbf{y})$ has two roots. One is less than the smallest zero of $f_2(\lambda\,;\,\mathbf{y}) := \mathbf{y}^H\,\mathbf{Q}_2(\lambda)\mathbf{y}$, and one is greater then the biggest root of $f_2(\cdot\,;\,\mathbf{y})$.
3) From $f_2{}'(\lambda\,;\,\mathbf{y}) > 0$ and $\lambda>0$ it follows that $q'(\lambda\,;\,\mathbf{y}) > 0$.

Proof

1) From the positive definiteness of $\mathbf{C}$ we get:

$$q(0\,;\,\mathbf{y}) = \mathbf{y}^H\,\mathbf{T}(0)\mathbf{y} = \mathbf{y}^H\mathbf{C}_{22}\,\mathbf{y} - \mathbf{y}^H\mathbf{C}_{12}^H(\mathbf{C}_{11})^{-1}\mathbf{C}_{12}\mathbf{y} = (-\mathbf{y}^H\mathbf{C}_{12}^H(\mathbf{C}_{11})^{-1}, \mathbf{y}^H)\begin{pmatrix}\mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}^H & \mathbf{C}_{22}\end{pmatrix}\begin{pmatrix}(\mathbf{C}_{11})^{-1}\mathbf{C}_{12}\mathbf{y} \\ \mathbf{y}\end{pmatrix} > 0$$

2) Since $\mathbf{Q}_2(\cdot)$ is hyperbolic the function $f_2(\cdot\,;\,\mathbf{y})$ has two distinct roots which are both positive. $\mathbf{Q}_1(\lambda) > 0$ for $\lambda>0$ and therefore;

$$q(\lambda\,;\,\mathbf{y}) := \mathbf{y}^H\,\mathbf{T}(\lambda)\mathbf{y} = \mathbf{y}^H(\lambda^2\mathbf{I} - \lambda\mathbf{B}_2 + \mathbf{C}_{22}\,)\mathbf{y} - \underbrace{\mathbf{y}^H\mathbf{C}_{12}^H\left(\mathbf{Q}_1(\lambda)\right)^{-1}\mathbf{C}_{12}\mathbf{y}}_{>0}$$

$$< \lambda^2\mathbf{y}^H\mathbf{y} - \lambda\mathbf{y}^H\mathbf{B}_2\mathbf{y} + \mathbf{y}^H\mathbf{C}_{12}\,\mathbf{y} = f_2(\lambda\,;\,\mathbf{y})$$

From $q(0\,;\,\mathbf{y}) > 0$ it follows that there exists one root of $q(\cdot\,;\,\mathbf{y})$ less the smallest root of the $f_2(\cdot\,;\,\mathbf{y})$, and from $\lim_{\lambda\to+\infty} q(\lambda\,;\,\mathbf{y}) = +\infty$ that there is a second roots greater than biggest rot of $f_2(\cdot\,;\,\mathbf{y})$. This proof does not demonstrate that $q(\cdot\,;\,\mathbf{y})$ has exactly two roots.

3) $q'(\lambda\,;\,\mathbf{y}) = 2\lambda\mathbf{y}^H\mathbf{y} - \mathbf{y}^H\mathbf{B}_2\mathbf{y} + \mathbf{y}^H\mathbf{C}_{12}^H\big(\mathbf{Q}_1(\lambda)\big)^{-2}(2\lambda\mathbf{I} + \mathbf{B}_1)\mathbf{C}_{12}\mathbf{y} = f_2'(\lambda\,;\,\mathbf{y}) + \mathbf{y}^H\mathbf{C}_{12}^H\big(\mathbf{Q}_1(\lambda)\big)^{-2}(2\lambda\mathbf{I} +$ $\mathbf{B}1\mathbf{C}12\mathbf{y}>0$ for $\lambda>\mathbf{y}H\mathbf{B}2\mathbf{y}2\mathbf{y}H\mathbf{y}$o

From here follows that $q(\lambda\,;\,\mathbf{y})$ is strictly monotonically inereasing forλ greater than the minimum $\lambda = \frac{\mathbf{y}^H\mathbf{B}_2\mathbf{y}}{2\mathbf{y}^H\mathbf{y}}$ of $f_2(\cdot\,;\,\mathbf{y})$ and therefore $q(\cdot\,;\,\mathbf{y})$ has exactly one root wich is larger that the biggest root of $f_2(\cdot\,;\,\mathbf{y})$. The uniques of the root less than the smailest root of of $f_2(\cdot\,;\,\mathbf{y})$ is still not clear.

Let us look at some other features of the function $q(\cdot\,;\,\mathbf{y})$. Funkcija $q(\lambda\,;\,\mathbf{y})$ has *2(n-k)* positive poles while $\mathbf{B}_2$ has $k \times k$ format. This means that this function for every $\mathbf{y} \in \mathbb{C}^k\backslash\{\mathbf{0}\}$ has 2(n-k+1) zeros. The assumption is that *2(n-k)* of them are negative. This claim has not yet been proven. If we could prove the existence of the *2(n-k)* negative zero from the previous theorem, it would immediately follow that $q(\cdot\,;\,\mathbf{y})$ has a unique root less than the smallest root of $f_2(\cdot\,;\,\mathbf{y})$.

In the following we assume that $q(\cdot\,;\,\mathbf{y})$ has a unique root less than the smallest root of $f_2(\cdot\,;\,\mathbf{y})$.

Definition1 Let the functional $t_-(\mathbf{y})$ and $t_+(\mathbf{y})$ be defined as the smallest and largest root of $q(\lambda\,;\,\mathbf{y})$, respecively and $W_\pm \coloneqq t_\pm(\mathbb{C}^k\backslash\{\mathbf{0}\})$ where matrix $\mathbf{B}_2$ ima format $k \times k$.

Lemma 1 $\max W_- < \min W_+$.

Proof

Let:

$$\mathrm{p}_{2\pm}(\mathrm{y}) = -\frac{\mathbf{y}^H\mathbf{B}_2\mathbf{y}}{2\mathbf{y}^H\mathbf{y}} \pm \sqrt{\left(\frac{\mathbf{y}^H\mathbf{B}_2\mathbf{y}}{2\mathbf{y}^H\mathbf{y}}\right)^2 - \frac{\mathbf{y}^H\mathbf{C}_2\mathbf{y}}{2\mathbf{y}^H\mathbf{y}}} \tag{12}$$

and

$$J_\pm \coloneqq t_\pm(\mathbb{C}^k\backslash\{\mathbf{0}\}) \tag{13}$$

Then the hyperbolicity of $\mathbf{Q}_2$ implies $\max J_- < \min J_+$ and from $t_-(\mathbf{y}) < \max J_- < \min J_+ < t_+(\mathbf{y})$ for every $\mathbf{y} \in \mathbb{C}^k$ we obtain $\max W_- < \min W_+$.

It is now obvious that the following theorem holds

Theorem 2 All positive eigenvalues of $\mathbf{Q}(\cdot)$ are either minimax values of t_- or maxmin values of t_+ . Since $q(\lambda\,;\,\mathbf{y})$ has fixed vector $\mathbf{y}$ *2(n-k+1)* zeros, finding the appropriate zero from $q(\lambda\,;\,\mathbf{y})$in safeguarded iteration will be done numerically.

Remark 1 For λ<0 we consider the rational eigenvalue problem;

$$\mathbf{T_1}(\lambda) \coloneqq \mathbf{Q}_1(\lambda) - \mathbf{C}_{12}\big(\mathbf{Q}_2(\lambda)\big)^{-1}\mathbf{C}_{12}^H \tag{14}$$

Then all preceding results hold on an analogous way.

IV. CONLUSION

Schur complement matrix plays important role in linear algebra. By taking the advantage of the Schur complement of gyroscopically stabilized quadratic pencils we get function $q(\cdot\,;\,\mathbf{y})$ that helps us construct the functionals $t_-(\mathbf{y})$ and $t_+(\mathbf{y})$. Functionals $t_-(\mathbf{y})$ and $t_+(\mathbf{y})$ are very important for variational characterization. For this reason we presented properties of the function $q(\cdot\,;\,\mathbf{y})$.

In future research we want to prove that function $q(\cdot\,;\,\mathbf{y})$ has *2(n-k)* negative zeros and two positive zeros. If we would be able to prove the existence of *2(n-k)* negative zero it would mean that $q(\cdot\,;\,\mathbf{y})$ has a unique root less than the smallest root of $f_2(\cdot\,;\,\mathbf{y})$ and this would enable the application of variational characterization fully with the new functionals $t_-(\mathbf{y})$ and $t_+(\mathbf{y})$

REFERENCES

[1] A. Kostić, Eigenvalue Problems, in V. N. Katsikis (Ed.), *Applied linear algebra in action*, 3 (Rijeka: InTech, 2016) 57-83.

[2] L. Barkwell, P. Lancaster, and A. S. Markus, Gyroscopically stabilized systems: a class pf quadratic eigenvalue problems with real spectrum, *Can. J. Math.*, *44*, 1992, 42-53.

[3] P. Lancaster, A. S. Markus, and F. Zhou, A wider class of stable gyroscopic systems, *Linear Algebra Appl.*, *370*, 2003, 257-267.

[4] H. Voss, A minmax principle for nonlinear eigenproblems depending continuously on the eigenparameter,*Numer. Lin. Algebra Appl.*, *16*, 2009, 899 – 913.