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Shift Strategy for Gyroscopically Stabilized Quadratic Systems

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ABSTRACT: *In this paper we consider a special class of the quadratic eigenvalue problems, gyroscopically stabilized quadratic problems. All eigenvalue of the gyroscopically stabilized quadratic problems are real and lying in four disjoint intervals. Since the bounds of these intervals are not exactly known we cannot apply variational characterization for this kind of problems in full. The aim of this paper is to use a shift strategy to determine the interval at which variational characterization can be applied, for negative eigenvalues the negative type .*

KEYWORDS: *Eigenvalue, gyroscopic stabilization, minimax principle, shift strategy, variational characterization*

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I. INTRODUCTION

A quadratic matrix pencil:

$$\mathbf{Q}(\lambda) := \lambda^2\mathbf{I} + \lambda\mathbf{B} + \mathbf{C}, \quad \mathbf{B} = \mathbf{B}^{\boldsymbol{H}}, \det\mathbf{B} \neq 0, \mathbf{C} = \mathbf{C}^{\boldsymbol{H}} > 0 \tag{1}$$

is gyroscopically stabilized if for some *k>0* it holds that:

$$|\mathbf{B}| > k\mathbf{I} + k^{-1}\mathbf{C} \tag{2}$$

where $|\mathbf{B}|$ denotes the positive squere root of $\mathbf{B}^2$. Without loss of generality we can assume that $\mathbf{B} = \begin{pmatrix} \mathbf{B_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_2} \end{pmatrix}$, where $\mathbf{B_1}$ and $\mathbf{B_2}$ are diagonal, $\mathbf{B_1} > 0$ and $\mathbf{B_2} < 0$.

Gyroscopically stabilized quadratic systems have application in technical disciplines. For $\mathbf{G}^H = -\mathbf{G}$ the free motions of a conservative, time-invariant linear system oscillating about an unstable equilibrium under action of a gyroscopic force are governed by:

$$\ddot{u}(t) + \mathbf{G}\dot{u}(t) - \mathbf{C}u(t) = 0 \tag{3}$$

Making the substitution $u(t) = \mathbf{x}\exp(\mu t)$ with $\mathbf{x}$ independent of *t*, and then the rotation of the parameter $\lambda = -i\mu$ leads to the eigenvalue problem $\mathbf{Q}(\lambda)\mathbf{x} = \mathbf{0}$ where $\mathbf{B} = i\mathbf{G}$ is clearly indefinite.

Barkwell, Lancaster and Markus studied (1) in [1]. They proved that the quadratic eigenvalue problem $\mathbf{Q}(\lambda)\mathbf{x} = \mathbf{0}$ has *2n* real eigenvalues. If $in(\mathbf{B}) = (p, n-p, 0)$ then there are 4 disjoint intervals $\Delta_j := [\alpha_j, \beta_j], \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 0 < \alpha_3 < \beta_3 < \alpha_4 < \beta_4$, Δ_1 contains p eigenvalues each of negative type, Δ_2 contains p eigenvalues each of positive type, Δ_3 contains $n-p$ eigenvalues each of negative type and Δ_4 contains $n-p$ eigenvalues each of positive type.

The important features of gyroscopically stabilized quadratic systems are considered in [1,2]

An important tool for finding eigenvalues is variational characterization of the nonlinear eigenvalue problems [3]. Although it is well known that eigenvalues of gyroscopically stabilized quadratic systems lie in four disjoint intervals, in practice the bounds of these intervals are not known. This makes variational characterization for this type of problem only partially feasible.

Kostić, Šikalo and Kustura in [4] present the shift strategy for nonoverdamped quadratic eigenproblems in which all the matrices participating in the corresponding quadratic pencil are Hermitian matrices and the matrix located beside $\lambda^2$is also a positive definite one. The applicationof the shift strategy is to extend the range at which variational characterization can be applied.

Since indefinite from matrix B does not affect the shift strategy the idea of this paper is to apply the mentioned strategy to the gyroscopically stabilized quadratic systems.

The remainder of the paper is organized in the following manner:
Section 2 of this paper presents the basic results about gyroscopically stabilized quadratic systems. In Section 3 we consider gyroscopically stabilized quadratic problems and an interval in which the variational characterization and shift strategy can be applied . We give conclusion and indications for further research in Section 4.

II. BASIC RESULTS

In this section we bring basic results for this type of problems. In [2] have been proved the following lemme:
Lemma 1. Condition (2) is equivalent to:

$$\rho\left(|\mathbf{B}|^{-1}(k\mathbf{I}+k^{-1}\mathbf{C})\right)<1 \text{ for some } k>0 \tag{4}$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix, or equivalently:

$$\min_{\omega>0}\rho\left(|\mathbf{B}|^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right)<1 \tag{5}$$

Lemma2

$$\rho\left(\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right)<\rho\left(|\mathbf{B}|^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right) \tag{6}$$

Let:

$$\rho_{+}\left(\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right)=max\left\{\lambda>0\middle|\,\lambda\in\sigma\left(\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right)\right\} \tag{7}$$

$$\rho_{-}\left(\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right)=max\left\{\lambda>0\middle|-\lambda\in\sigma\left(\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right)\right\} \tag{8}$$

Since the spectrum of $\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})$ is real.

$$\rho\left(\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right)=\max\left\{\rho_{+}\left(\boldsymbol{B}^{-1}(\omega\boldsymbol{I}+\omega^{-1}\boldsymbol{C})\right),\rho_{-}\left(\boldsymbol{B}^{-1}(\omega\boldsymbol{I}+\omega^{-1}\boldsymbol{C})\right)\right\} \tag{9}$$

With:

$$\rho(\mathbf{B},\mathbf{C}):=\max\left\{\min_{\omega>0}\rho_{+}\left(\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right),\min_{\omega>0}\rho_{-}\left(\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right)\right\} \tag{10}$$

we have:

$$\rho(\mathbf{B},\mathbf{C})\le\min_{\omega>0}\rho\left(\mathbf{B}^{-1}(\omega\mathbf{I}+\omega^{-1}\mathbf{C})\right) \tag{11}$$

Assume that (11) is satisfied. Then there exist $k_1>0$ such that:

$$\rho_{+}\left(\boldsymbol{B}^{-1}(\omega\boldsymbol{I}+\omega^{-1}\boldsymbol{C})\right)<1 \tag{12}$$

and $k_2>0$ such that:

$$\rho_{-}\left(\boldsymbol{B}^{-1}(\omega\boldsymbol{I}+\omega^{-1}\boldsymbol{C})\right)<1 \tag{13}$$

In [2] next theorem is proved

Theorem1. Let $\mathbf{Q}(\lambda)=\lambda^2\mathbf{I}+\lambda\mathbf{B}+\mathbf{C}$ with $\mathbf{B}=\mathbf{B}^{\boldsymbol{H}}, \det\mathbf{B}\neq 0$ and $\mathbf{C}=\mathbf{C}^{\boldsymbol{H}}>0$. Let $k_1>0$ and $k_2>0$ satisfy (12) and (13) , and let $in(\mathbf{B})=(p,n-p,0)$**.** Then all eigenvalues of $\mathbf{Q}(\lambda)\mathbf{x}=\mathbf{0}$ are real with definite type, and there are $\boldsymbol{p}$ eigenvalues of negative type in $(-\infty,-k_1)$, p eigenvalues if positive type in $(-k_1,0)$, *n-p* eigenvalues of negative type in $(0,k_2)$ and *n-p* eigenvalues of positive type in $(k_2,+\infty)$.

In the proof of the Theorem1 in [1], the existence of k_1 and k_2 is proven existentially but not constructively.
So k_1 and k_2 are not specifically known to us, which makes it difficult to apply variational characterization.

Let us briefly summarize the variational characterization for this type of problem. With this aim we consider the equation:

$$f(\lambda\,;\,\mathbf{x}):=\mathbf{x}^{\mathbf{H}}\mathbf{Q}(\lambda)\mathbf{x}=0 \tag{14}$$

Then for $\mathbf{x}\neq\mathbf{0}$ the two complex roots of (14) are with:

$$p_{+}(\mathbf{x}):=-\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}}+\sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}}\right)^2-\frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}} \tag{15}$$

$$p_{-}(\mathbf{x}):=-\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}}-\sqrt{\left(\frac{\mathbf{x}^H\mathbf{B}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}}\right)^2-\frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}} \tag{16}$$

Between these complex roots (15) and (16) there are all eigenvalues of the corresponding eigenvalue problem $\mathbf{Q}(\lambda)\mathbf{x}=\mathbf{0}$. $\mathbf{Q}(\lambda)\mathbf{x}=\mathbf{0}$ has *2n* real eigenvalues. Eigenvalues obtained by the functional (15) are called eigenvalues of positive type. Eigenvalues obtained by the functional (15) are called eigenvalues of negative type. For the eigenvector $\mathbf{x}\neq\mathbf{0}$ there is $\mathbf{Q}(\lambda)\mathbf{x}=\mathbf{0}$ and therefore $f(\lambda\,;\,\mathbf{x}):=\mathbf{x}^{\mathbf{H}}\mathbf{Q}(\lambda)\mathbf{x}=0$.

Let:

$$p_{-}^{+}(\mathbf{x}):=\begin{cases}p_{-}(\mathbf{x}) \text{ if } p_{-}(\mathbf{x})>0\\ \infty \text{ else}\end{cases}\qquad p_{+}^{+}(\mathbf{x}):=\begin{cases}p_{+}(\mathbf{x}) \text{ if } p_{+}(\mathbf{x})>0\\ 0 \text{ else}\end{cases} \tag{17}$$

$$p_{-}^{-}(\mathbf{x}):=\begin{cases}p_{-}(\mathbf{x}) \text{ if } p_{-}(\mathbf{x})<0\\ \infty \text{ else}\end{cases}\qquad p_{+}^{-}(\mathbf{x}):=\begin{cases}p_{+}(\mathbf{x}) \text{ if } p_{+}(\mathbf{x})<0\\ 0 \text{ else}\end{cases} \tag{18}$$

If $\max\{p_{-}^{+}(\mathbf{x})\}<\min\{p_{+}^{+}(\mathbf{x})\}$ then all eigenvalues in Δ_3 and Δ_4 are minmax and maxmin values of $p_{-}^{+}$and p_{+}^{+}, respectively. If $\max\{p_{-}^{-}(\mathbf{x})\}<\min\{p_{+}^{-}(\mathbf{x})\}$ then all eigenvalues in Δ_1 and Δ_2 are minmax and maxmin values of $p_{-}^{-}$and p_{+}^{-}, respectively.
From these variational characterizations of eigenvalues one obtains a Sylvester Theorem for gyroscopically stabilized quadratic eigenvalue problems in an obvious way.

III. SHIFT STRATEGY

Without loss of generality, we only consider negative eigenvalues for gyroscopically stabilized quadratic eigenvalue problems. This means that we will limit ourselves in our considerations to the interval in Δ_1 and Δ_2. Let $0 < \lambda_1(\mathbf{C}) < \lambda_2(\mathbf{C}) <. \cdots < \lambda_{n-1}(\mathbf{C}) < \lambda_n(\mathbf{C})$ be eigenvalues of the eigenvalue problems **Cx=μx** $(\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0})$ and $\lambda_1(\mathbf{B}) < \lambda_2(\mathbf{B}) < \cdots < \lambda_{l-1}(\mathbf{B}) < 0 < \lambda_l(\mathbf{B}) < \cdots < \lambda_{n-1}(\mathbf{B}) < \lambda_n(\mathbf{B})$ are eigenvalues of the eigenvalue problems **Bx=νx** $(\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0})$.

Obviously $\left(-\lambda_n(\mathbf{B}), -\frac{\lambda_n(\mathbf{B})}{2}\right) \subseteq \Delta_1$ and $\left(-\frac{\lambda_l(\mathbf{B})}{2}, 0\right) \subseteq \Delta_2$ and to these two intervals we can apply variational characterizations with the purpose of determining the negative eigenvalues of a given type. There is a logical question of what happens in the interval $\left(-\frac{\lambda_n(\mathbf{B})}{2}, -\frac{\lambda_l(\mathbf{B})}{2}\right)$. To expand the interval to which we can apply variational characterizations for the negative eigenvalues of the negative type we will introduce the shift strategy. Shift strategy is based on joining gyroscopically stabilized quadratic pencil to a corresponding hyperbolic quadratic pencil. Let :

$$\mathbf{Q_2}(\lambda) = \lambda^2\mathbf{I} + \lambda\mathbf{D} + \mathbf{C} - 1.5 \cdot \lambda_{\boldsymbol{n}}(\mathbf{C})\,\mathbf{I} \tag{19}$$

where $\mathbf{B} = \begin{pmatrix} \mathbf{B_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_2} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} \mathbf{B_1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B_2} \end{pmatrix}$, $\mathbf{B_1} > 0$ and $\mathbf{B_2} < 0$ be a quadratic pencil is a corresponding hyperbolic quadratic pencil that has two distinct real roots. Duffin in [5] has proved that the hyperbolic quadratic eigenvalue problems satisfies conditions of the variational characterization of the eigenvalues. Obliviously the quadratic pencil (19) is hyperbolic. Appropriate functioanls for hyperbolic pencil (19) are:

$$\bar{p}_+(\mathbf{x}) \coloneqq -\frac{\mathbf{x}^H\mathbf{D}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}} + \sqrt{\left(\frac{\mathbf{x}^H\mathbf{D}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} + 1.5 \cdot \lambda_{\boldsymbol{n}}(\mathbf{C})} \tag{20}$$

$$\bar{p}_-(\mathbf{x}) \coloneqq -\frac{\mathbf{x}^H\mathbf{D}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}} - \sqrt{\left(\frac{\mathbf{x}^H\mathbf{D}\mathbf{x}}{2\mathbf{x}^H\mathbf{x}}\right)^2 - \frac{\mathbf{x}^H\mathbf{C}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} + 1.5 \cdot \lambda_{\boldsymbol{n}}(\mathbf{C})} \tag{21}$$

The ranges $\bar{J}_+ \coloneqq \bar{p}_+(\mathbb{C}^n \backslash \{\mathbf{0}\})$ and $\bar{J}_- \coloneqq \bar{p}_-(\mathbb{C}^n \backslash \{\mathbf{0}\})$ are disjoint real intervals with $\max\bar{J}_- < \min\bar{J}_+$.
Theorem 2 is proved similar to the corresponding theorem in [4].

Theorem 2 Let $\bar{p}_-(\mathbf{x}) = \max\bar{J}_-$ then for every $\mathbf{y} \in \mathbb{C}^n$ for which $\left(\frac{\mathbf{y}^H\mathbf{B}\mathbf{y}}{2\mathbf{y}^H\mathbf{y}}\right)^2 - \frac{\mathbf{y}^H\mathbf{C}\mathbf{y}}{\mathbf{y}^H\mathbf{y}} > 0$ is:

$$p_+(\mathbf{y}) \coloneqq -\frac{\mathbf{y}^H\mathbf{B}\mathbf{y}}{2\mathbf{y}^H\mathbf{y}} + \sqrt{\left(\frac{\mathbf{y}^H\mathbf{B}\mathbf{y}}{2\mathbf{y}^H\mathbf{y}}\right)^2 - \frac{\mathbf{y}^H\mathbf{C}\mathbf{y}}{\mathbf{y}^H\mathbf{y}}} \geq \bar{p}_-(\mathbf{x}) \tag{22}$$

Now it is obvious that stands following theorem.

Theorem 3 Let $b = \max\{-\frac{\lambda_n(\mathbf{B})}{2}, \max\bar{J}_-\}$. Then $(-\lambda_n(\mathbf{B}), b) \subseteq \Delta_1$ and to $(-\infty, b)$ we can apply the variational characterization in order to determine the negative eigenvalues of the negative type.

Example1 In numerical experiment, the matrices B,D and C were generated by following MATLAB statements:
randn('state',0);
b=[1000000;5000;2000;-500;-6000]; **d**=[1000000;5000;2000;500;6000];
B=diag(**b**); **D**=diag(**d**);
I=eye(5);**C**=randn(5); **C=C'*C**;
Suitable gyroscopically stabilized quadratic eigenvalue problem has three negative eigenvalues of the negative type as follows:
$\lambda_1 = -999999.9, \lambda_2 = -4999.9$i $\lambda_3 = -1999.9$. By using the shift strategy we obtain the interval (-1000000, -5.00.0255) to which we can apply the variational characterization . We note that all of the negative eigenvalues of the negative type are covered by the obtained interval

IV. CONLUSION

In this paper we presented a shift strategy for gyroscopically stabilized quadratic eigenvalue problems. In order to justify the shift strategy and the introduction of a corresponding hyperbolic quadratic pencil, we considered the properties of gyroscopically stabilized quadratic eigenvalue problems. Although all of the eigenvalues of these problems lie in four disjoint intervals depending on their positivity and their type, the application of the variational characterization is not simple, since these four intervals are not known in advance.

In future research we will deal with an improvement in the interval in which variational characterization can be applied. We will try to use more of the matrix properties that enter the gyroscopically stable quadratic pencil. Here we primarily refer to the use of the eigenvalues of the matrix **C** as well as the eigenvalues of the problem **Bx** = μ**Cx**.

REFERENCES

[1] L. Barkwell, P. Lancaster, and A. S. Markus, Gyroscopically stabilized systems: a class pf quadratic eigenvalue problems with real spectrum, *Can. J. Math., 44*, 1992, 42-53.
[2] P. Lancaster, A. S. Markus, and F. Zhou, A wider class of stable gyroscopic systems, *Linear Algebra Appl., 370*, 2003, 257-267.
[3] H. Voss, A minmax principle for nonlinear eigenproblems depending continuously on the eigenparameter,*Numer. Lin. Algebra Appl*., *16*, 2009, 899 – 913.
[4] A. Kostić, Š. Šikalo, and M. Kustura, Shift Strategy for Non-overdamped Quadratic Eigen-Problems, *J. Stat. Math*., *2*, 2016, 08-15.
[5] R. J. Duffin, A minimax theory for overdamped networks, *J. Rat. Mech. Anal*., *4*, 1955, 221 – 233.