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Oscillation of Solutions to Neutral Delay and Advanced Difference Equations with Positive and Negative Coefficients

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ABSTRACT: *In this article we give infinite-sum conditions for the oscillation of all solutions of the following first order neutral delay and advanced difference equations with positive and negative coefficientsof the forms*

$$\Delta[x(n)-p(n)x(n-\tau)]+\sum_{i=1}^{m}q_i(n)\,x\,(n-\sigma_i)-\sum_{j=1}^{k}r_j(n)\,x(n-\rho_j)=0; \qquad n=0,1,2,\ldots, \qquad (*)$$

and

$$\Delta[x(n)-p(n)x(n+\tau)]+\sum_{i=1}^{m}q_i(n)\,x\,(n+\sigma_i)-\sum_{j=1}^{k}r_j(n)\,x(n+\rho_j)=0; \qquad n=0,1,2,\ldots, \qquad (**)$$

where $\{p(n)\}$ *is a sequence of nonnegative real numbers,* $\{q_i(n)\}$ *and* $\{r_j(n)\}$ *are sequences of positive real numbers,* τ,σ_i *and* ρ_j *are positive integers. We derived sufficient conditions for oscillation of all solutions of* $(*)$ *and* $(**)$.

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I. INTRODUCTION

We consider the following first order neutral delay difference equation of the form

$$\Delta[x(n)-p(n)x(n-\tau)]+\sum_{i=1}^{m}q_i(n)\,x\,(n-\sigma_i)-\sum_{j=1}^{k}r_j(n)\,x(n-\rho_j)=0; \qquad n=0,1,2,\ldots, \qquad (1.1)$$

and the first order neutral advanced difference equation of the form

$$\Delta[x(n)-p(n)x(n+\tau)]+\sum_{i=1}^{m}q_i(n)\,x\,(n+\sigma_i)-\sum_{j=1}^{k}r_j(n)\,x(n+\rho_j)=0;\ n=0,1,2,\ldots, \qquad (1.2)$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$.

Throughout the paper we assume the following conditions:

(H_1) $\{p(n)\}$ is a sequence of nonnegative real numbers;

(H_2) $\{q_i(n)\}$(i=1,2,…,m) are sequences of positive real numbers;

(H_3) $\{r_j(n)\}$(j=1,2,…,k) are sequences of positive real numbers;

(H_4) τ,σ_i $(i=1,2,\ldots,m)$ and ρ_j $(j=1,2,\ldots,k)$ are nonnegative integers;

Let $n^*=\max\{\tau,\sigma_i,\rho_j\}$ for i=1,2,…,m and j=1,2,…,k. A solution of (1.1) on $N(n_0)=\{n_0,n_0+1,\ldots\}$ is defined as a real sequence $\{x(n)\}$ defined $for\ n\ge n_0-n^*$ and which satisfies (1.1) for $n\epsilon N(n_0)$. A solution $\{x(n)\}$ of (1.1) on $N(n_0)$ is said to be oscillatory if for every positive integer $N_0>n_0$, there exist $n\ge N_0$ such that $x(n)x(n+1)\le 0$, otherwise $\{x(n)\}$ is said to be non oscillatory.

Furthermore, unless otherwise stated, when we write a functional inequality it indicates that it holds for all sufficiently large values of n.

The utilization of the theory of difference equations is briskly advancing to various areas such as control theory, numerical analysis, finite mathematics and computer science. Especially, the relation between the theory of difference equations and computer science has become more significance in the past few years for the basic theory of difference equations we refer to the monographs by Agarwal [1], Györi and Ladas [4], Agarwal and Wong [2] and Lakshmikantham and Trigiante [5].

In [7], Öcalan et al. established some oscillation criteria for the difference equation (1.1) and also derive sufficient conditions for the existence of positive solution for the equation (1.1). In [6], the authors determined sufficient conditions for the oscillation of all the solutions of the equations (1.1) and (1.2) under the conditions that $0 \le p(n) \le p < 1$ and $\{p(n)\}$ is monotonically.

In this paper, our aim is to determine sufficient conditions for oscillation of all solutions of the equations (1.1) and (1.2). The results obtained are discrete analogues of the well known results due to [3].

II. SOME USEFUL LEMMAS.

Lemma 2.1. C*onsider the delay difference inequality*

$$\Delta x(n) + \sum_{i=1}^{r} R_i(n)\, x(n-\tau_i) \le 0; \qquad n = 0,1,2,\ldots, \tag{2.1}$$

where $\{R_i(n)\}$*, i=1,2,…,r aresequences of positive real numbers. Suppose that*

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\tau_i} R_i(s) > 0 \qquad \mathit{for\ some}\ i \tag{2.2}$$

and $\{x(n)\}$ *is an eventually positive solution of (2.1), then for the same i,*

$$\liminf_{n\to\infty} \frac{x(n-\tau_i)}{x(n)} < \infty. \tag{2.3}$$

Proof: From (2.1), we have

$$\Delta x(n) + R_i(n) x(n-\sigma_i) \le 0. \tag{2.4}$$

In view of the assumption that there exist a constant $d > 0$ and a sequence $\{n_k\}$ of integers such that $n_k \to \infty$ *as* $k \to \infty$ and

$$\sum_{s=n_k}^{n_k+\tau_i} R_i(s) \ge d, \quad k = 1,2,3, \ldots.$$

Then there exists $\xi_k \epsilon \{n_k, \ n_k+1, \ldots, n_k+\sigma\}$ for every k such that

$$\sum_{s=n_k}^{\xi_k} R_i(s) \ge \frac{d}{2} \text{and} \sum_{s=n_k}^{n_k+\tau_i} R_i(s) \ge \frac{d}{2}. \tag{2.5}$$

Summing the inequality (2.4) from n_k to ξ_k and from ξ_k to $n_k+\tau_i$, we see that

$$x(\xi_k+1) - x(n_k) + \sum_{s=n_k}^{\xi_k} R_i(s) x(s-\tau_i) \le 0 \tag{2.6}$$

and

$$x(n_k+\tau_i+1) - x(\xi_k) + \sum_{s=\xi_k}^{n_k+\tau_i} R_i(s) x(s-\tau_i) \le 0. \tag{2.7}$$

By omitting the first terms in (2.6) and (2.7) and by using the decreasing nature of $\{x(n)\}$ and (2.5), we find

$$-x(n_k) + \frac{d}{2} x(\xi_k - \tau_i) \le 0 \tag{2.8}$$

and

$$-x(\xi_k) + \frac{d}{2}x(n_k) \le 0. \qquad (2.9)$$

From (2.7) and (2.8), we have

$$\frac{x(\xi_k - \tau_i)}{x(\xi_k)} \le \left(\frac{d}{2}\right)^2.$$

This completes the proof.

Lemma 2.2. If (2.1) has an eventually positive solution, then

$$\sum_{s=n}^{n+\tau_i} R_i(s) < 1, \quad i = 1,2,\dots,r. \qquad (2.10)$$

Proof: Assume that $\{x(n)\}$is an eventually positive solution of (2.1). Then from (2.1), we have

$$\Delta x(n) + R_i(n)x(n-\tau_i) \le 0, \qquad i = 1,2,\dots,r. \qquad (2.11)$$

Summing the above inequality from *n* to$n+\tau_i$, we have

$$x(n+\tau_i+1) - x(n) + \sum_{s=n}^{n+\tau_i} R_i(s)x(n-\tau_i) \le 0$$

or

$$-x(n) + \sum_{s=n}^{n+\tau_i} R_i(s)x(n-\tau_i) \le 0. \qquad (2.12)$$

Using decreasing nature of $\{x(n)\}$, we have

$$x(n)\left(-1 + \sum_{s=n}^{n+\tau_i} R_i(s)\right) \le 0,$$

which implies that

$$\sum_{s=n}^{n+\tau_i} R_i(s) \le 1, \qquad i = 1,2,\dots,r.$$

This completes the proof.

By applying the same procedure as we followed in the proof of the Lemma 2.2, we can prove the following lemma for advanced argument.

Lemma 2.3. *If the advanced difference inequality has an eventually positive solution*

$$\Delta x(n) - \sum_{i=1}^{r} R_i(n)x(n+\tau_i) \ge 0; \quad n = 0,1,2,\dots, \qquad (2.13)$$

then

$$\sum_{s=n-\tau_i}^{n-1} R_i(s) \le 1, \qquad i = 1,2,\dots,r.$$

III. OSCILLATION OF SOLUTIONS.

Our aim in this section is to establish infinite-sum conditions for oscillation of all solutions of (1.1) and (1.2).

Theorem 3.1.*Suppose that*

(H_5)*there exist a positive integer* $l \le m$ *and a partition of the set* $\{1,2,\dots,k\}$ *into l disjoint subsets* $J_1, J_2, \dots, J_l$ *such that* $j \epsilon J_i$ *implies* $\rho_j \le \sigma_i$;

(H_6) $g_i(n) = q_i(n) - \sum_{u\epsilon J_i} r_u(n - \sigma_i + \rho_u) \geq 0$ *and are not identically zero for i=1,2,…,l,*$g_i(n) = q_i(n)$ *for* $i = l + 1, l + 2, \ldots, m;$

(H_7) $\sigma_m = max\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ *and* $\sum_{i=1}^{m} \sum_{s=n+1}^{n+\sigma_i} g_i(s) > 0$ *for* $n \geq n_0$ *for some* $n_0 \geq 0.$

(H_8) $\sum_{i=1}^{l} \sum_{u\epsilon J_i} \left(\sum_{s=n-\sigma_i+\rho_u}^{n-1} r_u(s)\right) \leq 1 - p(n).$

Suppose further that

$(H_9)$$\limsup_{n\to\infty} \sum_{s=n}^{n+\sigma_m} g_m(s) > 0;$

$(H_{10})$$\sum_{n=0}^{\infty} \left(\sum_{i=1}^{m} g_i(n)\right) ln\left[e \sum_{i=1}^{m} \sum_{s=n+1}^{n+\sigma_i} g_i(s)\right] = +\infty;$

Then every solution of (1.1) is oscillatory.

Proof: Assume the contrary. Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1).Set z(n) as

$$z(n) = x(n) - p(n)x(n - \tau) - \sum_{i=1}^{l} \sum_{u\epsilon J_i} \sum_{s=n-\sigma_i+\rho_u}^{n-1} r_u(s)x(s - \rho_u). \quad (3.1)$$

$$\Delta z(n) = \Delta[x(n) - p(n)x(n - \tau)] - \sum_{i=1}^{l} \sum_{u\epsilon J_i} \left[\sum_{s=n+1-\sigma_i+\rho_u}^{n} r_u(s)x(s - \rho_u) - \sum_{s=n-\rho_i+\sigma_u}^{n-1} r_u(s)x(s - \rho_u)\right]$$

$$= - \sum_{i=1}^{m} q_i(n)x(n - \sigma_i) + \sum_{j=1}^{k} r_j(n)x(n - \rho_j)$$

$$- \sum_{i=1}^{l} \sum_{u\epsilon J_i} [r_u(n)x(n - \rho_u) - r_u(n - \sigma_i + \rho_u)x(n - \sigma_i)]$$

$$= - \sum_{i=1}^{m} q_i(n)x(n - \sigma_i) + \sum_{i=1}^{l} \sum_{u\epsilon J_i} r_u(n - \sigma_i + \rho_u)x(n - \sigma_i)$$

or

$$\Delta z(n) = - \sum_{i=1}^{m} g_i(n)x(n - \sigma_i) \leq 0 \quad (3.2)$$

or

$$\Delta z(n) + \sum_{i=1}^{m} g_i(n)z(n - \sigma_i) \leq 0. \quad (3.3)$$

This shows that $\{z(n)\}$ is a nonincreasing sequence. Also by Lemma 2.1 in [3], we can easily show that $z(n) > 0$, eventually.

Define a sequence $\{u(n)\}$ by

$$u(n) = \frac{-\Delta z(n)}{z(n)}. \quad (3.4)$$

We can easily prove that

$$u(n) \geq \sum_{i=1}^{m} g_i(n) exp\left(\sum_{s=n-\sigma_i}^{n-1} u(s)\right). \quad (3.5)$$

Let

$$A(n) = \sum_{i=1}^{m} \sum_{s=n+1}^{n+\sigma_i} g_i(s). \quad (3.6)$$

Using the inequality

$$e^{rx} \geq x + \frac{\ln(er)}{r}, \qquad x, r > 0,$$

we have from (3.5) and (3.6),

$$u(n) \geq \sum_{i=1}^{m} g_i(n) \exp\left(\frac{A(n)}{A(n)} \sum_{s=n-\sigma_i}^{n-1} u(s)\right)$$

$$\geq \sum_{i=1}^{m} g_i(n)\left(\frac{1}{A(n)} \sum_{s=n-\sigma_i}^{n-1} u(s) + \frac{\ln(eA(n))}{A(n)}\right),$$

or

$$u(n) \sum_{i=1}^{m} \sum_{s=n+1}^{n+\sigma_i} g_i(s) - \sum_{i=1}^{m} g_i(n) \sum_{s=n-\sigma_i}^{n-1} u(s) \geq \sum_{i=1}^{m} g_i(n) \ln\left(e \sum_{i=1}^{m} \sum_{s=n+1}^{n+\sigma_i} g_i(s)\right). \tag{3.7}$$

Hence for $\eta > N$

$$\sum_{n=N}^{\eta-1} u(n)\left(\sum_{i=1}^{m} \sum_{s=n+1}^{n+\sigma_i} g_i(s)\right) - \sum_{n=N}^{\eta-1} \sum_{i=1}^{m} g_i(n)\left(\sum_{s=n-\sigma_i}^{n-1} u(s)\right) \geq \sum_{n=N}^{\eta-1}\left(\sum_{i=1}^{m} g_i(n)\right) \ln\left(e \sum_{i=1}^{m} \sum_{s=n+1}^{n+\sigma_i} g_i(s)\right). \tag{3.8}$$

By interchanging the order of summation, we have

$$\sum_{n=N}^{\eta-1} \sum_{i=1}^{m} g_i(n)\left(\sum_{s=n-\sigma_i}^{n-1} u(s)\right) \geq \sum_{i=1}^{m} \sum_{n=N}^{\eta-\sigma_i-1} u(n) \sum_{s=n+1}^{n+\sigma_i} g_i(s). \tag{3.9}$$

Using (3.9) in (3.8), we obtain

$$\sum_{i=1}^{m} \sum_{n=\eta-\sigma_i}^{\eta-1} u(n) \sum_{s=n+1}^{n+\sigma_i} g_i(s) \geq \sum_{n=N}^{\eta-1}\left(\sum_{i=1}^{m} g_i(n)\right) \ln\left(e \sum_{i=1}^{m} \sum_{s=n+1}^{n+\sigma_i} g_i(s)\right). \tag{3.10}$$

On the other hand by Lemma 2.2, we have

$$\sum_{s=n+1}^{n+\sigma_i} g_i(s) < 1, \qquad i = 1,2,\dots,m, \quad eventually. \tag{3.11}$$

Then by (3.10) and (3.11), we see that

$$\sum_{i=1}^{m} \sum_{n=\eta-\sigma_i}^{\eta-1} u(n) \geq \sum_{n=N}^{\eta-1}\left(\sum_{i=1}^{m} g_i(n)\right) \ln\left(e \sum_{s=n+1}^{n+\sigma_i} \sum_{i=1}^{m} g_i(s)\right)$$

or

$$\sum_{i=1}^{m} \ln\left(\frac{z(\eta-\sigma_i)}{z(\eta)}\right) \geq \sum_{n=N}^{\eta-1}\left(\sum_{i=1}^{m} g_i(n)\right) \ln\left(e \sum_{s=n+1}^{n+\sigma_i} \sum_{i=1}^{m} g_i(s)\right). \tag{3.12}$$

In view of (H_{10}), we have

$$\lim_{n\to\infty} \prod_{i=1}^{m} \frac{z(n-\sigma_i)}{z(n)} = +\infty. \tag{3.13}$$

This implies that

$$\lim_{n\to\infty} \frac{z(n-\sigma_m)}{z(n)} = +\infty. \tag{3.14}$$

However by Lemma2.1, we have

$$\lim_{n\to\infty} inf \frac{y(n-\sigma_m)}{y(n)} < \infty.$$

This is a contradiction to (3.14) and completes the proof.

Theorem 3.2. *Assume that $\rho_j > 1, i = 1,2, \dots, k$. Suppose that*

(H_{11}) there exist a positive integer $l \leq k$ and a partition of the set {1,2,…,m} into l disjoint subsets $I_1, I_2, \dots, I_l$ such that $i \epsilon I_j$ implies $\rho_j \geq \sigma_i$;

(H_{12}) $h_j(n) = r_j(n) - \sum_{i\epsilon I_j} q_i(n + \rho_j - \sigma_i) \geq 0$ for $j = 1,2, \dots, l$ and $h_j(n) = r_j(n)$ for j=l+1,l+2,…,k.

Suppose further that
(H_{13}) $\sum_{n=0}^{\infty} \sum_{j=1}^{k} h_j(n) = +\infty$;

(H_{14}) $\sum_{n=0}^{\infty} \left(\sum_{j=1}^{k} h_j(n)\right) ln \left(e \sum_{j=1}^{k} \sum_{s=n-\sigma_j+1}^{n-1} h_j(s)\right) = +\infty$.

Then every solution of (1.2) is either oscillatory or $\liminf_{n\to\infty} x(n) = 0$.

Proof. Assume the contrary. Without loss of generality, we may assume that $\{x(n)\}$ is an eventually positive solution of (1.2) such that

$$\liminf_{n\to\infty} x(n) > 0. \qquad (3.15)$$

Set

$$z(n) = x(n) - p(n)x(n+\tau) - \sum_{j=1}^{l} \sum_{i\epsilon I_j} \sum_{s=n}^{n+\rho_j-\sigma_i-1} q_i(s)x(s-\sigma_i). \qquad (3.16)$$

Then from (1.2) and (3.16), we obtain

$$\Delta z(n) = -\sum_{i=1}^{m} q_i(n)x(n+\sigma_i) + \sum_{j=1}^{k} r_j(n)x(n+\rho_j) - \sum_{j=1}^{l} \sum_{i\epsilon I_j} \left(q_i(n+\rho_j-\sigma_i)x(n+\rho_j) - q_i(n)x(n+\sigma_i)\right)$$

$$= \sum_{j=1}^{l} r_j(n)x(n+\rho_j) - \sum_{j=1}^{l} \sum_{i\epsilon I_j} q_i(n+\rho_j-\sigma_i) + \sum_{j=l+1}^{k} r_j(n)x(n+\rho_j)$$

or

$$\Delta z(n) = \sum_{j=1}^{k} h_j(n)x(n+\rho_j) \geq 0 \qquad (3.17)$$

or

$$\Delta z(n) - \sum_{j=1}^{k} h_j(n)z(n+\rho_j) \geq 0. \qquad (3.18)$$

Clearly we see from (3.17) that $\{z(n)\}$ is an nondecreasing sequence. In view of (H_{13}) and (3.15) and from (3.17), we obtain $z(n) \to +\infty$ as $n \to \infty$. Since $\{z(n)\}$ is increases to $+\infty$, we have $z(n) > 0$, eventually.

Set

$$v(n) = \frac{\Delta z(n)}{z(n+1)}. \qquad (3.19)$$

Then we can easily show that

$$0 \leq v(n) < 1 \qquad (3.20)$$

and

$$v(n) \geq \sum_{j=1}^{k} h_j(n) exp\left(\sum_{s=n+1}^{n+\rho_j-1} v(s)\right). \qquad (3.21)$$

Let

$$B(n) = \sum_{j=1}^{k} \sum_{s=n-\rho_j+1}^{n-1} h_j(s). \qquad (3.22)$$

Using the inequality

$$e^{rx} \geq x + \frac{\ln(er)}{r}; \quad x, r > 0,$$

we have from (3.21) and (3.22)

$$v(n) \geq \sum_{j=1}^{k} h_j(n)\, exp\left(\frac{B(n)}{B(n)} \sum_{s=n+1}^{n+\rho_j-1} v(s)\right)$$

$$\geq \sum_{j=1}^{k} h_j(n)\left(\frac{1}{B(n)} \sum_{s=n+1}^{n+\rho_j-1} v(s) + \frac{ln(e\,B(n))}{B(n)}\right)$$

or

$$v(n) = \sum_{j=1}^{k} \sum_{s=n-\rho_j+1}^{n-1} h_j(s) - \sum_{j=1}^{k} h_j(n) \sum_{s=n+1}^{n+\rho_j-1} v(s)$$

$$\geq \sum_{j=1}^{k} h_j(n)\, ln\left(e \sum_{j=1}^{k} \sum_{s=n-\rho_j+1}^{n-1} h_j(s)\right). \qquad (3.23)$$

Hence for $\mu > N$ with N sufficiently large, we have

$$\sum_{n=N}^{\mu} v(n)\left(\sum_{j=1}^{k} \sum_{s=n-\rho_j+1}^{n-1} h_j(s)\right) - \sum_{n=N}^{\mu} \sum_{j=1}^{k} h_j(n) \sum_{s=n+1}^{n+\rho_j-1} v(s)$$

$$\geq \sum_{n=N}^{\mu} \sum_{j=1}^{k} h_j(n)\; ln\left(e \sum_{j=1}^{k} \sum_{s=n-\rho_j+1}^{n-1} h_j(s)\right) \qquad (3.24)$$

or

$$\sum_{j=1}^{k}\left(\sum_{n=N}^{\mu} v(n) \sum_{s=n-\rho_j+1}^{n-1} h_j(s)\right) - \sum_{j=1}^{k} \sum_{n=N}^{\mu} h_j(n) \sum_{s=n+1}^{n+\rho_j-1} v(s)$$

$$\geq \sum_{n=N}^{\mu} \sum_{j=1}^{k} h_j(n)\; ln\left(e \sum_{j=1}^{k} \sum_{s=n-\rho_j+1}^{n-1} h_j(s)\right). \qquad (3.25)$$

We can easily show that

$$\sum_{n=N}^{\mu} h_j(n) \sum_{s=n+1}^{n+\rho_j-1} v(s) \geq \sum_{n=N+\rho_j-1}^{\mu} v(n) \sum_{s=n-\rho_j+1}^{n-1} h_j(s). \qquad (3.26)$$

Using (3.26) in (3.25), we obtain,

$$\sum_{j=1}^{k} \sum_{n=N}^{N+\rho_j-2} v(n) \sum_{s=n-\rho_j+1}^{n-1} h_j(s) \geq \sum_{n=N}^{\mu}\left(\sum_{j=1}^{k} h_j(n)\right) ln\left(e \sum_{j=1}^{k} \sum_{s=n-\rho_j+1}^{n-1} h_j(s)\right). \qquad (3.27)$$

From this and (H_{14}), we have

$$\sum_{j=1}^{k} \sum_{n=N}^{N+\rho_j-2} v(n) \sum_{s=n-\rho_j+1}^{n-1} h_i(s) = +\infty. \tag{3.28}$$

On the other hand, by Lemma2.3 and by (3.20), we have

$$\sum_{j=1}^{k} \sum_{n=N}^{N+\rho_j-2} v(n) \sum_{s=n-\rho_j+1}^{n-1} h_j(s) < \sum_{j=1}^{k} (\rho_j - 1). \tag{3.29}$$

This contradicts (3.28) and completes the proof.

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