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On Coincidence Points in Pseudocompact Tichonov Spaces and Common Fixed Points in Pseudocompact Topological Spaces

Uday Dolas
Department of Mathematics, C.S.A.P.G.College, Sehore-466001, M.P., INDIA

I. INTRODUCTION

Harinath[2] earlier established some fixed point theorems in Pseudocompact tichonov space. Jain and Dixit[3] established also some fixed point theorems in Pseudocompact tinconov spaces, which generalize the results of Fisher[1], Harinath[2] and Liu Zeqing[5] also established some coincidences point theorems in Pseudocompact tichonov space.

Certain common fixed point theorems for pairs of selfmaps on a Pseudocompact topological space and pairs as well as families of selfmaps on a. compact metric space are obtained by S.V.R. Naidu and K.P.R. Rao[4] and also certain sufficient conditions for the existence of coincidence points for at least one pair of four maps (two of which are multi-functions) on a compact metric space are discussed by them.

We define F to be a non-negative real valued function on X x X such that $F(x,y) = 0 \Leftrightarrow x = y$

1.1 Some coincidence point theorems in Pseudocompact tichonov spaces.

A topological space X is said to Pseudocompact iff every real valued continuous function, on X is bounded. By tichonov space we mean a completely regular Hausdorff space it may be noted that compact space is Pseudocompact. if X is an arbitrary tichonov space. Then X is Pseudocompact iff every real valued continuous function over X is bounded and assumed its bounds.

Throughout this section unless otherwise stated, X stands for a Pseudocompact tichonov space.

We improve the results of Fisher [1], Harinath [2], Jain and Dixit [3] in this section.

Now we establish the following.

Theorem 1.1

Let f, g and h be three self mapping on X, such that

(I) $f(x) \cup g(x) \subseteq h(x)$,

(II) the functions a and b defined on X by $a(x) = F(fx, hx)$ and $b(x) = F(hx, gx)$ are continuous on X and

(III) $F(fx, gy) < \max \{ F(hy, hx), F(fx,hx), F(hy,gy), ½ [F(fx,hy)+F(hy,hx)],$

$$\left. \frac{F(fx, \square x).F(\square y. gy)}{f(\square y, \square x)} \right\}$$

for all x,y in X and $hx \neq hy$, then f and h or g and h have a coincidence point.

Proof:

Since X is a Pseudocompact tichonov space it follows from (II) that there exist two points u and v in X such that

$a(u) = \inf \{a(x): x \in X\}$

and $b(v) = \inf \{b(x): x \in X\}$

we may assume, without loss of generality

$a(u) \leq b(v)$.

Since $f(x) \subseteq h(x)$, there exists a point w in X, such that

$fu = hw$.

We now assert that u is coincidence point of f and h. If not, let us suppose that $fu \neq hu$ i.e. $hu \neq hw$. Then using (III) we have

$F(hw,gw) < \max. \{ f(fu,hu), f(fu,hu), f(hw,gw), ½ [F(fu,hw) + F(hw,hu)].$

$$\left. \frac{F(fu,\square u).F(\square w,gw)}{F(\square w,\square u)} \right\}$$

i.e.

$F(hw,gw) < \max. \{F(fu,hu), F(fu,hu), f(hw,gw), ½ [F(hw,hw) + F(fu,hu)],$

$\frac{F(fu, \square u).F(\square w.gw)}{F(fu, \square u)}\}$

So that b(w) < max {a(u), a(u), b(w), ½ a(u), b(w) },
which implies that b(w) < max {a(u), b(w)} = a(u).
Since a(u) $\leq$ b(v) = inf {b(x) : x $\in$ X} $\leq$ b (w),

it follows that a(u) $\leq$ b(w) < a(u),
which is a contradiction and hence fu=hu.
This comples the proof.
As a consequence of Theorem 1.1 we have the following
Theorem 1.2
Let F be continuous, f, g, and h be three continuous self mapping on X satisfying (I) and (III), then f and h or g and h have a coincidence point.
Remark 1.
Theorem 1.2 extends, Theorem 1 and 3 of Harinath [2] and Theorem 2 of Jain and Dixit (3)
As a particular case of Theorem 1.2, we have following:
Corollary 1.3.
Let (X,d) be a compact metric space f,g and h be three continuous self mapping on X satisfying (I) and (III), then f and h or g and h have a coincidence point
Remark 2.
Corollary 1.3 generalizes fisher's Theorem 1 and 2 of [1]
Corollary 1.4
Let f.g and h be three self mapping on X. if in the Theorem 1.1, the condition (III) is replaced by any one of the following conditions (IV), (V), (VI), (VII), and (VIII), then also we arrive at the same conclusion as in Theorem 1.1.

(IV) F(fx, gy) <max{F(hx,hy), F(hx,fx), F(hy,gy), F(hy,fx), [F(hx,hy) F(hx,fx)] ½ }
(V) $[f(fx,gy)]^2$ < max{F(hx,hy). F(hx,fx), F(hy,fx). F(hx,gx), $[F(hx,fx)]^2$, $[F(hy,gy)^2$}
(VI) $[F(fx,gy)]^2$<max{F(hx,hy). F(hx,fx), F(hy,gy). F(fx,gy), F(hx,fy). F(hy,fx), F(hy,fx)}
(VII) F(fx,gy) <max {F(hy,hx), F(fx,hx), F(hy,gy), ½[F(fx,hy) + F(hy,hx)],

$$\left.\frac{[F(\square y, \square x).F(\square y.gy) + F(fx, \square x).F(fx, gy)]}{2\,F(\square y, \square x)}\right\}$$

and
(viii) F(fx,gy) < max {F(hy,hx), F(fx,hx), F(hy,gy), ½ [F(fx,hy) + F(hy,hx)],

$$\left.\frac{[F(\square x, \square y).F(\square y.gy) + F(fx, \square x).F(\square y, \square x)]}{F(\square x, gy) + F(\square y, gy)}\right\}$$

We define a mapping $\emptyset : (R^+)^5 \rightarrow R^+$ such that

(i) $\emptyset$ is non-decreasing in each coordinate variable

and (ii) k(t) = $\emptyset$(t,t,t,t,t) <t,for each t> 0.

Theorem 1.4
Let f,g and h be three self mappings on X such that
(IX) f(x) ∪ g(x) ⊆ h(x),

(X) the functions a and b defined on X by a(x) = F (fx,hx) and b(x) = F (hx,gx) are continuous on X and
(XI) F (fx,gy) < [$\emptyset$ {F^2 (hy,hx), F(hx,hy). F(hx,fx), F(fx,fy). F(hy,gy)],

F(hy,gy). F (fx,gy), F^2 (fx,gy)}]^{½}
for all x,y in X and hx ≠ hy, then f and h or g and h have a coincidence point.

Proof:
Proceeding as in the proof of Theorem 1.1, we have

$$b(w) < [\emptyset\{a^2(u), a(u).\ a(u), a(w).\ b(w), b(w).\ b(w), b^2(w)\}]^{1/2}$$
$$\leq [\emptyset\{b^2(w), b^2(w), b^2(w), b^2(w).\ b^2(w)\}]^{1/2}$$
$$= [k(b^2(w))]^{1/2}$$

$< \quad [b^2 (w)]^{1/2}$

$= \quad b(w)$,

Which is a contradiction

Hence f and g or g and h have a coincidence point.

1.2 Common fixed point theorem in Pseudocompact topological sapce

Inspired by the theorem of S.V.R. Naidu and K.P.R. Rao [4], we now prove the following common fixed point theorem in Pseudocompact topological space, for a pair of self maps, when the maps together satisfy certain generalized contractive conditions with reference to a metric-type function on the space.

We begin with the following known definitions.

Definition 1.

A topological space is said to be Pseudocompact if every real-valued continuous function on it is bounded.

We note that ever real-valued continuous function on a Pseudocompact topological space attrains its bounds.

Definition 2.

A pair of self maps f and g on a metric space (X,d) are said to be weakly commutive if d (fgx,gfx) $\leq$ d (gx,fx) for all x $\in$ X.

Theorem 2.1

Let f and g be weakly commuting continuous self maps on a Pseudocompact topological space X such that,

(XII) $f(x) \subseteq g(x)$,

(XIII) the function $\emptyset$ defined on X by $\emptyset(x) = F(fx,gx)$ is continuous

and,

(XIV) $F(fx,fy) < \{F(gx,gy), F(fx,gx), F(fy,gy), \frac{1}{2}[F(gx,gy) + F(fy,gy)], \frac{F(gx,gy).F(fx.gx)}{F(fx,fy)}\}$

Whenever x,y in X are such that fx ≠ fy and gx ≠ gy. Then f and g have a coincidence point. Also f and g have a unique common fixed point.

Proof:

Since X is pseudo-compact, from (XIII) it follows that there exits a point v in X, such that

$$\emptyset(v) = \min\{\emptyset(x) : x \in X\}$$

From (XII), there exists a point w in X, such that fv = gw.

We now assert that w or v is a coincidence point of f and g. if not,

let us suppose that fw ≠ gw and fv ≠ gv, that is fv ≠ fw and gv ≠ gw.

Taking x = w and y= v in inequality (XIV), we get

$$F(fw,fv) < \max\{F(gw,gv), F(fw,gw), F(fv,gv), \tfrac{1}{2}[F(gw,gv) + F(fv,gv)], \frac{F(gw,gv).F(fw.gw)}{F(fw,fv)}\}$$

i.e.

$$F(fw,gw) < \max\{F(fv,gv), F(fw,gw), F(fv,gv), \tfrac{1}{2}[F(fv,gv) + F(fv,gv)], \frac{F(fv,gv).F(fw.gw)}{F(fw,gw)}\}$$

This implies that

$$\emptyset(w) < \max\{\emptyset(v), \emptyset(w), \emptyset(v), \tfrac{1}{2}[\emptyset(v)+ \emptyset(v)], \frac{\emptyset(v).\emptyset(w)}{\emptyset(w)}\}$$

i.e. $\emptyset(w) < \max\{\emptyset(v), \emptyset(w), \emptyset(v), \emptyset(v), \emptyset(v),\} = \emptyset(v)$,

but $\emptyset(v) = \min\{\emptyset(x) : x \in X\} \leq \emptyset(w)$.

Therefore from the above inequality we get $\emptyset$ (w) < $\emptyset$ (w),

Which is a contradiction,
Hence either fv= fw or gv = gw. Since fv= gw so that fw = gw or gv = fv.
In follows from this that either v or w is a coincidence point of f and g.
suppose that z is a coincidence point of f and g.
Since fz = gz and so by the weak commutativity of f and g,
we have fgz = gfz = g(gz) = g^2z.
if possible, suppose that $g^2z \neq gz$, then $fgz \neq fz$.
Taking x = gz and y=z in the inequality (XIV) we have

$$F(fgz, fz) < \max \left\{F(g^2z, gz), F(fgz,g^2z), F(fz,gz), \tfrac{1}{2} [F(g^2z,gz)+F(fz,gz)], \frac{F(g2z,gz).F(fgz,g2z)}{F(fgz,fz)}\right\}$$

i.e.

$$F(g^2z,gz) < \max \left\{F(g^2z,gz), F(g^2z,g^2z), F(gz,gz), \tfrac{1}{2} [F(g^2 z,gz) + F(gz,gz)], \frac{F(g2,gz).F(g2z,g2z)}{F(g2z,gz)}\right\}$$

So that,

$$F(g^2z,gz) < \max \{F(g^2z,gz), 0,0, \tfrac{1}{2} F(g^2z, gz),0\}$$
$$=F(g^2z,gz),$$

Which is a contradiction. Hence $g^2z = gz$.
Therefore f(gz)= gz and g(gz) = gz
This further implies that gz is a common fixed point of f and g.

Uniqueness:
Let w and w' be two common fixed point of f and g. so that by (XIV) we have

$$F(fw,fw') < \max\left\{F(gw,gw'), F(fw,gw), F(fw',gw'), \tfrac{1}{2} [F(gw,gw') + F(fw',gw')], \frac{F(gw,gw').F(fw.gw)}{F(fw,fw')}\right\}$$

i.e.

$$F(fw,fw') < \max\left\{F(w,w'), F(w,w), F(w',w'), \tfrac{1}{2} [F(w,w') + F(w',w')], \frac{F(w,w').F(w.w)}{F(w.w')}\right\}$$

This implies that
$F(w,w') < \max \{F(w,w'), 0,0 , \tfrac{1}{2} F(w,w'), 0\} = F(w,w')$

Which is a contradiction,
So that w= w'
Hence f and g have unique common fixed point.

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