International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 4 Issue 9 || November. 2016 || PP-01-08

Second Order Parallel Tensors and Ricci Solitons in S-space form

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ABSTRACT: *In this paper, we prove that a symmetric parallel second order covariant tensor in (2m+s)-dimensional S-space form is a constant multiple of the associated metric tensor. Then we apply this result to study Ricci solitons for S-space form and Sasakian space form of dimension 3.*

KEYWORDS: *Einstein metric, η-Einstein manifold, Parallel second order covariant tensor, Ricci soliton, S-space form.*

I. INTRODUCTION

A Ricci soliton is a generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0 \qquad (1.1)$$

where S is a Ricci tensor of M and L_V denotes the Lie derivative operator along the vector field V. Metrics satisfying (1.1) are interesting and useful in physics and are often referred as quasi-Einstein. A Ricci soliton is said to be shrinking, steady and expanding when λ is negative, zero and positive respectively.

In 1923, L.P. Eisenhart [1] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1926, H. Levy [2] proved that a second order parallel symmetric non-degenerated tensor in a space form is proportional to the metric tensor. In ([3], [4], [5]) R. Sharma generalized Levy's result and also studied a second order parallel tensor on Kaehler space of constant holomorphic sectional curvature as well as contact manifolds. Later Debasish Tarafdar and U.C. De [6]) proved that a second order symmetric parallel tensor on a P-Sasakian manifold is a constant multiple of the associated metric tensor, and that on a P-Sasakian manifold there is no non-zero parallel 2-form. Note that the Eisenhart problem have also been studied in [7] on P-Sasakian manifolds with a coefficient k, in [8] on α -Sasakian manifold, in [9] on $N(k)$ quasi Einstein manifold, in [10] on f-Kenmotsu manifold, in [11] on Trans-Sasakian manifolds and in [12] on (k, μ) -contact metric manifolds. Also the authors C.S. Bagewadi and Gurupadavva Ingalahalli ([13]), [14]) studied Second order parallel tensors on α -Sasakian and Lorentzian α -Sasakian manifolds. Recently C.S. Bagewadi and Sushilabai Adigond [15] studied L.P. Eisenhart problem to Ricci solitons in almost $C(\alpha)$ manifolds.

On the other hand, as a generalization of both almost complex (in even dimension) and almost contact (in odd dimension) structures, Yano introduced in [16] the notion of framed metric structure or f -structure on a smooth manifold of dimension $2n + s$, i.e a tensor field of type $(1,1)$ and rank $2n$ satisfying $f^3 + f = 0$. The existence of such a structure is equivalent to the tangent bundle $U(n) \times O(s)$. for manifolds with an f -structure f, D.E. Blair [17] has introduced the S -manifold as the analogue of the Kaehler structure in the almost complex case and of the quasi-Sasakian structure in the almost contact case and many authors [18], [19], [20] have studied the geometry of submanifolds of S -space form.

Motivated by the above studies in this paper we study second order parallel tensor on S -space form. As an application of this notion we study Ricci pseudo-symmetric S -space form. Also, we study Ricci solitons for $(2m + s)$ -dimensional S -space form and Sasakian space form of dimension 3 and obtain some interesting results.

II. PRELIMINARIES

Let N be a $(2n+s)$ -dimensional framed metric manifold (or almost r -contact metric manifold) with a framed metric structure $(f, \xi_\alpha, \eta_\alpha, g)$, $\alpha = \{1,2,\ldots, s\}$ where f is a $(1,1)$ tensor field defining an f -structure of rank $2n$, $\xi_1, \xi_2, \ldots, \xi_s$ are vector fields; $\eta_1, \eta_2, \ldots, \eta_s$ are 1 -forms and g is a Riemannian metric on N such that

$$f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha, \quad f(\xi_\alpha) = 0, \quad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \quad \eta_\alpha \circ f = 0 \quad (2.1)$$

$$g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \quad g(X, \xi_\alpha) = \eta_\alpha(X) \quad (2.2)$$

An framed metric structure is normal, if

$$[f, f] + 2\sum_\alpha d\eta_\alpha \otimes \xi_\alpha = 0 \quad (2.3)$$

where $[f, f]$ is Nijenhuis torsion of f .

Let F be the fundamental 2 -form defined by $F(X, Y) = g(fX, Y)$, $X, Y \in TN$. A normal framed metric structure is called S -structure if the fundamental form F is closed. that is $\eta_1 \wedge \eta_1 \wedge \ldots\ldots \wedge (d\eta_\alpha)^n \neq 0$ for any α , and $d\eta_1 = \ldots\ldots = d\eta_s = F$. A smooth manifold endowed with an S -structure will be called an S -manifold. These manifolds were introduced by Blair [17]. If $s = 1$, a framed metric structure is an almost contact metric structure, while S -structure is an Sasakian structure. If $s = 0$, a framed metric structure is an almost Hermitian structure, while an S -structure is Kaehler structure.

If a framed metric structure on N is an S -structure, then it is known that

$$(\nabla_X f)(Y) = \sum_\alpha \{g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X\} \quad (2.4)$$

$$\nabla_X \xi_\alpha = -fX, \quad X, Y \in TN, \quad \alpha = 1,2,\ldots, s \quad (2.5)$$

The converse also to be proved. In case of Sasakian structure (i.e $s = 1$) (2.4) implies (2.5). for $s > 1$, examples of S -structures given in [4], [5] and [6].

A plane section in T_pN is a f -section if there exists a vector $X \in T_pN$ orthogonal to $\xi_1, \xi_2, \ldots, \xi_s$ such that $\{X, fX\}$ span the section. The sectional curvature of a f -section is called a f -sectional curvature. If N is an S -manifold of constant f -sectional curvature k , then its curvature tensor has the form

$$R(X,Y)Z = \sum_{\alpha,\beta} \{\eta_\alpha(X)\eta_\beta(Z) f^2 Y - \eta_\alpha(Y)\eta_\beta(Z) f^2 X - g(fX, fZ)\eta_\beta(Y)\xi_\beta \quad (2.6)$$

$$+ g(fY, fZ)\eta_\alpha(X)\xi_\beta\} + \frac{1}{4}(k+3s)\{-g(fY, fZ) f^2 X + g(fX, fZ) f^2 Y\}$$

$$+ \frac{1}{4}(k-s)\{g(X, fZ) fY - g(Y, fZ) fX + 2g(X, fY) fZ\}$$

for all $X, Y, Z, W \in TN$. Such a manifold $N(k)$ will be called an S -space form. The euclidean space E^{2n+s} and hyperbolic space H^{2n+s} are examples of S -space forms. When $s = 1$, an S -space form reduces to a Sasakian space form and if $s = 0$ then it reduces to complex-space-form.

From (2.6) when $X = \xi_\alpha$ and $Z = \xi_\alpha$, we have the following.

$$R(\xi_\alpha, Y)Z = \sum_\alpha [g(Y, Z)\xi_\alpha - \eta_\alpha(Z)Y] \quad (2.7)$$

$$R(X, Y)\xi_\alpha = s\sum_\alpha [\eta_\alpha(Y)X - \eta_\alpha(X)Y] \quad (2.8)$$

Further from (2.5), we have

$$(\nabla_X \eta_\alpha)(Y) = g(X, fY) \quad (2.9)$$

Definition 2.1 *A S -manifold $(M^n, f, \eta_\alpha, \xi_\alpha, g)$ is to be η -Einstein if the Ricci tensor S of M is of the form*

$$S(X,Y) = ag(X,Y) + b\sum_{\alpha=1}^{s}\eta_\alpha(X)\eta_\alpha(Y) \quad (2.10)$$

where a, b are constants on M.

Let M be a $(2m+s)$ -dimensional S -space form then from (2.6), the Ricci tensor S is given by

$$S(X,Y) = \frac{4s + (k+3s)(2m-1) + 3(k-s)}{4} g(X,Y) + \frac{(2m+s-2)(4-k-3s) - 3(k-s)}{4}\eta_\alpha(X)\eta_\alpha(Y) \quad (2.11)$$

In (2.11), taking $Y = \xi_\alpha$ and $X = Y = \xi_\alpha$ we have

$$S(X,\xi_\alpha) = A\sum_\alpha \eta_\alpha(X) \quad (2.12)$$

$$S(\xi_\alpha,\xi_\alpha) = B \quad (2.13)$$

$$QX = AX \quad (2.14)$$

where

$$A = \frac{1}{4}[-3s^3 - (6m+k-13)s^2 + (14m - 2mk - k - 10)s + (2m+2)k] \quad (2.15)$$

$$B = \frac{1}{4}[-3s^4 - (6m+k-13)s^3 + (14m - 2mk - k - 10)s^2 + (2mk + 2k)s] \quad (2.16)$$

Remark 2.2 *If we take $m = 1$ and $s = 1$ in $(2m+s)$ -dimensional S -space form then it reduces to Sasakian-spce-form of dimension 3.*

In this case equations (2.8), (2.12), (2.13) and (2.14) reduces to

$$R(X,Y)\xi = [\eta(Y)X - \eta(X)Y] \quad (2.17)$$

$$S(X,\xi) = 2\eta(X) \quad (2.18)$$

$$QX = 2X \quad (2.19)$$

where $\xi_1 = \xi$ and $\eta_1 = \eta$.

III. PARALLEL SYMMETRIC SECOND ORDER TENSORS AND RICCI SOLITONS IN S-SPACE FORM

Let h be a symmetric tensor field of $(0,2)$ type which we suppose to be parallel with respect to ∇ i.e $\nabla h = 0$. Applying Ricci identity

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0 \quad (3.1)$$

We obtain the following fundamental relation

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0 \quad (3.2)$$

Replacing $Z = W = \xi_\alpha$ in (3.2) and by virtue of (2.8), we have

$$2\eta_\beta(\xi_\alpha)\sum_{\alpha,\beta}[h(f^2Y,\xi_\alpha)\eta_\alpha(X) - h(f^2X,\xi_\alpha\eta_\alpha(Y)] = 0 \quad (3.3)$$

by the symmetry of h.

Put $X = \xi_\alpha$ in (3.3) and by virtue of (2.1), we have

$$2\eta_\beta(\xi_\alpha)\sum_\alpha h(f^2Y,\xi_\alpha)\eta_\alpha(\xi_\alpha) = 0 \quad (3.4)$$

and supposing $2\eta_\beta(\xi_\alpha) \neq 0$. it results

$$\sum_{\alpha} h(Y,\xi_\alpha) = \sum_{\alpha} \eta_\alpha(Y) h(\xi_\alpha,\xi_\alpha) \qquad (3.5)$$

Differentiating (3.5) covariantly with respect to Z, we have

$$\sum_{\alpha} [(\nabla_Z h)(Y,\xi_\alpha) + h(\nabla_Z Y,\xi_\alpha) + h(Y,\nabla_Z \xi_\alpha)] \quad (3.6)$$

$$= \sum_{\alpha} [\{(\nabla_Z \eta_\alpha)Y + \eta_\alpha(\nabla_Z)Y\} h(\xi_\alpha,\xi_\alpha) + \eta_\alpha(Y)\{(\nabla_Z h)(\xi_\alpha,\xi_\alpha) + 2h(\nabla_Z \xi_\alpha,\xi_\alpha)\}]$$

By using the parallel condition $\nabla h = 0$ and (2.5) in (3.6), we have

$$\sum_{\alpha} h(Y,\nabla_Z \xi_\alpha) = \sum_{\alpha} (\nabla_Z \eta_\alpha)(Y) h(\xi_\alpha,\xi_\alpha) \quad (3.7)$$

Using (2.5) in (2.9) in (3.7), we get

$$-h(Y, fZ) = g(Z, fY) \sum_{\alpha} h(\xi_\alpha,\xi_\alpha) \qquad (3.8)$$

Replacing X by ϕX in (3.8), we have

$$h(Y,Z) = g(Y,Z) \sum_{\alpha} h(\xi_\alpha,\xi_\alpha) \qquad (3.9)$$

Using the fact that $\nabla h = 0$, we have from the above equation $h(\xi_\alpha,\xi_\alpha)$ is a constant. Thus, we can state the following theorem.

Theorem 3.1 *A symmetric parallel second order covariant tensor in S -space form is a constant multiple of the metric tensor.*

Corollary 3.2 *A locally Ricci symmetric $(\nabla S = 0)$ S -space form is an Einstein manifold.*

Remark 3.3 *The following statements for S -space form are equivalent.*

1. Einstein
2. locally Ricci symmetric
3. Ricci semi-symmetric
4. Ricci pseudo-symmetric i.e $R \cdot S = L_S Q(g,S)$.

where L_S is some function on $U_S = \{x \in M : S \neq \frac{r}{n} gatx\}$.

Proof. The statements $(1) \to (2) \to (3)$ and $(3) \to (4)$ is trivial. Now, we prove the statement $(4) \to (1)$ is true.

Here $R \cdot S = L_S Q(g,S)$ means

$$(R(X,Y) \cdot S)(U,V) = L_S[S((X \wedge Y)U,V) + S(U,(X \wedge Y)V] = 0 \qquad (3.10)$$

Putting $X = \xi_\alpha$ in (3.10), we have

$$S(R(\xi_\alpha,Y)U,V) + S(U,R(\xi_\alpha,Y)V) = L_S[((\xi_\alpha \wedge Y)U,V) + S(U,(\xi_\alpha \wedge Y)V)]$$

(3.11)

By using (2.7) in (3.11), we obtain

$$[L_S + 1] \sum_{\alpha} [g(Y,U)S(V,\xi_\alpha) - S(Y,V)\eta_\alpha(U) + g(Y,V)S(U,\xi_\alpha) - S(Y,U)\eta_\alpha(V)] = 0 \quad (3.12)$$

In view of (2.12), we obtain

$$[L_S + 1] \sum_{\alpha} [A\eta_\alpha(V)g(Y,U) - S(Y,V)\eta_\alpha(U) - Ag(Y,V)\eta_\alpha(U) - S(Y,U)\eta_\alpha(V)] = 0 \quad (3.13)$$

Putting $U = \xi_\alpha$ in (3.13) and by using (2.1) and (2.12), we get

$$[L_S + 1][s \cdot S(Y,V) - As \cdot g(Y,V)] = 0 \qquad (3.14)$$

If $L_S + 1 \neq 0$, then (3.14) reduces to

$$S(Y,V) = Ag(Y,V) \qquad (3.15)$$

where A is given by equation (2.15).
Therefore we conclude the following.

Proposition 3.4 *A Ricci pseudo-symmetric S -space form is an Einstein manifold if $L_S \neq -1$.*

Corollary 3.5 *A Ricci pseudo-symmetric Sasakian space form is an Einstein manifold if $L_S \neq -1$.*

Proof. If we take $s = 1$ in (2.15), we get

$$A = 2m \qquad (3.16)$$

Put this in (3.15), we have

$$S(Y,V) = 2mg(Y,V) \qquad (3.17)$$

Hence the proof.

Corollary 3.6 *Suppose that on a regular S -space form, the $(0,2)$ type field $L_V g + 2S$ is parallel where V is a given vector field. Then (g,V) yield a Ricci soliton. In particular, if the given S -space form is Ricci semi-symmetric with $L_V g$ parallel. we have the same conclusion.*

Proof: Follows from theorem (3.1) and corollary (3.2).

If V be the linear span of $\xi_1, \xi_2, ..., \xi_s$ i.e $V = c_1\xi_1 + c_2\xi_2 + ... + c_s\xi_s = \sum_{i=1}^{s} c_i\xi_i$ where $c_i \in F$ for $i = 1,2,..,s$ then Ricci soliton $(g, \xi_1, \xi_2, ..., \xi_s, \lambda)$ along V is given by

$$\left(\sum_{i=1}^{s} c_i L_{\xi_i}\right) g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0$$

We are interested in expressions for $\left(\sum_{i=1}^{s} c_i L_{\xi_i} g + 2S\right)$

A straight forward computation gives

$$\left(\sum_{i=1}^{s} c_i L_{\xi_i} g\right)(X,Y) = 0 \qquad (3.18)$$

from equation (1.1), we have $h(X,Y) = -2\lambda g(X,Y)$ and then putting $X = Y = \xi_\alpha$ for $\alpha = 1,2,..., s$, we have

$$h(\xi_\alpha, \xi_\alpha) = -2\lambda s \qquad (3.19)$$

where

$$h(\xi_\alpha, \xi_\alpha) = \left(\sum_{i=1}^{s} c_i L_{\xi_i} g\right)(\xi_\alpha, \xi_\alpha) + 2S(\xi_\alpha, \xi_\alpha) \qquad (3.20)$$

By using (2.13) and (3.18), we have

$$h(\xi_\alpha, \xi_\alpha) = \frac{1}{2} B \qquad (3.21)$$

Equating (3.19) and (3.21), we get

$$\lambda = \frac{1}{16}[3s^3 + (6m + k - 13)s^2 - (14m - 2mk - k - 10)s - (2mk + 2k)] \qquad (3.22)$$

Hence we state the following:

Theorem 3.7 *A Ricci soliton $(g, \xi_1, \xi_2, ..., \xi_s, \lambda)$ in an $(2m + s)$ -dimensional S -space form is given by equation (3.22)*

Corollary 3.8 *If we take $m = 1$, $s = 1$ in (3.22) then the $(2m + s)$ -dimensional S -space form is reduces to Sasakian-space-form [17] of dimension 3. In this case, $\lambda = -\frac{1}{2}$. Hence Ricci soliton is shrinking in Sasakian-space-form of dimension 3.*

Corollary 3.9 *If we take $s = 1$ in (3.22) then the $(2m + s)$ -dimensional S -space form is reduces to Sasakian-space-form [17] of dimension $(2m + 1)$. In this case, $\lambda = -\frac{1}{2}m$. Hence Ricci soliton is shrinking in $(2m + 1)$ -dimensional Sasakian-space-form.*

Corollary 3.10 *If we take $s = 0$ in (3.22) then the $(2m + s)$ -dimensional S -space form is reduces to complex-space-form [17] of dimension $2m$. In this case, $\lambda = -\frac{k(m+1)}{8}$. Hence Ricci soliton in $2m$ -dimensional complex-space-form is shrinking if $k > 0$, steady if $k = 0$ and expanding if $k < 0$.*

Theorem 3.11 *If an $(2m + s)$ -dimensional S -space form is η -Einstein then the Ricci soliton in S -space form with constant scalar curvature r is give by*

$$\lambda = \frac{1}{8}[3s^3 + (6m + k - 13)s^2 - (14m - 2mk - k - 10)s - (2mk + 2k)]$$

Proof. First we prove that S -space form is η -Einstein. from equation (2.10), we have

$$S(X,Y) = ag(X,Y) + b\sum_{\alpha}\eta_\alpha(X)\eta_\alpha(Y)$$

Now, by simple calculation we find the values of a and b. Let $\{e_i\}, i = 1,2,\ldots\ldots, (2m + s)$ be an orthonormal basis of the tangent space at any point of the manifold. then putting $X = Y = e_i$ in (2.10) and taking summation over i , we get

$$r = a(2m + s) + bs \quad (3.23)$$

Again putting $X = Y = \xi_\alpha$ in (2.10) then by using (2.13), we have

$$B = as + bs^2 \quad (3.24)$$

Then from (3.23) and (3.24), we obtain

$$a = \left[\frac{r}{(2m + s - 1)} - \frac{B}{(2m + s - 1)}\right], \quad b = \left[\frac{r}{s(2m + s - 1)} - \frac{B(2m + s)}{s^2(2m + s - 1)}\right]$$

(3.25)

Substituting the values of a and b in (2.10), we have

$$S(X,Y) = \left[\frac{r}{(2m + s - 1)} - \frac{B}{(2m + s - 1)}\right]g(X,Y) \quad (3.26)$$
$$- \left[\frac{r}{s(2m + s - 1)} - \frac{B(2m + s)}{s^2(2m + s - 1)}\right]\sum_{\alpha}\eta_\alpha(X)\eta_\alpha(Y)$$

The above equation shows that S -space form is an η -Einstein manifold.

Now, we have to show that the scalar curvature r is constant. For an $(2n + s)$ -dimensional S -space form the symmetric parallel covariant tensor $h(X,Y)$ of type $(0,2)$ is given by

$$h(X,Y) = \left(\sum_{i=1}^{s} c_i L_{\xi_i} g\right)(X,Y) + 2S(X,Y) \quad (3.27)$$

By using (3.18) and (3.26) in (3.27), we have

$$h(X,Y) = \left[\frac{2r}{(2m + s - 1)} - \frac{2B}{(2m + s - 1)}\right]g(X,Y) \quad (3.28)$$

Differentiating the above equation covariantly w.r.t Z , we get

$$(\nabla_Z h)(X,Y) = \left[\frac{2\nabla_Z r}{(2m + s - 1)}\right]g(X,Y) - \left[\frac{2\nabla_Z r}{s(2m + s - 1)}\right]\sum_{\alpha}\eta_\alpha(X)\eta_\alpha(Y)$$

$$-\left[\frac{2r}{s(2m+s-1)}-\frac{2B(2m+s)}{s^{2}(2m+s-1)}\right]\sum_{\alpha}[g(X,\nabla_{Z}\xi_{\alpha})\eta_{\alpha}(Y)+g(Y,\nabla_{Z}\xi_{\alpha})\eta_{\alpha}(X)]$$
(3.29)

Substituting $Z=\xi_{\alpha}$, $X=Y=(span\ \xi_{\alpha})^{\perp}$, $\alpha=1,2..,\ s$ in (3.29) and by virtue of $\nabla h=0$, we have

$$\nabla_{\xi_{\alpha}} r = 0 \qquad (3.30)$$

This shows that r is constant scalar curvature. From equation (1.1) and (3.27), we have $h(X,Y)=-2\lambda g(X,Y)$ and then putting $X=Y=\xi_{\alpha}$ for $\alpha=1,2,...,\ s$, we obtain

$$h(\xi_{\alpha},\xi_{\alpha})=-2\lambda s \qquad (3.31)$$

Again, putting $X=Y=\xi_{\alpha}$ in (3.28), we get

$$h(\xi_{\alpha},\xi_{\alpha})=B \qquad (3.32)$$

Equating (3.31) and (3.32), we have

$$\lambda=\frac{1}{8}[3s^{3}+(6m+k-13)s^{2}-(14m-2mk-k-10)s-(2mk+2k)] \qquad (3.33)$$

Hence the proof.

Corollary 3.12 *If we take $m=1$, $s=1$ in (3.33) then the $(2m+s)$-dimensional S-space form is reduces to Sasakian-space-form [17] of dimension 3. In this case, $\lambda=-1$. Hence Ricci soliton is shrinking in Sasakian-space-form of dimension 3.*

Corollary 3.13 *If we take $s=1$ in (3.33) then the $(2m+s)$-dimensional S-space form is reduces to Sasakian-space-form [17] of dimension $(2m+1)$. In this case, $\lambda=-m$. Hence Ricci soliton is shrinking in $(2m+1)$-dimensional Sasakian-space-form.*

Corollary 3.14 *If we take $s=0$ in (3.33) then the $(2m+s)$-dimensional S-space form is reduces to complex-space-form [17] of dimension $2m$. In this case, $\lambda=-\frac{k(m+1)}{4}$. Hence Ricci soliton in $2m$-dimensional complex-space-form is shrinking if $k>0$, steady if $k=0$ and expanding if $k<0$.*

IV. RICCI SOLITON IN SASAKIAN SPACE FORM OF DIMENSION 3

In this section, we compute an expression for Ricci tensor for 3-dimensional S-space form. The curvature tensor for 3-dimensional Riemannian manifold is given by

$$R(X,Y)Z=g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y-\frac{r}{2}[g(Y,Z)X-g(X,Z)Y] \qquad (4.1)$$

Put Z = ξ in (4.1) and by using (2.17) and (2.19), we have

$$[\eta(Y)X-\eta(X)Y]=[\eta(Y)QX-\eta(X)QY]+\left[2-\frac{r}{2}\right][\eta(Y)X-\eta(X)Y] \qquad (4.2)$$

Again put $Y=\xi$ in (4.2) and using (2.1), (2.2) and (2.4), we get

$$QX=\left[\frac{r}{2}-1\right]X-\left[\frac{r}{2}-3\right]\eta(X)\xi \qquad (4.3)$$

By taking inner product with respect to Y in (4.3), we get

$$S(X,Y)=\left[\frac{r}{2}-1\right]g(X,Y)-\left[\frac{r}{2}-3\right]\eta(X)\eta(Y) \qquad (4.4)$$

This shows that Sasakian space form of dimension 3 is η-Einstein manifold. where r is the scalar curvature. For a Sasakian space form of dimension 3, we have

$$h(X,Y) = (L_{\xi} g)(X,Y) + 2S(X,Y) \qquad (4.5)$$

By using (3.15) and (4.4) in (4.5), we get

$$h(X,Y) = [r-2]\,g(X,Y) - [r-6]\,\eta(X)\eta(Y) \qquad (4.6)$$

Differentiating (4.6) covariantly with respect to Z, we obtain

$$(\nabla_Z h)(X,Y) = (\nabla_Z r)g(X,Y) - (\nabla_Z r)\eta(X)\eta(Y) - [r-6][\,g(X,\nabla_Z \xi)\eta(Y) + g(Y,\nabla_Z \xi)\eta(X)] \qquad (4.7)$$

Substituting $Z = \xi$, $X = Y \in (span\ \xi)^{\perp}$ in (4.7) and by virtue of $\nabla h = 0$, we have

$$\nabla_{\xi} r = 0 \qquad (4.8)$$

Thus, r is a constant scalar curvature.

From equation (1.1) and (3.5), we have $h(X,Y) = -2\lambda g(X,Y)$ and then putting $X = Y = \xi$, we get

$$h(\xi,\xi) = -2\lambda \qquad (4.9)$$

Again, putting $X = Y = \xi$, in (4.6), we get

$$h(\xi,\xi) = 4 \qquad (4.10)$$

In view of (4.9) and (4.10), we have

$$\lambda = -2 \qquad (4.11)$$

Therefore, λ is negative. Hence we state the following theorem:

Theorem 4.1 *An η -Einstein Sasakian-space form of dimension 3 admits Ricci soliton (g,ξ,λ) with constant scalar curvature r is shrinking.*

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