International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 4 Issue 8 || October. 2016 || PP-01-03

Numerical Evaluation of Complex Integrals of Analytic Functions

P.M.Mohanty(1) and M.Acharya(2)
(1)*S.C.S. College,Puri,Odisha.*
(2) *ITER, SOA University,Bhubaneswar.*

ABSTRACT: *A nine point degree nine quadrature rule with derivatives has been formulated for the numerical evaluation of integral of analytic function along a directed line segment in the complex plane. The truncation error associated with the method has been analyzed using the Taylors' series expansion and also some particular cases have been discussed for enhancing the degree of precision of the rule and reducingthe number of function evaluations. The methods have been verified by considering standard examples.*
KEYWORDS: *Quadrature rules, Degree of precision, truncation error*
***AMS Classification**: 65D 30*

I. INTRODUCTION

Birkhoff and Young [3], Lether [4],Tosic [7],Senapati etal [6],Acharya and Nayak[2] and Acharya, Acharya and Nayak[1] have constructed quadrature rules for the numerical evaluation of one dimensional integral of an analytic function which is given by

$$I(f) = \int_L f(z)dz \qquad (1)$$

where$f(z)$ is an analytic function in the disk

$\Omega = \{z : |z - z_0| \leq \rho, \rho > |h|\}$ (2)

andL is a directed line segment from the point $z_0 - h$ to $z_0 + h$. Milovanovic [5] has constructed a generalized quadrature rule of degree nine and more for the numerical evaluation of the integral I(f). Most of the rules cited above can be obtained as particular limiting cases of this rule which is proposed to be constructed.

The object of the present paper is to formulate a nine point degree nine quadrature rule with derivatives for approximating numerically the integral I(f) and to find out the truncation error associated with the rule by Taylor's series expansion. Studying the different relations between coefficients and error functions of the rule the degree of precision has been increased from nine to thirteen and number of function evaluations has been reduced.

II. GENERATION OF THE RULE

Let us consider the set of nodes

$S = \{z_0, z_0 \pm sh, z_0 \pm ish, z_0 \pm th, z_0 \pm ith\}$(3)

The nine point rule using the above set of nodes is proposed in the following form:

$$R(f; s, t) = Af(z_0) + B\{f(z_0 + sh) + f(z_0 - sh)\} + C\{f(z_0 + ish) + f(z_0 - ish)\}$$
$$+Dth\{f'(z_0 + th) - f'(z_0 - th)\} + Eith\{f'(z_0 + ith) - f'(z_0 - ith)\} \qquad (4)$$

Where A, B, C, D, E are coefficients and s, t are free parameters between (0,1].

Since the rule $R(f; s, t)$ is symmetric it is exact for all odd monomials $f(z) = (z - z_0)^{2\mu+1},\ \mu = 0,1,2,3$ .... We make the rule exact for even monomials $f(z) = (z - z_0)^{2\mu}, \mu = 0,1,2,3,4$ which gives us the following equations:

$$\left.\begin{aligned} A + 2B + 2C &= 2h, \\ Bs^2 - Cs^2 + 2Dt^2 - 2Et^2 &= h/3, \\ Bs^4 + Cs^4 + 4Dt^4 + 4Et^4 &= h/5, \\ Bs^6 - Cs^6 + 6Dt^6 - 6Et^6 &= h/7, \\ Bs^8 + Cs^8 + 8Dt^8 + 8Et^8 &= h/9. \end{aligned}\right\} (5)$$

Solving the above system of equations by determinant method, we have

$$\left.\begin{aligned} A &= 2h\{1 - B_1/s^2\}, \\ B &= (B_1 + B_2)h/2s^2, \\ C &= (B_1 - B_2)h/2s^2, \\ D &= (C_1 + C_2)h/12t^2, \\ E &= (C_1 - C_2)h/12t^2. \end{aligned}\right\} (6)$$

where

$$\left.\begin{array}{l} B_1 = (18t^4 - 5)/(90s^2t^4 - 45\,s^6), B_2 = (7t^4 - 1)/(21t^4 - 7\,s^4), \\ C_1 = (5 - 9s^4)/(60t^6 - 30t^2s^4), C_2 = (3 - 7s^4)/(21t^4 - 7\,s^4), \end{array}\right\} (7)$$

and $t/s \neq 1/3, t/s \neq 1/2$.

Theorem 1: The degree of precision of the rule $R(f; s, t)$ is at least nine for all values of $s, t \in (0,1]$ except for the cases $t/s \neq 1/3, t/s \neq 1/2$.

III. ANALYSIS OF ERROR

The error $E(f; s, t)$associated with the rule $R(f; s, t)$ is given by

$E(f; s, t) = I(f) - R(f; s, t)$(8)

As f is assumed to be analytic inside the disk Ω, $f(z)$ can be expanded in Taylor's series about $z_0$inside Ω. The Taylor's series expansion of $f(z)$ is given by

$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n$,$a_n = \frac{f^n(z_0)}{n!}$(9)

Setting the equ.(9) in equ.s (1),(4) and (8), we obtain after simplification

$E(f; s, t) = \beta_1(s, t)a_{10}h^{11} + \beta_2(s, t)a_{12}h^{13} + \beta_3(s, t)a_{14}h^{15} + O(h^{17})$(10)

where$\beta_1(s, t)$, $\beta_2(s, t)$and $\beta_3(s, t)$ are error functions and given by

$$\left.\begin{array}{r} \beta_1(s, t) = 2B_2s^8 + 10C_2t^8/3 - 2/11, \\ \beta_2(s, t) = 2B_1s^{10} + 4C_1t^{10} - 2/13, \\ \beta_3(s, t) = 2B_2s^{12} + 14C_2t^{12}/3 - 2/15. \end{array}\right\} (11)$$

Theorem 2: The truncation error $E(f; s, t)$ associated with the rule $R(f; s, t)$ satisfies the order relation $E(f; s, t) = O(|h^{11}|)$ provided $\beta_1(s, t)$ is non-zero.

3.1 Some particular cases:

In this article we attempt to derive some rules of higher degree of precision and with lesser number of function evaluations establishing different relation between the coefficients A, B, C, D, E and first error function $\beta_1(s, t)$.

i) If $B + C = 1$, then $A = 0$. It gives a relation between s and t i.e. $t^4 = \frac{45s^8 - 5}{90s^4 - 18}$ from which we get the domain of $s \in (0, (1/5)^{.25}) \cup ((1/9)^{.125}, 1]$. Again if $\beta_1(s, t) = 0$, we have three sets of solutions for s within the specified domain. These solutions for s and their corresponding values oft are given below:

s	t
0.862860537063815	0.725348447933857
0.797814404937058	0.599594039837057
0.529450590383445	0.810774376526716

(12)

It is evident that the rule $R(f; s, t)$ is an eight point rule of degree of precision eleven.

ii) If $A = 0$ and $E = 0$, then solving it we have two sets of solutions for s and their corresponding values of t are as follows:

s	t
0.498954104984763	0.789542087859687
0.795280016073590	0.591303696513965

(13)

It is noted that for these values of s and t the rule $R(f; s, t)$is a six point degree nine rule.

iii) If $C = 0$ and $E = 0$, solving it by generalized Newton Raphson method, then we have two sets of solutions for s andt are as follows:

s	t
0.607598845506436	0.826070992994919
0.893894114497005	0.340452712661936

(14)

It is observed that for these values of s and t the rule $R(f; s, t)$is a five point degree nine rule.

iv) If $E = 0$ and $\beta_1(s, t) = 0$, solving it by generalized Newton Raphson method, then we have following sets of solutions for s and t are as follows:

s	t
0.862190731946722	0.723799494986752
0.904635786593100	0.371161935610790
0.648262853694959	0.850112195194705

(15)

It is noted that for these values of s and t the rule $R(f; s, t)$is a seven point rule of degree of precision eleven.

v) Lastly if $\beta_1(s, t) = 0$ and $\beta_2(s, t) = 0$ solving it by above method we have the following four sets of solutions for s and t:

s	t	
0.918955582192060	0.434783319295906	(16)
0.862577405054147	0.724694468901017	
0.893372168151982	0.786442903798219	
0.670976509948225	0.863213540937855	

It is noted that for these values of s and t the rule $R(f; s, t)$ is a nine point rule and its degree of precision raised from nine to thirteen. It is noted that setting
$s = 1, t \to \infty$, the rule reduces to the rule due to Birkhoff and Young [3],
$s = \sqrt{0.6}, t \to \infty$, it reduces to the rule due to Lether [4],
$s = (3/7)^{1/4}, t \to \infty$, it reduces to the rule due to Tosic [7],
$s = t = 1$, the rule reduces to the modified BY rule due to Acharya and Nayak [2]
$s = t$, it reduces to the rule due to Acharya, Acharya and Nayak [1].

IV. NUMERICAL VERIFICATIONS

For the purpose of numerical verification we consider the integral $J(z)$ given by

$$J(z) = \int_{-1+i}^{1+i} e^z dz \quad (17)$$

The computed values of the integral for different values of s, t and its absolute error are given in the following table. For computation one pair of s, t has been taken from equns. (12)-(16) arbitrarily.

Table

s	t	|error|
0.607598845506436	0.826070992994919	1.33×10^{-09}
0.498954104984763	0.789542087859687	3.62×10^{-09}
0.529450590383445	0.810774376526716	3.54×10^{-11}
0.648262853694959	0.850112195194705	8.26×10^{-12}
0.893372168151982	0.786442903798219	3.0×10^{-14}

It is observed that the rule whose degree of precision is thirteen is of higher accuracy than other rules discussed above.

ACKNOWLEDGEMENT

Authors are grateful to Prof. B.P.Acharya for constructive suggestions.

REFERENCES

[1]. Acharya,B.P.,Acharya,M.and Nayak,M., Approximate evaluation of integrals of analytic functions, Facta Universities, 23,(2008),63-68.
[2]. Acharya,B.P. and Nayak,P., Numerical evaluation of integrals of analytic functions, Intern.J.Computer Math., 59,(1996),245-249.
[3]. Birkhoff,G.andYoung,D., Numerical quadrature of analytic and harmonic functions, J.Math.Phys. 29,(1950),217-221.
[4]. Lether,F.G., On Birkhoff -Young quadrature of analytic functions, J.Comp. Appl.Math., 2,(1976),81-84.
[5]. Milovanovic,G.V., Generalized quadrature formulae for analytic functions, Appl. Math. Computations, 218, (2012), 8537-8551.
[6]. Senapati,S.,Acharya,M. and Acharya,B.P., On Numerical quadrature of analytic functions,J.(theo&appl.), Appl. Math. Sci., 5,(2011), 3411-3420.
[7]. Tosic, D., A modification of the Birkhoff –Youngquadrature formula for analytical functions,Univ.Beograd Publ.Elektrotehn.Fak.Ser.Mat.Fiz.,no. 601-no.633,(1978),73-77.