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The Odd Generalized Exponential Log Logistic Distribution

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Abstract: *We propose a new lifetime model, called the odd generalized exponential log logistic distribution (OGELLD).We obtain some of its mathematical properties. Some structural properties of the new distribution are studied. The maximum likelihood method is used for estimating the model parameters and the Fisher's information matrix is derived. We illustrate the usefulness of the proposed model by applications to real lifetime data.*

Keywords:*Generalised exponential distribution, Log logistic distribution, Maximum likelihood estimation*.
2010 *Mathematics subject classification: 62E10, 62F10*

I. Introduction

Statistical distributions are very useful in describing the real world phenomena.In the analysis of lifetime data, the log-logistic distribution is widely used in practice and it is an alternative to the log-normal distribution, since it presents a failure rate function that increases, reaches a peak after some finite period and then declines gradually.According to Collet (2003), the properties of the log-logistic distribution make it an attractive alternative to the log-normal and Weibull distributions in the analysis of survival data. This distribution can exhibit a monotonically decreasing failure rate function for some parameter values. Ahmad *et al.*(1988), suggested thatit shares some properties of the log-normal and normal distributions, i.e., if T has a log-logistic distribution, then Y = log(T) has a logistic distribution. According to Kleiber and Kotz (2003),some applications of the log-logistic distribution are discussed in economy to model the wealth, income.Ashkar and Mahdi (2006) given that, ithas application in hydrology to model stream flow data. Collet (2003) suggested the log-logistic distribution is useful for modelling the time following heart transplantation.It is known that exponential have only constant hazard rate function and generalized exponential distribution can have only monotone increasing or decreasing hazard rate. There are always urge among the researchers for developing new and more flexible distributions. As a result, many new distributions have come up and studied. Recently, Tahir *et al.* (2015) propose a new class of distributions called the odd generalized exponential (OGE) family and study each of the OGE- Weibull (OGE-W) distribution, the OGE-Fréchet (OGE-Fr)distribution and the OGE-Normal (OGE-N) distribution. These models areflexible because of the hazard shapes: increasing, decreasing, bathtub and upside subset of down bathtub.

A random variable X is said to have generalized exponential (GE) distribution with parameters λ, θ if the cumulative distribution function (CDF) is given by

$$F(x) = \left[1 - e^{-\lambda x}\right]^{\theta}, \quad x > 0, \lambda > 0, \theta > 0. \quad (1)$$

The odd generalized exponential family suggested by Tahir *et al.* (2015) is defined as follows. If $G(x;\varepsilon)$ is the CDF of any distribution and thus the survival function is $\overline{G}(x;\varepsilon) = 1 - G(x;\varepsilon)$, then the OGE-X is defined by replacing x in CDF of GE in equation (1) by $\frac{G(x;\varepsilon)}{\overline{G}(x;\varepsilon)}$ to get the CDF of the new distribution as follows:

$$F(x;\lambda,\varepsilon,\theta) = \left[1 - e^{-\lambda \frac{G(x;\varepsilon)}{\overline{G}(x;\varepsilon)}}\right]^{\theta}, \quad x > 0, \lambda > 0, \varepsilon > 0, \theta > 0. \quad (2)$$

In this paper, we define a new distribution using generalized exponential distribution and log-logistic distribution and named it as "The odd generalized exponential log logistic distribution (OGELLD)" from a new family of distributions proposed by Tahir *et al.* (2015). The paper is organized as follows. The new distribution is developed in Section 2 and also we define the CDF, density function, reliability function and hazard functions of the odd generalized exponentiallog logisticdistribution (OGELLD). A comprehensive account of statistical properties of the new distribution is providedin Section 3.In Section 4, wediscuss the distribution of the order statistics for OGELLD. In section 5, maximum likelihood estimation and Fisher's information matrix are derived for the parameters.A real life data set has been analyzed and compared with other fitted distributionsin Section 6 and also the concluding remarks are presented in Section 7.

2. The Probability Density and Distribution Function of the OGELLD

In this section, we define new four parameter distribution called odd generalized exponential log-logistic distribution (OGELLD) with parameters $\sigma, \lambda, \theta, \gamma$ onlineof El-Damceseet *al.* (2015). The probability density function(pdf), cumulative distribution function (CDF), reliability function*R(x)*and hazard function*h(x)*of the new model OGELLDare respectively defined as:

$$f(x;\Theta) = \frac{\gamma\theta}{\lambda\sigma}\left(\frac{x}{\sigma}\right)^{\theta-1}\left[1-e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\right]^{\gamma-1} e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}} \quad (3)$$

$$F(x;\Theta) = \left[1-e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\right]^{\gamma}, \quad x>0, \sigma, \lambda, \theta, \gamma > 0 \quad (4)$$

$$R(x) = 1-\left[1-e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\right]^{\gamma} \quad (5)$$

$$h(x) = \frac{\frac{\gamma\theta}{\lambda\sigma}\left(\frac{x}{\sigma}\right)^{\theta-1}\left[1-e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\right]^{\gamma-1} e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}}{1-\left[1-e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\right]^{\gamma}} \quad (6)$$

where σ, λ are scale parameters and θ, γ are the shape parameters.

The graphs of *f(x)*,*F(x)* and *h(x)* are given below for different values of the parameters. The graphs of *h(x)* show that the proposed model is an increasing failure rate (IFR) model. Hence this model can be used for reliability studies.

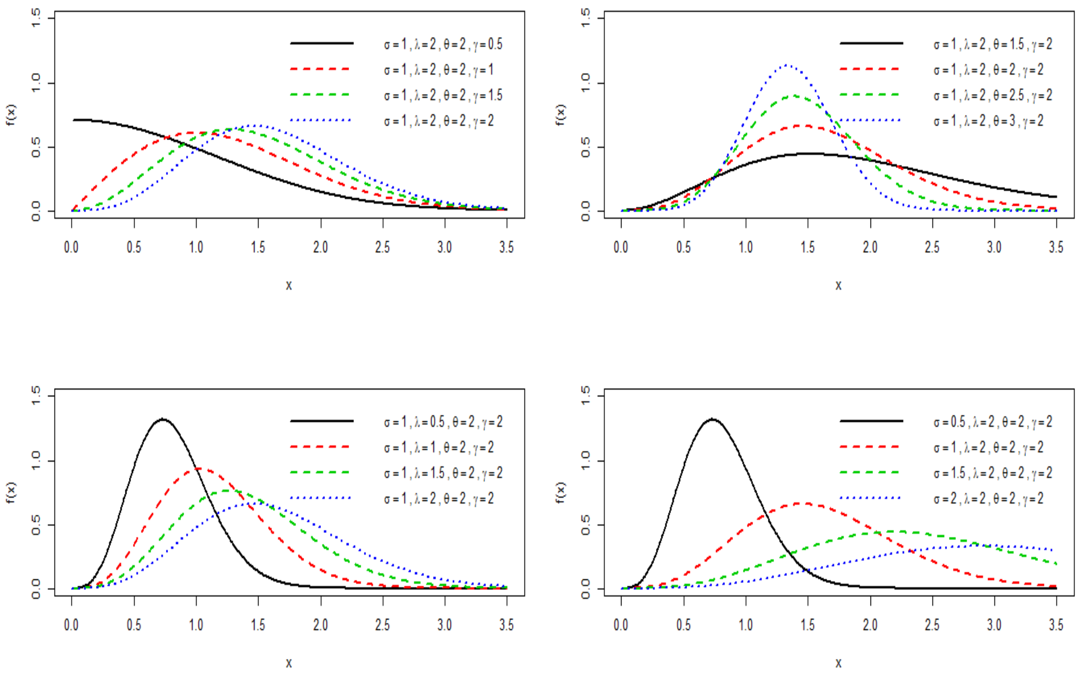

Figure-1: The probability density function of the OGELLD

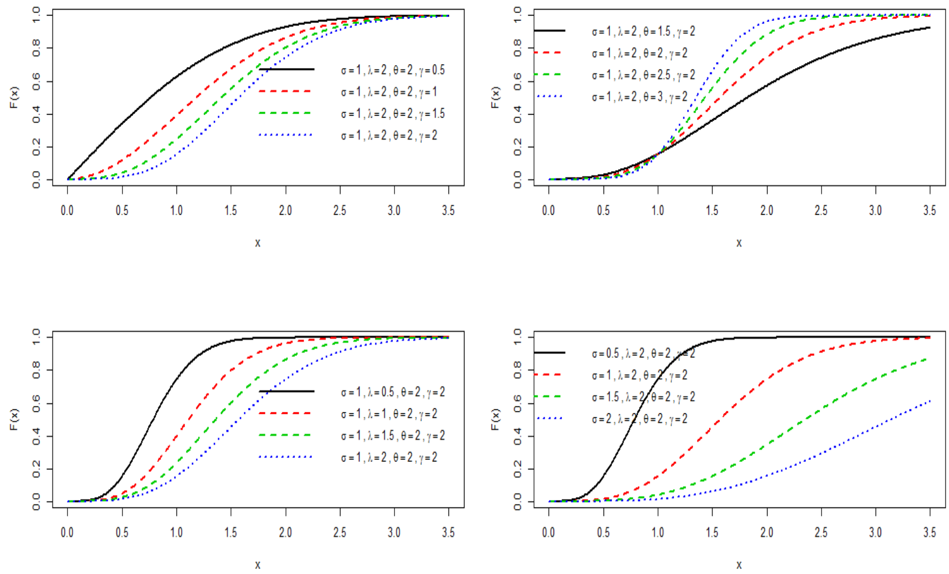

Figure-2: The cumulative distribution function of the OGELLD

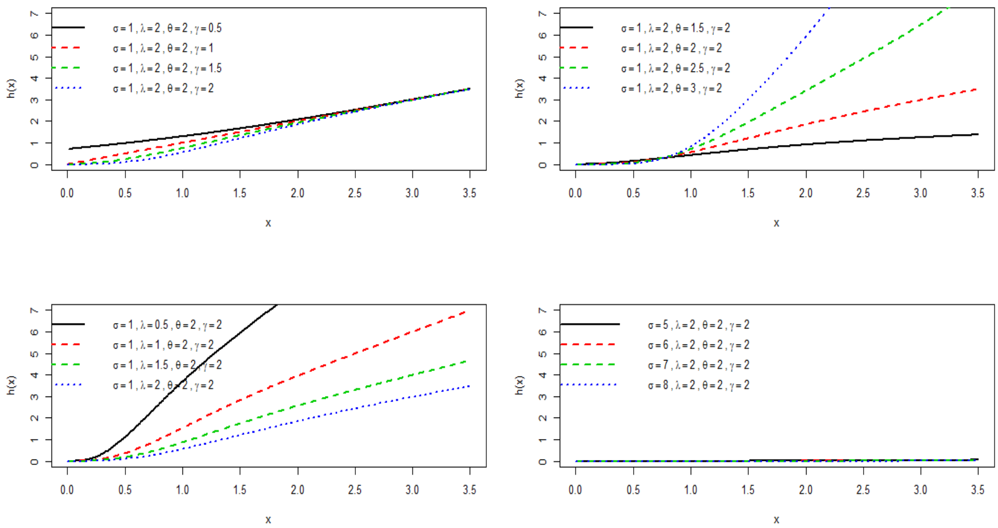

Figure-3: The hazard function of the OGELLD

3. Statistical properties

In this section, we study some statistical properties of OGELLD, especially quantile, median, mode and moments.

Limit of the Distribution Function:

Since the cdf of OGELLD is

$$F(x;\sigma,\lambda,\theta,\gamma)=\left[1-e^{-\frac{1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\right]^{\gamma},\quad x>0,\sigma,\lambda,\theta,\gamma>0$$

We have, $\lim_{x\to 0} F(x;\sigma,\lambda,\theta,\gamma)=0$ and $\lim_{x\to\infty} F(x;\sigma,\lambda,\theta,\gamma)=1$

Quantile and median of OGELLD

The 100_q percentile of $X \sim OGELL(\Theta)$ distribution is given by

$$x_q=\sigma\left[-\lambda\ln\left(1-q^{\frac{1}{\gamma}}\right)\right]^{\frac{1}{\theta}},\quad 0<q<1. \qquad (7)$$

Setting $q = 0.5$ in (7), we obtain the median of $X \sim OGELL(\sigma,\lambda,\theta,\gamma)$ distribution as follows:

$$\text{Median} = \sigma\left[-\lambda \ln\left(1-(0.5)^{\frac{1}{\gamma}}\right)\right]^{\frac{1}{\theta}} \quad (8)$$

The mode of OGELLD

$$\ln f(x;\Theta) = \ln\gamma + \ln\theta - \ln\lambda - \ln\sigma + (\theta-1)\left[\ln(x) - \ln\sigma\right] + (\gamma-1)\ln\left[1-e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\right] - \frac{1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}$$

$$\frac{d\ln f(x;\Theta)}{dx} = 0 \quad \Rightarrow \text{Mode} = \sigma\left[\left(\frac{1-\theta}{\theta}\right)\lambda\right]^{1/\theta} \quad (9)$$

The moments

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). In this subsection, we will derive the r^th moment of the $X \sim OGELL(\Theta)$ distribution as an infinite series expansion.

Theorem 3.1. *If* $X \sim OGELL(\Theta)$, where $(\Theta) = (\lambda,\sigma,\theta,\gamma)$, then the r^{th} moment of X is given by $\mu_r' = \sum_{i=0}^{\infty} \gamma\sigma^r\lambda^{r/\theta}(-1)^i\binom{\gamma-1}{i}\frac{\left(r/\theta\right)!}{(i+1)^{r/\theta+1}}$

Proof: The r^th moment of the random variable X with pdf *-f(x)* is defined by

$$E\left(x^r\right) = \mu_r' = \int_0^{\infty} x^r f(x;\Theta)\,dx \quad (10)$$

Substituting from (3) into (10), we get

$$E\left(x^r\right) = \mu_r' = \int_0^{\infty} x^r \frac{\gamma\theta}{\lambda\sigma}\left(\frac{x}{\sigma}\right)^{\theta-1}\left[1-e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\right]^{\gamma-1} e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\,dx. \quad (11)$$

Since $0 < 1-e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}} < 1 \text{ for } x > 0$, we have,

$$\left[1-e^{\frac{-1}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\right]^{\gamma-1} = \sum_{i=0}^{\infty}\binom{\gamma-1}{i}(-1)^i e^{\frac{-i}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}} \quad (12)$$

Substituting from (12) into (11), we obtain

$$\mu_r' = \sum_{i=0}^{\infty}\frac{\gamma\theta}{\lambda\sigma}(-1)^i\binom{\gamma-1}{i}\int_0^{\infty} x^r\left(\frac{x}{\sigma}\right)^{\theta-1} e^{\frac{-(i+1)}{\lambda}\left(\frac{x}{\sigma}\right)^{\theta}}\,dx = \sum_{i=0}^{\infty}\frac{\gamma}{\lambda}\sigma^r(-1)^i\binom{\gamma-1}{i}\int_0^{\infty} z^{r/\theta} e^{\frac{-(i+1)}{\lambda}z}\,dz$$

$$\text{where} \quad z = \left(\frac{x}{\sigma}\right)^{\theta}$$

$$\text{Therefore,} \quad \mu_r' = \sum_{i=0}^{\infty}\gamma\sigma^r\lambda^{r/\theta}(-1)^i\binom{\gamma-1}{i}\frac{\left(\frac{r}{\theta}\right)!}{(i+1)^{\frac{r}{\theta}+1}}$$

This completes the proof.

Moment generating function (MGF)

$$M_x(t) = E\left[e^{tx}\right] = \mathrm{E}\left[1+tx+\frac{t^2x^2}{2!}+\frac{t^3x^3}{3!}+\ldots+\frac{t^rx^r}{r!}+\ldots\right] = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r' = \sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}\gamma\sigma^r\lambda^{r/\theta}(-1)^i\binom{\gamma-1}{i}\frac{\left(\frac{r}{\theta}\right)!}{(i+1)^{\frac{r}{\theta}+1}} \quad (13)$$

Characteristic Function (CF)

$$\phi_x(t) = E\left[e^{itx}\right] = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\gamma\sigma^r\lambda^{r/\theta}(-1)^k\binom{\gamma-1}{k}\frac{\left(\frac{r}{\theta}\right)!}{(k+1)^{\frac{r}{\theta}+1}} \quad (14)$$

Cumulant Generating Function (CGF)

$$K_x(t) = \ln\left(M_x(t)\right) = \ln\left[\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}\gamma\sigma^r\lambda^{r/\theta}(-1)^i\binom{\gamma-1}{i}\frac{\left(\frac{r}{\theta}\right)!}{(i+1)^{\frac{r}{\theta}+1}}\right] \quad (15)$$

IV. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from OGELLD with probability density function $f(x;\Theta)$ andcumulative distribution $F(x;\Theta)$ given by (3), (4) respectively. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics obtained from this sample. The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x,\Theta) = \frac{1}{B(r,n-r+1)}\left[F(x,\Theta)\right]^{r-1}\left[1-F(x,\Theta)\right]^{n-r} f(x,\Theta) \quad (16)$$

$B(.,.)$ is the beta function. Since $0 < F(x,\Theta) < 1$ for *x>0*, we can use the binomial expansion of $\left[1-F(x,\Theta)\right]^{n-r}$ given as follows:

$$\left[1-F(x,\Theta)\right]^{n-r} = \sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\left[F(x,\Theta)\right]^i \quad (17)$$

Substituting from (17) into (16), we have

$$f_{r:n}(x,\Theta) = \frac{1}{B(r,n-r+1)} f(x,\Theta)\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\left[F(x,\Theta)\right]^{i+r-1} \quad (18)$$

Substituting from (3) and (4) into (18), we obtain

$$f_{r:n}(x;\lambda,\sigma,\theta,\gamma) = \sum_{i=0}^{n-r}\frac{(-1)^i\, n!}{i!(r-1)!(n-r-i)!} f\left(x;\lambda,\sigma,\theta,\gamma(r+i)\right) \quad (19)$$

Thus $f_{r:n}(x;\lambda,\sigma,\theta,\gamma)$ defined in (19) is the weighted average of the OGELLD.

V. Estimation and Inference

5.1 Maximum likelihood estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n,which is drawn from OGELL(Θ), where $\Theta = (\sigma,\lambda,\theta,\gamma)$, then the likelihood function *L* of this sample is

$$L \square \prod_{i=1}^{n} f(x_i;\lambda,\sigma,\theta,\gamma) = \prod_{i=1}^{n}\frac{\gamma\theta}{\lambda\sigma}\left(\frac{x_i}{\sigma}\right)^{\theta-1}\left[1-e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}\right]^{\gamma-1} e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}$$

The log-likelihood function is

$$\ln L = n(\ln\gamma + \ln\theta - \ln\lambda - n\theta\ln\sigma) + (\theta-1)\sum_{i=1}^{n}\ln x_i - \frac{1}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\theta} + (\gamma-1)\sum_{i=1}^{n}\ln\left[1-e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}\right]$$

The MLE's of $\sigma,\lambda,\theta,\gamma$ are the simultaneous solutions of the following equations using numerical iterative method:

$$\frac{\partial\ln L}{\partial\lambda} = \frac{-n}{\lambda} + \frac{1}{\lambda^2\sigma^\theta}\sum_{i=1}^{n}x_i^\theta - \frac{(\gamma-1)}{\lambda^2\sigma^\theta}\sum_{i=1}^{n}\frac{x_i^\theta e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}}{\left[1-e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}\right]} = 0 \quad (20)$$

$$\frac{\partial \ln L}{\partial \sigma} = \frac{-\theta n}{\sigma} + \frac{\theta}{\lambda \sigma^{\theta+1}} \sum_{i=1}^{n} x_i^{\theta} - \frac{\theta(\gamma-1)}{\lambda \sigma^{\theta+1}} \sum_{i=1}^{n} \frac{x_i^{\theta} e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}}{\left[1 - e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}\right]} = 0 \quad (21)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - n \ln \sigma + \sum_{i=1}^{n} \ln x_i - \frac{1}{\lambda} \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\theta} \ln\left(\frac{x_i}{\sigma}\right) + \frac{(\gamma-1)}{\lambda} \sum_{i=1}^{n} \frac{e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}} \left(\frac{x_i}{\sigma}\right)^{\theta} \ln\left(\frac{x_i}{\sigma}\right)}{\left[1 - e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}\right]} = 0 \quad (22)$$

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} \ln\left[1 - e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}\right] = 0 \quad (23)$$

The ML equations do not have explicit solutions and they have to be obtained numerically from equation (23), the MLE of γ can be obtained as follows.

$$\hat{\gamma} = \frac{-n}{\sum_{i=1}^{n} \ln\left[1 - e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}\right]} \quad (24)$$

Substituting from (24) into (20), (21) and (22), we get

$$\frac{-n}{\hat{\lambda}} + \frac{1}{\hat{\lambda}^2 \hat{\sigma}^{\hat{\theta}}} \sum_{i=1}^{n} x_i^{\hat{\theta}} - \frac{(\hat{\gamma}-1)}{\hat{\lambda}^2 \hat{\sigma}^{\hat{\theta}}} \sum_{i=1}^{n} \frac{x_i^{\hat{\theta}} e^{\frac{-1}{\hat{\lambda}}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{\theta}}}}{\left[1 - e^{\frac{-1}{\hat{\lambda}}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{\theta}}}\right]} = 0 \quad (25)$$

$$\frac{-\hat{\theta} n}{\hat{\sigma}} + \frac{\hat{\theta}}{\hat{\lambda} \hat{\sigma}^{\hat{\theta}+1}} \sum_{i=1}^{n} x_i^{\hat{\theta}} - \frac{\hat{\theta}(\hat{\gamma}-1)}{\hat{\lambda} \hat{\sigma}^{\hat{\theta}+1}} \sum_{i=1}^{n} \frac{x_i^{\hat{\theta}} e^{\frac{-1}{\hat{\lambda}}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{\theta}}}}{\left[1 - e^{\frac{-1}{\hat{\lambda}}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{\theta}}}\right]} = 0 \quad (26)$$

$$\frac{n}{\hat{\theta}} - n \ln \hat{\sigma} + \sum_{i=1}^{n} \ln x_i - \frac{1}{\hat{\lambda}} \sum_{i=1}^{n} \left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{\theta}} \ln\left(\frac{x_i}{\hat{\sigma}}\right) + \frac{(\hat{\gamma}-1)}{\hat{\lambda}} \sum_{i=1}^{n} \frac{e^{\frac{-1}{\hat{\lambda}}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{\theta}}} \left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{\theta}} \ln\left(\frac{x_i}{\hat{\sigma}}\right)}{\left[1 - e^{\frac{-1}{\hat{\lambda}}\left(\frac{x_i}{\hat{\sigma}}\right)^{\hat{\theta}}}\right]} = 0 \quad (27)$$

These equations cannot be solved analytically and statistical software can be used to solve the equations numerically. We can use iterative techniques to obtain the estimate $\hat{\gamma}$.

5.2 Asymptotic confidence bounds

In this subsection, we derive the asymptotic confidence intervals of the unknown parameters σ, λ, θ *and* γ. The simplest large sample approach is to assume that the MLEs $(\hat{\sigma}, \hat{\lambda}, \hat{\theta}$ *and* $\hat{\gamma})$ are approximately multivariate normal with mean $(\sigma, \lambda, \theta, \gamma)$ and covariance matrix I_0^{-1}, where I_0^{-1} is the inverse of the observed information matrix which is defined as follows:

$$I_0^{-1} = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \lambda^2} & \frac{\partial^2 \ln L}{\partial \sigma \partial \lambda} & \frac{\partial^2 \ln L}{\partial \theta \partial \lambda} & \frac{\partial^2 \ln L}{\partial \gamma \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \lambda \partial \sigma} & \frac{\partial^2 \ln L}{\partial \sigma^2} & \frac{\partial^2 \ln L}{\partial \theta \partial \sigma} & \frac{\partial^2 \ln L}{\partial \gamma \partial \sigma} \\ \frac{\partial^2 \ln L}{\partial \lambda \partial \theta} & \frac{\partial^2 \ln L}{\partial \sigma \partial \theta} & \frac{\partial^2 \ln L}{\partial \theta^2} & \frac{\partial^2 \ln L}{\partial \gamma \partial \theta} \\ \frac{\partial^2 \ln L}{\partial \lambda \partial \gamma} & \frac{\partial^2 \ln L}{\partial \sigma \partial \gamma} & \frac{\partial^2 \ln L}{\partial \theta \partial \gamma} & \frac{\partial^2 \ln L}{\partial \gamma^2} \end{bmatrix}^{-1} = \begin{bmatrix} var(\hat{\lambda}) & cov(\hat{\sigma},\hat{\lambda}) & cov(\hat{\theta},\hat{\lambda}) & cov(\hat{\gamma},\hat{\lambda}) \\ cov(\hat{\lambda},\hat{\sigma}) & var(\hat{\sigma}) & cov(\hat{\theta},\hat{\sigma}) & cov(\hat{\gamma},\hat{\sigma}) \\ cov(\hat{\lambda},\hat{\theta}) & cov(\hat{\sigma},\hat{\theta}) & var(\hat{\theta}) & cov(\hat{\gamma},\hat{\theta}) \\ cov(\hat{\lambda},\hat{\gamma}) & cov(\hat{\sigma},\hat{\gamma}) & cov(\hat{\theta},\hat{\gamma}) & var(\hat{\gamma}) \end{bmatrix} \quad (28)$$

The second partial derivatives included in I_0^{-1} are given as follows:

$$\frac{\partial^2 \ln L}{\partial \gamma^2} = \frac{-n}{\gamma^2};\ \frac{\partial^2 \ln L}{\partial \gamma \partial \lambda} = \frac{-1}{\lambda^2}\sum_{i=1}^{n}\frac{A_i}{B_i};\ \frac{\partial^2 \ln L}{\partial \gamma \partial \sigma} = \frac{-\theta}{\lambda\sigma}\sum_{i=1}^{n}\frac{A_i}{B_i};\ \frac{\partial^2 \ln L}{\partial \gamma \partial \theta} = \frac{-1}{\lambda}\sum_{i=1}^{n}\frac{A_i \ln\left(\frac{x_i}{\sigma}\right)}{B_i}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{n}{\lambda^2} - \frac{2}{\lambda^3}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\theta} - \frac{(\gamma-1)}{\lambda^4}\sum_{i=1}^{n}\frac{A_i}{B_i} + \frac{2(\gamma-1)}{\lambda^3}\sum_{i=1}^{n}\frac{A_i}{B_i} - \frac{(\gamma-1)}{\lambda^4}\sum_{i=1}^{n}\frac{C_i}{D_i}$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2} = \frac{n\theta}{\sigma^2} - \frac{(\theta^2+\theta)}{\lambda\sigma^2}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\theta} + \frac{\theta(\gamma-1)}{\lambda\sigma^2}\sum_{i=1}^{n}\frac{A_i}{B_i} - \frac{\theta^2(\gamma-1)}{\lambda^2\sigma^2}\sum_{i=1}^{n}\frac{C_i}{D_i} + \frac{\theta^2(\gamma-1)}{\lambda\sigma^2}\sum_{i=1}^{n}\frac{A_i}{B_i} - \frac{\theta^2(\gamma-1)}{\lambda^2\sigma^2}\sum_{i=1}^{n}\frac{E_i}{B_i}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-n}{\theta^2} - \frac{1}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\theta} F_i - \frac{(\gamma-1)}{\lambda^2}\sum_{i=1}^{n}\frac{E_iF_i}{B_i} + \frac{(\gamma-1)}{\lambda}\sum_{i=1}^{n}\frac{E_iF_i}{B_i} - \frac{(\gamma-1)}{\lambda^2}\sum_{i=1}^{n}\frac{C_iF_i}{D_i}$$

$$\frac{\partial^2 \ln L}{\partial \sigma \partial \lambda} = \frac{-\theta}{\lambda^2\sigma}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\theta} + \frac{\theta(\gamma-1)}{\lambda^2\sigma}\sum_{i=1}^{n}\frac{A_i}{B_i} - \frac{\theta(\gamma-1)}{\lambda^3\sigma}\sum_{i=1}^{n}\frac{E_i}{B_i} - \frac{\theta(\gamma-1)}{\lambda^3\sigma}\sum_{i=1}^{n}\frac{C_i}{D_i}$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \lambda} = \frac{1}{\lambda^2}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\theta}\ln\left(\frac{x_i}{\sigma}\right) - \frac{(\gamma-1)}{\lambda^2}\sum_{i=1}^{n}\frac{A_i\ln\left(\frac{x_i}{\sigma}\right)}{B_i} + \frac{(\gamma-1)}{\lambda^3}\sum_{i=1}^{n}\frac{E_i\ln\left(\frac{x_i}{\sigma}\right)}{B_i} + \frac{(\gamma-1)}{\lambda^3}\sum_{i=1}^{n}\frac{C_i\ln\left(\frac{x_i}{\sigma}\right)}{D_i}$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \sigma} = \frac{n}{\sigma} + \frac{\theta}{\lambda\sigma}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\theta}\ln\left(\frac{x_i}{\sigma}\right) + \frac{1}{\lambda\sigma}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\theta} + \frac{\theta(\gamma-1)}{\lambda^2\sigma}\sum_{i=1}^{n}\frac{E_i\ln\left(\frac{x_i}{\sigma}\right)}{B_i}$$
$$-\frac{\theta(\gamma-1)}{\lambda\sigma}\sum_{i=1}^{n}\frac{A_i\ln\left(\frac{x_i}{\sigma}\right)}{B_i} - \frac{(\gamma-1)}{\lambda\sigma}\sum_{i=1}^{n}\frac{A_i}{B_i} + \frac{\theta(\gamma-1)}{\lambda^2\sigma}\sum_{i=1}^{n}\frac{C_i\ln\left(\frac{x_i}{\sigma}\right)}{D_i}$$

where $A_i = \left(\frac{x_i}{\sigma}\right)^{\theta} e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}$, $B_i = \left(1 - e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}\right)$, $C_i = \left(\frac{x_i}{\sigma}\right)^{2\theta} e^{\frac{-2}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}$, $D_i = \left(1 - e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}\right)^2$,

$E_i = \left(\frac{x_i}{\sigma}\right)^{2\theta} e^{\frac{-1}{\lambda}\left(\frac{x_i}{\sigma}\right)^{\theta}}$, $F_i = \left(\ln\left(\frac{x_i}{\sigma}\right)\right)^2$

The asymptotic $(1-\alpha)\,100\%$ confidence intervals of $\hat{\sigma}, \hat{\lambda}, \hat{\theta}$ and $\hat{\gamma}$ are $\hat{\sigma} \pm z_{\frac{\alpha}{2}}\sqrt{\mathrm{var}(\hat{\sigma})}$, $\hat{\lambda} \pm z_{\frac{\alpha}{2}}\sqrt{\mathrm{var}(\hat{\lambda})}$, $\hat{\theta} \pm z_{\frac{\alpha}{2}}\sqrt{\mathrm{var}(\hat{\theta})}$ and $\hat{\gamma} \pm z_{\frac{\alpha}{2}}\sqrt{\mathrm{var}(\hat{\gamma})}$ respectively. Where $z_{\frac{\alpha}{2}}$ is the upper $\left(\frac{\alpha}{2}\right)^{th}$ percentile of the standard normal distribution.

VI. Data Analysis and Conclusions

In this section, we present the application of the proposed OELLD (and their sub-models: ELLog, LeLLog and LLogetc., distributions considered by Lemonte [2014]) for a real dataset to illustrate its potentiality. The following real data set corresponds to an uncensoreddata set from Nichols and Padgett (2006) on breaking stress of carbon fibres (inGba):

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19,3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15,2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98,2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12,1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68,2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82,2.05, 3.65.
We fitted the proposed OGELLD curve for the above datawhich is shown in the following graphs:

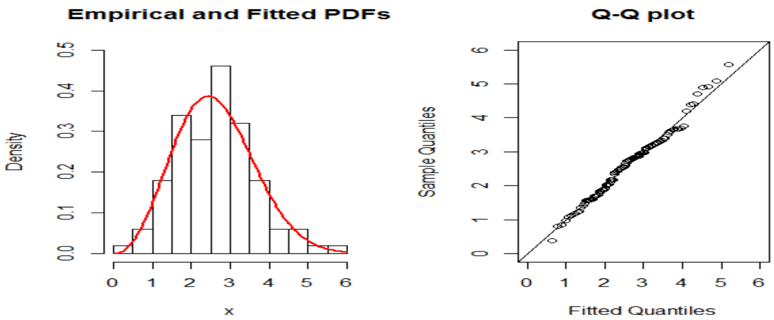

Figure-4: Estimated density and Q-Q plot for OGELLD.

We estimate the unknown parameters of proposed model by the maximum likelihood method. In order to compare the models considered by Lemonte(2014) with the proposed OGELLD model, we consider the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics. The statistics W^* and A^* are described in details in Chen and Balakrishnan (1995). In general, the smaller the values of these statistics, the better the fit to the data. Let H(x; θ) be the c.d.f., where the form of H is known but θ (a k-dimensional parameter vector) is unknown. To obtain the statistics W^* and A^*, we can proceed as follows: (i) Compute $v_i = H(x_i; \hat{\theta})$, where the x_i's are in ascending order, and then $y_i = \phi^{-1}(v_i)$, where $\phi(\cdot)$ is the standard normal c.d.f. and $\phi^{-1}(\cdot)$ its inverse; (ii) Compute $u_i = \phi\{(y_i - \bar{y})/s_y\}$, where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$;

(iii) Calculate $W^2 = \sum_{i=1}^{n}\{u_i - (2i-1)/(2n)\}^2 + 1/(12n)$ and $A^2 = -n - (1/n)\sum_{i=1}^{n}\{(2i-1)\ln(u_i) + (2n+1-2i)\ln(1-u_i)\}$ and then $W^* = W^2(1+0.5/n)$ and $A^* = A^2(1+0.75/n+2.25/n^2)$.

The followingTable lists the MLEs (and the corresponding standard errors in parentheses) of the parameters of all the models for the data set (breaking stress of carbon fibres). The statistics W^* and A^* are also listed in this table for the models. As can be seen from the figures of this table, the new model OGELLD proposed in this paper presents the smallest values of the statistics W^* and A^*than most of the other models, that is, the new model fits the breaking stress of carbon fibres data better than most of the other models considered. More information is provided by a visual comparison from the above graph of the histogram of the data with the fitted OGELLD density function. Clearly, the OGELLD provides a closer fit to the histogram. The Kaplan–Meier (K–M) estimate and the estimated survival function of the fitted OGELLDand Q-Q plot are shown in the Figure 4. OGELLD has four parameters. From this plot, note that the OGELLD model fits the data adequately and hence can be adequate for this data.

Distribution	**Estimates**				W^*	A^*
Beta log logistic (a,b,α, β)	0.09 (0.1700)	0.2254 (0.4452)	3.1486 (0.1851)	25.417 (46.670)	0.03867	0.27763
Exponentiated log logistic(a,α,β)	0.3339 (0.0998)	3.3815 (0.2270)	7.4714 (1.4975)		0.04627	0.3019
Weibull(α, β)	0.049 (0.0138)	2.7929 (0.2131)			0.06227	0.41581
LeLLog(b, α, β)	7.8795 (11.370)	5.6426 (3.3334)	3.0234 (0.3873)		0.06717	0.38989
KW (a, b, c, λ)	1.9447 (5.7460)	12.030 (146.64)	1.6217 (4.6401)	0.0561 (0.1776)	0.06938	0.40705

OGELLD(□, □, □, □)	**1.2576 (2.0610)**	**6.2003 (24.4799)**	**2.4089 (0.0607)**	**1.3173 (0.0598)**	**0.07036**	**0.41314**
BW (α, β, a, b)	0.1013 (0.3160)	2.4231(0.7389)	1.3080(0.6133)	0.8907(3.5611)	0.07039	0.41325
Exponentiated Weibull(α, β, a)	0.0928 (0.0904)	2.4091 (0.5930)	1.3168 (0.5969)		0.07036	0.41313
Marshall-Olkin Weibull(α, γ, λ)	0.6926 (0.8310)	3.0094 (0.7181)	0.0309 (0.0472)		0.07052	0.43016
Beta half-Cauchy (φ, a, b)	15.194 (20.687)	5.5944 (0.8087)	46.116 (70.775)		0.1386	0.70838
Log Logistic (α, β)	2.4984 (0.1051)	4.1179 (0.3444)			0.23903	1.2409
Gamma(λ, η)	5.9526 (0.8193)	2.2708 (0.3261)			0.14802	0.75721
Log-normal(μ, σ)	0.8774 (0.0444)	0.4439 (0.0314)			0.27734	1.48332
Birnbaum-Saunders(α, β)	0.4622 (0.0327)	2.366 (0.1064)			0.29785	1.61816

The estimated asymptotic variance-covariance matrix of the parameters $\sigma, \lambda, \theta \text{ and } \gamma$

$$I_0^{-1} = \begin{bmatrix} 424.7979 & -5.44.2915 & 0.1148 & -0.1072 \\ -5.44.2915 & 59926.8542 & 1.8202 & -1.8134 \\ 0.1148 & 1.8202 & 0.3685 & -0.3463 \\ -0.1072 & -1.8134 & -0.3463 & 0.3583 \end{bmatrix}$$

The approximate 95% two sided confidence intervals of the unknown parameters $\sigma, \lambda, \theta \text{ and } \gamma$ are (0, 5.2972), (0, 54.1810), (2.2899, 2.5278) and (1.1983, 1.4362) respectively.

VII. Concluding Remarks

In this paper, we have studied a new probability distribution called odd generalized exponential log logistic distribution. This is a particular case of distributions proposed byTahir*et al.*(2015). The structural properties of this distribution have been studied and inferences on parameters have also been mentioned. Theappropriateness of fitting the odd generalized exponential log logistic distribution has been established by analyzing a real life data set.

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