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Dual Spaces of Generalized Cesaro Sequence Space and Related Matrix Mapping

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ABTRACT: *In this paper we define the generalized Cesaro sequence spaces $ces(p,q,s)$. We prove the space $ces(p,q,s)$ is a complete paranorm space. In section-2 we determine its Kothe-Toeplitz dual. In section-3 we establish necessary and sufficient conditions for a matrix A to map $ces(p,q,s)$ to l_∞ and $ces(p,q,s)$ to c, where l_∞ is the space of all bounded sequences and c is the space of all convergent sequences. We also get some known and unknown results as remarks.*
KEYWORDS: *Sequence space, Kothe-Toeplitz dual, Matrix transformation.*

I. INTRODUCTION

Let ω be the space of all (real or complex) sequences and let l_∞, c and c_0 are respectively the Banach spaces of bounded sequences, convergent sequences and null sequences. Let $p=(p_k)$ be a bounded sequence of strictly positive real numbers. Then $l(p)$ was defined by Maddox [7] as

$$l(p)=\left\{x=(x_k)\in\omega:\sum_{k=1}^{\infty}|x_k|^{p_k}<\infty\right\}$$

with $0<p_k\le \sup_k p_k=H<\infty$.

In [9] Shiue introduce the Cesaro sequence space ces_p as

$$ces_p=\left\{x=(x_k)\in\omega:\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p<\infty\right\} for\ 1<p<\infty$$

$$and\quad ces_\infty=\left\{x=(x_k)\in\omega:\sup_{n\ge 1}\frac{1}{n}\sum_{k=1}^{n}|x_k|\right\} for\ \ p=\infty.$$

In [5] Leibowitz studied some properties of this space and showed that it is a Banach space. Lim [10] defined this space in a different norm as

$$ces_p=\left\{x=(x_k)\in\omega:\sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_{r}|x_k|\right)^p<\infty\right\} for\ 1<p<\infty$$

$$and\quad ces_\infty=\left\{x=(x_k)\in\omega:\sup_{r\ge 0}\frac{1}{2^r}\ |x_k|<\infty\right\} for\ p=\infty$$

where $\sum_r denotes$ a sum over the ranges $[2^r,2^{r+1})$, determined its dual spaces and characterize some matrix classes. Later in [11] Lim extended this space $ces_p\ to\ ces(p)$ for the sequence $p=(p_r)$ with $\inf p_r>0$ and defined as

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$$ces(p)=\left\{x=(x_k)\in\omega:\sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_{r}|x_k|\right)^{p_r}<\infty\right\}.$$

For positive sequence of real numbers $(p_n),(q_n)\ and\ Q_n=q_1+q_2+\cdots\ldots.+q_n$, Johnson and Mohapatra [14] defined the Cesaro sequence space $ces(p,q)\ \ as$

$$ces(p,q)=\left\{x=(x_k)\in\omega:\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k\ |x_k|\right)^{p_r}<\infty\right\}$$

and studied some inclusion relations. What amounts to the same thing defined by Khan and Rahman [4] as

$$ces(p,q) = \left\{x = (x_k) \in \omega: \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k|x_k|\right)^{p_r} < \infty\right\}$$

for $p = (p_r)$ with $\inf p_r > 0$, $Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots \ldots \ldots \ldots + q_{2^{r+1}-1}$ and $\sum_r$ denotes a sum over the ranges $[2^r, 2^{r+1})$. They determined it's Kothe –Toeplitz dual and characterized some matrix classes.

The main purpose of this paper is to define the generalized Cesaro sequence space $ces(p,q,s)$. We determine the Kothe-Toeplitz dual of $ces(p,q,s)$ and then consider the matrix mapping $ces(p,q,s)\ to\ l_\infty\ and\ ces(p,q,s)\ to\ c$.

In [2] Bulut and Cakar defined and studied the sequence space $l(p,s)$, in [3] Khan and Khan defined and investigated the Cesaro sequence space $ces(p,s)$, in [12] we defined and studied the Riesz sequence space $r^q(u,p,s)$ of non-absolute type and in [13] we defined and studied the generalized weighted Cesaro sequence space $ces(p,q,s)$. In the same vein we define generalized Cesaro sequence space $ces(p,q,s)$in the following way.

DEFINITION. For $s \geq 0$ we define

$$ces(p,q,s) = \left\{x = (x_k) \in \omega: \sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r q_k|x_k|\right)^{p_r} < \infty\right\}$$

where (q_k) is a bounded sequence of real numbers, $p = (p_r)$ with $\inf p_r > 0$, $Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots \ldots \ldots \ldots \ldots \ldots + q_{2^{r+1}-1}$ and $\sum_r$ denotes a sum over the range $2^r \leq k < 2^{r+1}$. With regard notation, the dual space of $ces(p,q,s)$, that is, the space of all continuous linear functional on $ces(p,q,s)$ will be denoted by $ces^*(p,q,s)$. We write

$$A_r(n) = \max_r\left(q_k^{-1}|a_{n,k}|\right)$$

where for each *n* the maximum with respect to k in $[2^r, 2^{r+1})$.
Throughout the paper the following well-known inequality (see [7] or [8]) will be frequently used. For any positive integer $E > 1$ and any two complex numbers a and b we have

$$|ab| \leq E(|a|^t E^{-t} + |b|^t) \qquad (1)$$

where $p > 1\ and\ \frac{1}{p} + \frac{1}{q} = 1$.

To begin with, we show that the space $ces(p,q,s)$ is a paranorm space paranormed by

$$g(x) = \left(\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r q_k|x_k|\right)^{p_r}\right)^{1/M} \qquad (2)$$

provided $H = \sup_r p_r < \infty\ and\ M = \max\{1, H\}$.
Clearly

$$g(\theta) = 0$$
$$g(-x) = g(x),$$

where $\theta = (0,0,0,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$
Since $p_r \leq M$, $M \geq 1$ $so\ for\ any\ x,y \in ces(p,q,s)\ we$ have by Minkowski's inequality

$$\left(\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r q_k|x_k+y_k|\right)^{p_r}\right)^{1/M}$$

$$\leq \left(\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r(q_k|x_k| + q_k|y_k|)\right)^{p_r}\right)^{1/M}$$

$$\leq \left(\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r q_k|x_k|\right)^{p_r}\right)^{1/M} + \left(\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r q_k|y_k|\right)^{p_r}\right)^{1/M}$$

which shows that g is subadditive.
Finally we have to check the continuity of scalar multiplication. From the definition of $ces(p,q,s)$, we have $\inf\ p_r > 0$. So, we may assume that $\inf\ p_r \equiv \rho > 0$. Now for any complex λ with $||\lambda|| < 1$, we have

$$g(\lambda x) = \left(\sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k |\lambda x_k|\right)^{p_r}\right)^{1/M}$$

$$= |\lambda|^{\frac{p_r}{M}} \left(\sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k |x_k|\right)^{p_r}\right)^{\frac{1}{M}}$$
$$\leq \sup_r \|\lambda\|^{\frac{p_r}{M}} g(x)$$
$$\leq \|\lambda\|^{\frac{\rho}{M}} g(x) \to 0 \ as\ \lambda \to 0$$

above. It is quite routine to show that $ces(p,q,s)$ is a metric space with the metric $d(x,y) = g(x-y)$ provided that $x, y \in ces(p,q,s)$, where g is defined by (2). And using a similar method to that in ([3],[4],[13])one can show that $ces(p,q,s)$ is complete under the metric mentioned.

II. KOTHE-TOEPLITZ DUALS

If X is a sequence space we define ([1], [6])

$$X^{|+|} = X^{\alpha} = \left\{a = (a_k) \in \omega : \sum_k |a_k x_k| < \infty,\ for\ every\ \ x \in X\right\}$$

$$X^{+} = X^{\beta} = \left\{a = (a_k) \in \omega : \sum_k a_k x_k\ is\ convergent\ \ for\ every\ \ x \in X\right\}$$

Now we are going to give the following theorem by which the generalized Kothe-Toeplitz dual $ces^{+}(p,q,s)$ will be determined.

Theorem 1: If $1 < p_r \leq \sup_r p_r < \infty\ and\ \frac{1}{p_r} + \frac{1}{t_r} = 1,\ for\ r = 0, 1, 2, \dots\dots$, then

$ces^{+}(p,q,s) = [ces(p,q,s)]^{\beta}$

$= \left\{a = (a_k) : \sum_{r=0}^{\infty} (Q_{2^r})^{s(t_r-1)} \left(Q_{2^r} \max_r (q_k^{-1} |a_k|)\right)^{t_r} E^{-t_r} < \infty,\ for\ some\ integer\ E > 1\right\}.$

Proof : Let $1 < p_r \leq \sup_r p_r < \infty\ and\ \frac{1}{p_r} + \frac{1}{t_r} = 1,\ for\ r = 0,1,2, \dots\dots$. Define

$\mu(t,s) =$

$$\left\{a = (a_k) : \sum_{r=0}^{\infty} (Q_{2^r})^{s(t_r-1)} \left(Q_{2^r} \max_r (q_k^{-1} |a_k|)\right)^{t_r} E^{-t_r} < \infty, for\ some\ integer\ E > 1\right\}. \quad (3)$$

We want to show that $ces^{+}(p,q,s) = \mu(t,s)$. Let $x \in ces(p,q,s)$ and $a \in \mu(t,s)$. Then using inequality (1) we get

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{r=0}^{\infty} \sum_{r} |a_k x_k|$$
$$= \sum_{r=0}^{\infty} \sum_{r} q_k^{-1} |a_k|\ q_k\ |x_k|$$
$$\leq \sum_{r=0}^{\infty} \max_r (q_k^{-1} |a_k|) \sum_{r} q_k\ |x_k|$$
$$= \sum_{r=0}^{\infty} Q_{2^r} \max_r (q_k^{-1} |a_k|) (Q_{2^r})^{\frac{s}{p_r}} \frac{1}{Q_{2^r}} (Q_{2^r})^{-\frac{s}{p_r}} \sum_{r} q_k |x_k|$$
$$\leq E \sum_{r=0}^{\infty} \left\{\left(Q_{2^r} \max_r (q_k^{-1} |a_k|)\right)^{t_r} (Q_{2^r})^{\frac{s\,t_r}{p_r}} E^{-t_r} + (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k |x_k|\right)^{p_r}\right\}$$
$$= E \left\{\sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |a_k|)\right)^{t_r} (Q_{2^r})^{s(t_r-1)} E^{-t_r} + \sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k |x_k|\right)^{p_r}\right\}$$
$$< \infty$$

which implies that the series $\sum_{k=1}^{\infty} a_k x_k$ convergent.

Therefore,

$a \in dual\ of\ ces(p,q,s) = ces^{+}(p,q,s)$. This shows, $\mu(t,s) \subset ces^{+}(p,q,s)$

Conversely, suppose that $\sum a_k x_k$ is convergent for all $x \in ces(p,q,s)$ but $a \notin \mu(t,s)$. Then

$$\sum_{r=0}^{\infty}(Q_{2^r})^{s(t_r-1)}\left(Q_{2^r}\max_r(q_k{}^{-1}\,|a_k|)\right)^{t_r}E^{-t_r}=\infty$$

for every integer $E>1$.

So, we can define a sequence $0=n(0)<n(1)<n(2)<\cdots\ldots\ldots\ldots\ldots$, such that $\gamma=0,1,2,\ldots\ldots\ldots\ldots$, we have

$$M_\gamma=\sum_{r=n(\gamma)}^{n(\gamma+1)-1}(Q_{2^r})^{s(t_r-1)}\left(Q_{2^r}\max_r(q_k{}^{-1}\,|a_k|)\right)^{t_r}(\gamma+2)^{-t_r/p_r}>1$$

Now we define a sequence $x=(x_k)$ in the following way:

$$x_{N(r)}=Q_{2^r}^{t_r}\left|a_{N(r)}\right|^{t_r-1}sgn\,a_{N(r)}(Q_{2^r})^{s(t_r-1)}(\gamma+2)^{-t_r}M_\gamma^{-1}$$

for $n(\gamma)\le r\le n(\gamma+1)-1$, $\gamma=0,1,2,\ldots\ldots\ldots\ldots\ldots\ldots$, and $x_k=0$ for $k\ne N(r)$, where $N(r)$ is such that

$$\left|a_{N(r)}\right|=\max_r(q_k{}^{-1}\,|a_k|),$$ the maximum is taken with respect to k in $[2^r,2^{r+1})$.

Therefore .

$$\sum_{k=2^{n(\gamma)}}^{2^{n(\gamma+1)-1}}a_kx_k=\sum_{r=n(\gamma)}^{n(\gamma+1)-1}\left(Q_{2^r}\left|a_{N(r)}\right|\right)^{t_r}(Q_{2^r})^{s(t_r-1)}(\gamma+2)^{-t_r}M_\gamma^{-1}$$

$$=M_\gamma^{-1}(\gamma+2)^{-1}\sum_{r=n(\gamma)}^{n(\gamma+1)-1}\left(Q_{2^r}\left|a_{N(r)}\right|\right)^{t_r}(Q_{2^r})^{s(t_r-1)}(\gamma+2)^{-t_r/p_r}$$

$$=M_\gamma^{-1}M_\gamma\ (\gamma+2)^{-1}$$

$$=(\gamma+2)^{-1}$$

It follows that

$$\sum_{k=1}^{\infty}a_kx_k=\sum_{\gamma=0}^{\infty}(\gamma+2)^{-1}$$

diverges.

Moreover

$$\sum_{r=n(\gamma)}^{n(\gamma+1)-1}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r q_k|x_k|\right)^{p_r}$$

$$=\sum_{r=n(\gamma)}^{n(\gamma+1)-1}(Q_{2^r})^{-s}\left(Q_{2^r}^{s(t_r-1)}\,Q_{2^r}^{(t_r-1)}\left|a_{N(r)}\right|^{(t_r-1)}(\gamma+2)^{-t_r}M_\gamma^{-1}\right)^{p_r}$$

$$=\sum_{r=n(\gamma)}^{n(\gamma+1)-1}(Q_{2^r})^{-s}\,Q_{2^r}^{(s+1)(t_r-1)p_r}\left|a_{N(r)}\right|^{(t_r-1)p_r}(\gamma+2)^{-t_rp_r}M_\gamma^{-p_r}$$

$$=\sum_{r=n(\gamma)}^{n(\gamma+1)-1}(Q_{2^r})^{-s}\,Q_{2^r}^{(s+1)t_r}\left|a_{N(r)}\right|^{t_r}(\gamma+2)^{-t_rp_r}M_\gamma^{-p_r}$$

$$=(\gamma+2)^{-2}M_\gamma^{-1}\sum_{r=n(\gamma)}^{n(\gamma+1)-1}Q_{2^r}^{s(t_{r}-1)}(Q_{2^r}\left|a_{N(r)}\right|)^{t_r}(\gamma+2)^{2-t_r-p_r}M_\gamma^{1-p_r}$$

$$=(\gamma+2)^{-2}M_\gamma^{-1}\sum_{r=n(\gamma)}^{n(\gamma+1)-1}Q_{2^r}^{s(t_{r}-1)}(Q_{2^r}\left|a_{N(r)}\right|)^{t_r}(\gamma+2)^{2-t_r/p_r}M_\gamma^{1-p_r}(\gamma+2)^{2-t_r-p_r+t_r/p_r}$$

$$=(\gamma+2)^{-2}M_\gamma^{-1}M_\gamma\ M_\gamma^{1-p_r}(\gamma+2)^{1-p_r}$$

$$=(\gamma+2)^{-2}M_\gamma^{-p_r/t_r}(\gamma+2)^{-p_r/t_r}$$

$$=\frac{(\gamma+2)^{-2}}{M_\gamma^{p_r/t_r}(\gamma+2)^{p_r/t_r}}<(\gamma+2)^{-2}<\infty.$$

Therefore

$$\sum_{r=0}^{\infty} (Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k\,|x_k|\right)^{p_r} \leq (\gamma+2)^{-2} < \infty$$

That is, $x \in ces(p,q,s)$ which is a contradiction to our assumption.
Hence $a \in \mu(t,s)$. That is, $\mu(t,s) \supset ces^{+}(p,q,s)$.
Then combining the two results, we get $ces^{+}(p,q,s) = \mu(t,s)$.
The continuous dual of $ces(p,q,s)$ is determined by the following theorem.

Theorem 2: Let $1 < p_r \leq \sup_r p_r < \infty$. Then continuous dual $ces^{*}(p,q,s)$ is isomorphic to $\mu(t,s)$, which is defined by (3)

Proof: It is easy to check that each $x \in ces(p,q,s)$ can be written in the form

$$x = \sum_{k=1}^{\infty} x_k e_k\,, where\;\; e_k = (0,0,0,\dots\dots\dots 0,1,0,\dots\dots\dots\dots\dots\dots..)$$

and the 1 appears at the k-th place. Then for any $f \in ces^{*}(p,q,s)$ we have

$$f(x) = \textstyle\sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} x_k\, a_k \quad . \tag{4}$$

where $f(e_k) = a_k$. By theorem 1, the convergence of $\sum a_k x_k$ for every x in $ces(p,q,s)$ implies that $a \in \mu(t,s)$.
If $x \in ces(p,q,s)$ and if we take $a \in \mu(t,s)$, then by theorem 1, $\sum a_k x_k$ converges and clearly defines a linear functional on $ces(p,q,s)$. Using the same kind of argument as in theorem 1, it is easy to check that

$$\sum_{k=1}^{\infty}|a_k x_k| \leq E\left(\sum_{r=0}^{\infty} Q_{2^r}^{s(t_r-1)}\left(Q_{2^r}\max_r (q_k^{-1}\,|a_k|)\right)^{t_r} E^{-t_r} + 1\right) g(x)$$

whenever $g(x) \leq 1$, where $g(x)$ is defined by (2). Hence $\sum a_k x_k$ defines an element of $ces^{*}(p,q,s)$.
Furthermore, it is easy to see that representation (4) is unique. Hence we can define a mapping

$$T\colon\; ces^{*}(p,q,s) \to \mu(t,s).$$

By $T(f) = (a_1, a_2, \dots\dots\dots\dots\dots\dots)$ where the a_k appears in representation (4). It is evident that T is linear and bijective. Hence $ces^{*}(p,q,s)$ is isomorphic to $\mu(t,s)$.

III. MATRIX TRANSFORMATIONS

In the following theorems we shall characterize the matrix classes $(ces(p,q,s), l_\infty)$ and $(ces(p,q,s), c)$. Let $A = (a_{n,k})\; n,k = 1,2,\dots\dots..$ be an infinite matrix of complex numbers and X, Y two subsets of the space of complex sequences. We say that the matrix A defines a matrix transformation from X into Y and denote it by $A \in (X,Y)$ if for every sequence $x = (x_k) \in X$ the sequence $A(x) = A_n(x)$ is in Y, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k$$

provided the series on the right is convergent.

Theorem 3: Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p,q,s), l_\infty)$ if and only if there exists an integer $E > 1$, such that $U(E,s) < \infty$, where

$$U(E,s) = \sup_n \sum_{r=0}^{\infty}\left(Q_{2^r} A_r(n)\right)^{t_r} (Q_{2^r})^{\,s(t_r-1)}\, E^{-t_r}$$

and $\frac{1}{p_r} + \frac{1}{t_r} = 1,\; r = 0,1,2,\dots\dots\dots\dots\dots$

Proof: Sufficiency: Suppose there exists an integer $E > 1$, such that $U(E,s) < \infty$. Then by inequality (1), we have

$$\begin{aligned}
\sum_{k=1}^{\infty}|a_{n,k}x_k| &= \sum_{r=0}^{\infty}\sum_{r} |a_{n,k}|\,|x_k| = \sum_{r=0}^{\infty}\sum_{r}\frac{|a_{n,k}|}{q_k}\, q_k\,|x_k| \\
&\leq \sum_{r=0}^{\infty}\max_r \frac{|a_{n,k}|}{q_k}\sum_{r} q_k|x_k| \\
&= \sum_{r=0}^{\infty}(Q_{2^r})^{\frac{s}{p_r}}\;\; Q_{2^r}\max_r\frac{|a_{n,k}|}{q_k}\;\; (Q_{2^r})^{-\frac{s}{p_r}}\,\frac{1}{Q_{2^r}}\sum_{r} q_k\,|x_k| \\
&\leq E\sum_{r=0}^{\infty}\left\{(Q_{2^r})^{\frac{s\,t_r}{p_r}}\,(Q_{2^r}\,A_r(n))^{t_r}\,E^{-t_r} + \left((Q_{2^r})^{-\frac{s}{p_r}}\,\frac{1}{Q_{2^r}}\sum_{r} q_k\,|x_k|\;\right)^{p_r}\right\}
\end{aligned}$$

$$\leq E\left\{\sum_{r=0}^{\infty}(Q_{2^r})^{s\,(t_r-1)}\,(Q_{2^r}\,A_r(n))^{t_r}\,E^{-t_r}+\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\,\left(\frac{1}{Q_{2^r}}\sum_{r}q_k\,|x_k|\right)^{p_r}\right\}$$
$< \infty$.

Therefore, $A \in (ces(p,q,s), l_\infty)$.

Necessity: Suppose that $A \in (ces(p,q,s), l_\infty)$, but

$$\sup_n \sum_{r=0}^{\infty}\left(Q_{2^r}A_r(n)\right)^{t_r}\,(Q_{2^r})^{s\,(t_r-1)}\,E^{-t_r} = \infty \; for\ every\ integer\ E > 1.$$

Then $\sum_{k=1}^{\infty} a_{n,k}x_k$ converges for every n and $x \in ces(p,q,s)$,
whence $\left(a_{n,k}\right)_{k=1,2,\ldots\ldots} \in ces^+(p,q,s)$ for every n. By theorem 1, it follows that each A_n defined by

$$A_n(x) = \sum_{k=1}^{\infty} a_{n,k}x_k$$

is an element of $ces^*(p,q,s)$. Since $ces(p,q,s)$ is complete and since $\sup_n |A_n(x)| < \infty$ on $ces(p,q,s)$, by the uniform boundedness principle there exists a number L independent of n and a number $\delta < 1$, such that

$$|A_n(x)| \leq L \qquad (5)$$

for every n and $x \in S[\theta,\delta]$, where $S[\theta,\delta]$ is the closed sphere in $ces(p,q,s)$ with centre at the origin θ and radius δ.

Now choose an integer $G > 1$, such that

$$G\delta^M > L.$$

Since

$$\sup_n \sum_{r=0}^{\infty}(Q_{2^r}A_r(n))^{t_r}\,(Q_{2^r})^{s\,(t_r-1)}\,G^{-t_r} = \infty$$

there exists an integer $m_0 > 1$, such that

$$R = \sum_{r=0}^{m_0}(Q_{2^r}A_r(n))^{t_r}\,(Q_{2^r})^{s\,(t_r-1)}\,G^{-t_r} > 1 \qquad (6)$$

Define a sequence $x = (x_k)$ as follows:

$$x_k = 0 \;\; if \;\; k \geq 2^{m_0+1}$$

$$x_{N(r)} = Q_{2^r}^{t_r}\delta^{M/p_r}\left(sgn\ a_{n,N(r)}\right)\left|a_{n,N(r)}\right|^{t_r-1}R^{-1}G^{-t_r/p_r}(Q_{2^r})^{s\,(t_r-1)}$$

and $x_k = 0 \;\; if \; k \neq N(r)$ for $0 \leq r \leq m_0$, where $N(r)$ is the smallest integer such that

$$\left|a_{n,N(r)}\right| = \max_r \frac{|a_{n,k}|}{q_k}$$

Then one can easily show that $g(x) \leq \delta$ but $|A_n(x)| > L$, which contradicts (5). This complete the proof of the theorem.

Theorem 4. Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p,q,s), c)$ if and only if

(i) $a_{n,k} \to \alpha_k(n \to \infty, k\ is\ fixed)$ and

(ii) there exists an integer $E > 1$, such that $U(E,s) < \infty$, where

$$U(E,s) = \sup_n \sum_{r=0}^{\infty}\left(Q_{2^r}A_r(n)\right)^{t_r}\,(Q_{2^r})^{s\,(t_r-1)}E^{-t_r} \text{ and } \frac{1}{p_r}+\frac{1}{t_r} = 1,\; r = 0,1,2,\ldots\ldots\ldots\ldots\ldots$$

Proof: Necessity. Suppose $A \in (ces(p,q,s), c)$. Then $A_n(x)$exists for each $n \geq 1$ *and* $\lim_{n\to\infty} A_n(x)$ exists for every $x \in ces(p,q,s)$. Therefore by an argument similar to that in theorem 3 we have condition (ii). Condition (i) is obtained by taking $x = e_k \in ces(p,q,s)$, where e_k is a sequence with 1 at the k-th place and zeros elsewhere.

Sufficiency. The conditions of the theorem imply that

$$\sum_{r=0}^{\infty}\left(Q_{2^r}\max_r \frac{|\alpha_k|}{q_k}\right)^{t_r}(Q_{2^r})^{s\,(t_r-1)}E^{-t_r} \le U(E,s) < \infty \qquad (7)$$

By (7) it is easy to check that $\sum_k \alpha_k x_k$ is absolutely convergent for each $x \in ces(p,q,s)$. For each $x \in ces(p,q,s)$ and $\varepsilon > 0$, we can choose an integer $m_0 > 1$, such that

$$g_{m_0}(x) = \sum_{r=m_0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r q_k\,|x_k|\right)^{p_r} < \varepsilon^M$$

Then by the proof of theorem 2 and by inequality (1), we have

$$\sum_{k=2^{m_0}}^{\infty}|a_{n,k}-\alpha_k||x_k| \le E\left(\sum_{r=m_0}^{\infty}(Q_{2^r})^{s\,(t_r-1)}\,(Q_{2^r}B_r(n))^{t_r}\,E^{-t_r}+1\right)\left(g_{m_0}(x)\right)^{1/M}$$

$$< E(2U(E,s)+1)\varepsilon,$$

where $B_r(n) = \max_r \frac{|a_{n,k}-\alpha_k|}{q_k}$ and

$$\sum_{r=m_0}^{\infty}(Q_{2^r})^{s\,(t_r-1)}\left(Q_{2^r}B_r(n)\right)^{t_r}E^{-t_r} \le 2U(E,s) < \infty$$

It follows immediately that

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}a_{n,k}\,x_k = \sum_{k=1}^{\infty}\alpha_k x_k$$

This shows that $A \in (ces(p,q,s),c)$ which proved the theorem.

Corollary 1. Let $1 < p_r \le \sup_r p_r < \infty$. Then $A \in (ces(p,q,s),c_0)$ if and only if

(i) $a_{n,k} \to 0\ (n\to\infty,\ k\ is\ fixed)$

(ii) there exists an integer $E > 1$ such that $U(E,s) < \infty$, where

$$U(E,s) = \sup_n\sum_{r=0}^{\infty}(Q_{2^r}A_r(n))^{t_r}\,(Q_{2^r})^{s\,(t_r-1)}E^{-t_r}\ and\ \frac{1}{p_r}+\frac{1}{t_r}=1, \qquad r=0,1,2,\dots\dots.$$

Remarks:

(1) If $s=0$ then we get the results of Khan and Rahman [4]
(2) If $s=0,\ q_n=1$ for every n then we get the results of Lim [11]
(3) When $s=0,\ q_n=1\ and\ p_n=p$ for all n then the results of Lim [10] follows.
(4) If $s \ge 1$ then specializing the sequences (p_n) and (q_n) we get many unknown results.
(5)

REFERENCES

[1] [1] B. Choudhury and S. K. Mishra, On Kothe-Toeplitz duals of certain sequence spaces and their matrix transformations, Indian J. pure appl. Math, 24(15), 291-301, May 1993.
[2] [2] E. Bulut and O Cakar, The sequence space $l(p,s)$ and related matrix transformation, communication de la faculatedes science d L'universite D'Ankara Tome, 28(1979), 33-44.
[3] [3] F.M. KHAN and M.A. KHAN, The sequence space $ces(p,s)$ and related matrix transformations, Research Ser.Mat.,21(1991); 95-104.
[4] [4] F.M. KHAN and M.F. RAHMAN, Infinite matrices and Cesaro sequence spaces, Analysis Mathematica, 23(1997), 3-11.
[5] [5] G. M. Leibowitz, A note on the Cesaro sequence spaces, Tamkang J. of Math.,2(1971),151-157
[6] [6] H. Kizmaz, Canadian Math. Bull. 24(2)(1981),169-176.
[7] [7] I,J. MADDOX, continuous and Köthe-Toeplitz dual of certain sequence spaces, Proc. Camb. phil. Soc., 65(1969),431-435.
[8] [8] I,J. MADDOX, Elements of Functional Analysis, Cambridge University Press Cambridge, second edition, 1988.
[9] [9] J.S. shiue, On the Cesaro sequence spaces, Tamkang J. of Math. 1(1970), 19-25.
[10] [10] K.P. LIM, Matrix transformation in the Cesaro sequence spaces, KyungpookMath. J. , 14(1974),221-227
[11] [11] K.P. LIM, Matrix transformation on certain sequence space, Tamkang J. of Math. 8(1977), 213-220.
[12] [12] M. F. Rahman and A.B.M. Rezaul Karim, Generalized Riesz sequence space of Non-absolute Type and Some Matrix Mapping. Pure and Applied Mathematics Journal.(2015); 4(3): 90-95.
[13] [13] M.F. Rahman and A.B.M. Rezaul Karim, Dual spaces of Generalized Weighted Cesaro sequence space and related Matrix Mapping. Bulletin of Mathematics and Statistics Research ,vol.4.Issue.1.2016(January-March).
[14] [14] P.D. Johnson Jr. and R.N. Mohapatra, density of finitely non-zero sequences in some sequence spaces, Math. Japonica 24, No. 3(1979), 253-262.