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A Fuzzy Mean-Variance-Skewness Portfolioselection Problem.

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Abstract*—A fuzzy number is a normal and convex fuzzy subsetof the real line. In this paper, based on membership function, we redefine the concepts of mean and variance for fuzzy numbers. Furthermore, we propose the concept of skewness and prove some desirable properties. A fuzzy mean-variance-skewness portfolio se-lection model is formulated and two variations are given, which are transformed to nonlinear optimization models with polynomial ob-jective and constraint functions such that they can be solved analytically. Finally, we present some numerical examples to demonstrate the effectiveness of the proposed models.*

Key words*—Fuzzy number, mean-variance-skewness model, skewness.*

I. INTRODUCTION

The modern portfolio theory is an important part of financial fields. People construct efficient portfolio to increasereturn and disperse risk. In 1952, Markowitz [1] published the seminal work on portfolio theory. After that most of the studies are centered around the Markowitz's work, in which the invest-ment return and risk are respectively regarded as the mean value and variance. For a given investment return level, the optimal portfolio could be obtained when the variance was minimized under the return constraint. Conversely, for a given risk level, the optimal portfolio could be obtained when the mean value was maximized under the risk constraint. With the development of financial fields, the portfolio theory is attracting more and more attention around the world.

One of the limitations for Markowitz's portfolio selection model is the computational difficulties in solving a large scale quadratic programming problem. Konno and Yamazaki [2] over-came this disadvantage by using absolute deviation in place of variance to measure risk. Simaan [3] compared the mean-variance model and the mean-absolute deviation model from the perspective of investors' risk tolerance. Yu *et al.* [4] proposed a multiperiod portfolio selection model with absolute deviation minimization, where risk control is considered in each period.The limitation for a mean-variance model and a mean-absolute deviation model is that the analysis of variance andabsolute deviation treats high returns as equally undesirable as low returns. However, investors concern more about the part in which the return is lower than the mean value. Therefore, it is not reasonable to denote the risk of portfolio as a vari-ance or absolute deviation. Semivariance [5] was used to over-come this problem by taking only the negative part of variance. Grootveld and Hallerbach [6] studied the properties and com-putation problem of mean-semivariance models. Yan *et al.* [7] used semivariance as the risk measure to deal with the multi-period portfolio selection problem. Zhang *et al.* [8] considered a portfolio optimization problem by regarding semivariance as a risk measure. Semiabsolute deviation is an another popular downside risk measure, which was first proposed by Speranza [9] and extended by Papahristodoulou and Dotzauer [10].
When mean and variance are the same, investors prefer a portfolio with higher degree of asymmetry. Lai [11] first con-sidered skewness in portfolio selection problems. Liu *et al.* [12] proposed a mean-variance-skewness model for portfolio selec-tion with transaction costs. Yu *et al.* [13] proposed a novel neural network-based mean-variance-skewness model by integrating different forecasts, trading strategies, and investors' risk preference. Beardsley *et al.* [14] incorporated the mean, vari-ance, skewness, and kurtosis of return and liquidity in portfolio selection model.
All above analyses use moments of random returns to mea-sure the investment risk. Another approach is to define the risk as an entropy. Kapur and Kesavan [15] proposed an entropy op-timization model to minimize the uncertainty of random return, and proposed a cross-entropy minimization model to minimize the divergence of random return from *a priori* one. Value at risk (VAR) is also a popular risk measure, and has been adopted in a portfolio selection theory. Linsmeier and Pearson [16] gave an introduction of the concept of VAR. Campbell *et al.* [17] devel-oped a portfolio selection model by maximizing the expected return under the constraints that the maximum expected loss satisfies the VAR limits. By using the concept of VAR, chance constrained programming was applied to portfolio selection to formalize risk and return relations [18]. Li [19] constructed an insurance and investment portfolio model, and proposed a method to maximize the insurers' probability of achieving their aspiration level, subject to chance constraints and institutional constraints.Probability theory is

widely used in financial fields, and many portfolio selection models are formulated in a stochastic en-vironment. However, the financial market behavior is also af-fected by several nonprobabilistic factors, such as vagueness and ambiguity. With the introduction of the fuzzy set theory [20], more and more scholars were engaged to analyze the portfolio

Selection models in a fuzzy environment . For example, Inuiguchi and Ramik [21] compared and the difference between fuzzy mathematical programming and stochastic programming in solving portfolio selection problem.Carlsson and Fuller [22] introduced the notation of lower and upper possibilistic means for fuzzy numbers. Based on these notations. Zhang and Nie[23] proposed the lower and upper possibilistic variances and covariance for fuzzy numbers and constructed a fuzzy mean variance model. Huang [24] proved some properties of semi variance for fuzzy variable,and presented two mean semivariancemosels.Inuiguchi et al.[25] proposed a mean-absolute deviation model,and introduced a fuzzy linear regression technique to solve the model.Liet al [26]defined the skewness for fuzzy variable with in the framework of credibility theory,and constructed a fuzzy mean –variance-skewness model. Cherubini and lunga [27] presented a fuzzy VAR to denote the liquidity in financial market .Gupta et al .[28] proposed a fuzzy multiobjective portfolio selection model subject to chance constraintsBarak et al.[29] incorporated liquidity into the mean variance skewness portfolio selection with chance constraints.Inuiguchi and Tanino [30] proposed a minimax regret approach.Li et al[31]proposed an expected regret minimization model to minimize the mean value of the distance between the maximum return and the obtained return .Huang[32] denoted entropy as risk.and proposed two kinds of fuzzy mean-entropy models.Qin et al.[33] proposed a cross-entropy miniomizationmodel.More studies on fuzzy portfolio selection can be found in [34]

Although fuzzy portfolio selection models have been widely studied,the fuzzy mean-variance –skewness model receives less attention since there is no good definition on skewness. In 2010,Li et al .[26]proposed the concept of skewness for fuzzy variables,and proved some desirable properties within the framework of credibility theory.However the arithmetic difficulty seriously hinders its applications in real life optimization problems.Some heuristic methods have to be used to seek the sub optimal solution, which results in bad performances on computation time and optimality. Based on the membershipfunction,this paper redefines the possibilistic mean (Carlsson and Fuller [22]) and possibilistic variance(Zhang and Nie[23]) ,and gives a new definition on Skewness for fuzzy numbers. A fuzzy mean-variance-skewnessportfolio selection models is formulated ,and some crisp equivalents are discussed, in which the optimal solution could be solved analytically.

This rest of this paper is Organized as follows.Section II reviews the preliminaries about fuzzy numbers.Section III redefines mean and variance,and proposes the definition of skewness for fuzzy numbers.Section IV constructs the mean –variance skewnessmodel,and proves some crisp equivalents.Section V lists some numerical examples to demonstrate the effectiveness of the proposed models.Section VI concludes the whole paper.

II. PRELIMINARIES

In this section , we briefly introduce some fundamental concepts and properties on fuzzy numbers, possibilistic means, and possibilistic variance

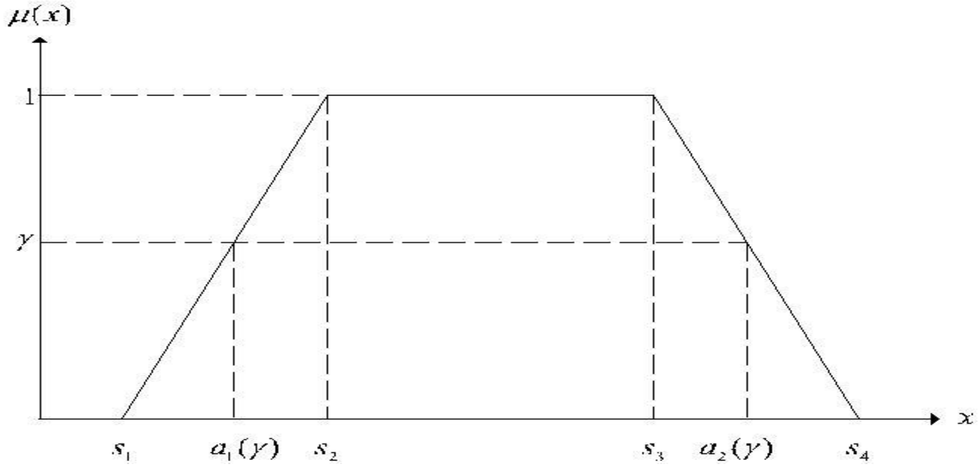

Fig 1.membership function of Trapezoidal fuzzy number $\eta = (s_1, s_2, s_3, s_4)$.

DefinationII.1 (Zadeh [20]) : A fuzzy subset $\tilde{A}$ in X is defined as $\tilde{A} = \{\langle x, \mu(x)\rangle : x \in X\}$, Where $\mu: X \to [0,1]$, and the real value $\mu(x)$ represents the degree of membership of $x \; in \; \tilde{A}$
Defination II.2 (Dubois and Prade [35]) : A fuzzy number ξ is a normal and convex fuzzy subset of $\Re$ .Here ,normality implies that there is a point x_0 such that $\mu(x_0) = 1$, and convexity means that

$$\mu(\alpha x_1 + (1-\alpha)x_2) \geq \min\big(\mu(x_1), \mu(x_2)\big) \dots\dots\dots\dots\dots..(1)$$

for any$\alpha \in [0,1] and \; x_1, x_2 \in \Re$

Defination II.3 (Zadeh [20]) : For any $\gamma \in [0,1]$ the $\gamma -$ level set of a fuzzy subset$\tilde{A}$ denoted by $[\tilde{A}]^{\gamma}$ is defined as $[\tilde{A}]^{\gamma} = \{x \in X: \mu(x) \geq \gamma\}$. If ξ is a fuzzy number there are an increasing function $a_1: [0,1] \to x$ and a decreasing function $a_2: [0,1] \to x$ such that $[\,\xi] = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0,1]$.Suppose that $\xi \; and \; \eta$ are two fuzzy numbers with $\gamma -$ level sets $[a_1(\gamma), a_2(\gamma)]$ and $[b_1(\gamma), b_2(\gamma)]$. For any$\lambda_1, \lambda_2 \geq 0$,if $[\lambda_1 \xi + \lambda_2 \eta] = [c_1(\gamma), c_2(\gamma)]$
we have

$$c_1(\gamma) = \lambda_1 a_2(\gamma) + \lambda_1 b_2(\gamma) \quad , \quad c_2(\gamma) = \lambda_1 a_2(\gamma) + \lambda_1 b_2(\gamma)$$

Let $\xi = (r_1, r_2, r_3,)$ be a triangular fuzzy number ,and let $\eta = (s_1, s_2\,, s_3, s_4)$ be a Trapezoidal fuzzy number (see Fig 1).
It may be shown that
$[\xi]^{\gamma} = [r_1 + (r_2 - r_1)\gamma, r_3 - (r_3 - r_2)\gamma\,]$and
$[\eta]^{\gamma} = [s_1 + (s_2 - s_1)\gamma, s_4 - (s_4 - s_3)\gamma\,]$.
Then ,For fuzzy numbers $\xi + \eta$, its level set is $[c_1(\gamma), c_2(\gamma)]$ with

$$c_1(\gamma) = r_1 + s_1 + \gamma(r_2 + s_2 - r_1 - s_1) \text{and}$$
$$c_2(\gamma) = r_3 + s_4 - \gamma(r_3 - r_2 + s_4 - s_3)$$

Defination II.4 (Carlsson and Fuller [22]): For a fuzzy number $\xi \; with \; \gamma$-level set
$[\xi]^{\gamma} = [a_1(\gamma), a_2(\gamma)] 0 < \gamma < 1$the lower and upper possibilistic means are defined as

$$E^-(\xi) = 2\int_0^1 \gamma a_1(\gamma) d\gamma \; E^+(\xi) = 2\int_0^1 \gamma a_2(\gamma) d\gamma$$

Theorem II.1: (Carlsson and Fuller [22]) Let$\xi_1, \xi_2, \xi_3, \dots\dots\dots..\xi_n$ be fuzzy numbers,and Let $\lambda_1, \lambda_2, \lambda_3, \dots\dots\dots..\lambda_n$ be nonnegative real numbers ,we have

$$E^-\left(\sum_{i=1}^{n} \lambda_i \xi_i\right) = \sum_{i=1}^{n} \lambda_i E^-(\xi)$$

$$E^+\left(\sum_{i=1}^{n} \lambda_i \xi_i\right) = \sum_{i=1}^{n} \lambda_i E^+(\xi)$$

Inspired by lower and upper possibilistic means.Zhang and Nie [23]introduced the lower and upper possibilistic variances and possibilisticcovariances of fuzzy numbers.

Defination II.5 (Zhang and Nie [23]): For a Fuzzy numbers ξ with lower possibilistic means $E^+(\xi)$,the lower and upper possibilistic variances are defined as

$$V^-(\xi) = 2\int_0^1 \gamma\big(a_1(\gamma) - E^-(\xi)\big)^2 d\gamma$$

$$V^+(\xi) = 2\int_0^1 \gamma\big(a_2(\gamma) - E^+(\xi)\big)^2 d\gamma$$

Definition II.6 (Zhang and Nie [23]): For a fuzzy number$\xi$with lower possibilistic mean $E^-(\xi)$and upper possibilistic mean $E^+(\xi)$), fuzzy number η with lower possibilistic mean $E^-(\eta)$and upper possibilistic mean$E^+(\eta)$, the lower and upperpossibilistic covariances between ξ and η are defined as

$$cov^{-}(\xi,\eta) = 2\int_0^1 \gamma\big(E^{-}(\xi) - a_1(\gamma)\big)\big(E^{-}(\eta) - b_1(\gamma)\big)d\gamma$$

$$cov^{+}(\xi,\eta) = 2\int_0^1 \gamma\big(E^{+}(\xi) - a_2(\gamma)\big)\big(E^{+}(\eta) - b_2(\gamma)\big)d\gamma$$

III. MEAN,VARIANCE AND SKEWNESS

In this section based on the membership functions,we redefine the mean and variance for fuzzy numbers ,and propose a defination of skewness.

Defination III.1: Let ξ be a fuzzy number with differential membership function $\mu(x)$. Then its mean is defined as

$E(\xi) = \int_{-\infty}^{+\infty} x\mu(x)\,|\mu'(x)|dx$.........................(2)

Mean value is one of the most important concepts for fuzzy number, which gives the center of its distribution

Example III.1: For a Trapezoidal fuzzy number $\eta = (s_1, s_2\,, s_3, s_4)$, the shape of function $\mu(x)|\mu'(x)|$is shown in Fig.2 according to defination 3.1, its mean value is

$$E(\eta) = \int_{s_1}^{s_2} x\frac{x - s_1}{s_2 - s_1}\;.\frac{1}{s_2 - s_1}\;dx + \int_{s_3}^{s_4} x\frac{s_1 - x}{s_4 - s_3}\;.\frac{1}{s_4 - s_3}\;dx$$

$$= \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

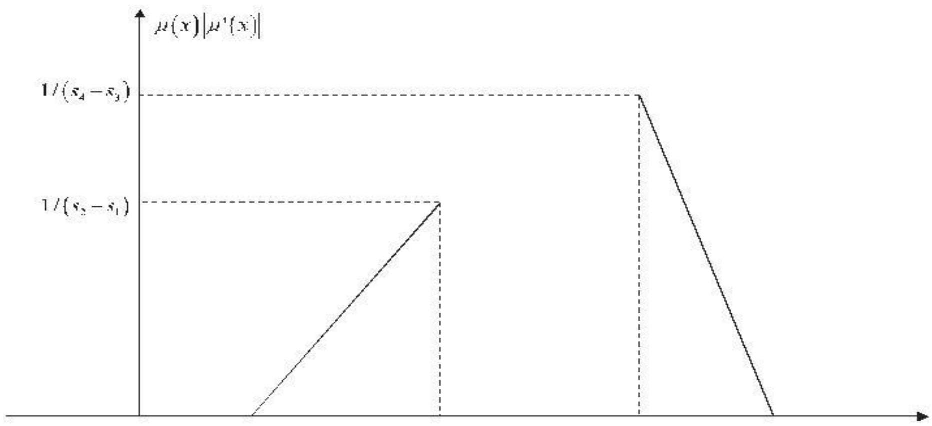

Fig 2. Shape of function $\mu(x)|\mu'(x)|$ for trapezoidal fuzzy number $\eta = (s_1, s_2\,, s_3, s_4)$

In perticular,if η is symmetric with $s_2 - s_1 = s_4 - s_3$we have $E(\eta) = \frac{s_2+s_3}{2}$ if η is a triangular fuzzy number (r_1, r_2, r_3) we have $E(\eta) = \frac{(r_1+4r_2+r_3)}{6}$

Theorem III.1: Suppose that a fuzzy number ξ has differentiable membership function $\mu(x)$ with $\mu(x) \to 0$ as $x \to -\infty$ and $x \to +\infty$ then we have

$\int_{-\infty}^{+\infty} \mu(x)\,|\mu'(x)|dx =$ 1.........................(3)

Proof: without loss of generality, we assume $\mu(x_0) = 1$, It is proved that

$$\int_{-\infty}^{+\infty} \mu(x)\,|\mu'(x)|dx = \int_{-\infty}^{x_0} \mu(x)\,\mu'(x)dx - \int_{x_0}^{+\infty} \mu(x)\,\mu'(x)dx$$

$$= \frac{1}{2}\int_{-\infty}^{x_0} d\mu^2(x) - \int_{x_0}^{+\infty} d\mu^2(x)$$

Further more , It follows from $\mu(x) \to 0 \ as \ x \to -\infty \ and \ x \to +\infty$ that

$$\int_{-\infty}^{+\infty} \mu(x)\,|\mu'(x)|dx$$
$$= \frac{1}{2}\left(\mu^2(x_0) - \lim_{x\to-\infty} \mu^2(x)\right) = \frac{1}{2}\left(\lim_{x\to+\infty} \mu^2(x) - \mu^2(x_0)\right)$$
$$= 1$$

The proof is complete

Let ξ be a fuzzy number with differentiable membership function μ.Equation (3) tells us that the counterpart of a probability density function for ξ is $f(x) = \mu(x)|\mu'(x)|$

Theorem III.2: Suppose that a fuzzy number ξ has differentiable membership function μ and $\mu(x) \to 0 \ as \ x \to -\infty$ and $x \to +\infty$ .If it has $\gamma - level\ set[\xi]^\gamma = [a_1(\gamma), a_2(\gamma)]$ then we have

$E(\xi) = \int_0^1 \gamma a_1(\gamma)d\gamma + \int_0^1 \gamma a_2(\gamma)d\gamma$.............(4)

Proof: Without loss of generality,we assume $\mu(x_0) = 1$ .According to Defination 3.1,we have

$$E(\xi) = \int_{-\infty}^{x_0} x\mu(x)\,\mu'(x)dx - \int_{x_0}^{+\infty} x\mu(x)\,\mu'(x)dx$$

Taking $x = a_1(\gamma)$,It follows from the integration by substitution that

$$\int_{-\infty}^{x_0} x\mu(x)\,\mu'(x)dx = \int_{-\infty}^{x_0} x\mu(x)\,d\mu(x)dx$$

$$= \int_0^1 a_1(\gamma)\mu\left(a_1(\gamma)d\mu\big(a_1(\gamma)\big)\right)$$
$$= \int_0^1 \gamma a_1(\gamma)d\gamma$$

Similarly taking $x = a_2(\gamma)$,It follows from the integration by substitution that

$$\int_{-\infty}^{x_0} x\mu(x)\,\mu'(x)dx = -\int_0^1 \gamma a_2(\gamma)d\gamma$$

The proof is complete

Remark III.1: Based on the above theorem, it is concluded that Definition 3.1 coincides with the lower and upper possibilistic means in the sense of $E = {E^- + E^+}/{2}$, which is also defined as the crisp possibilistic mean by Carlsson and Fuller [22]. In 2002, Liu and Liu [36] defined a credibilistic mean value for fuzzy variables based on credibility measures and Choquet integral, which does not coincide with the lower and upper possibilistic means. Taking triangular fuzzy number $\xi = (0,1,3)$for example, the lower possibilistic mean is2/3,the upper possibilistic mean is 10/3, and the mean is 2. However, its credibilistic mean is 2.5.

Theorem III.3:Suppose that ξ and η are two fuzzy numbers.For any nonnegative real numbers $\lambda_1 \ and \ \lambda_2$ we have

$$E(\lambda_1\xi + \lambda_2\eta) = \lambda_1 E(\xi) + \lambda_2 E(\eta)$$

Proof:For any $\gamma \in [0,1]$,denote$[\xi]^\gamma = [a_1(\gamma), a_2(\gamma)]$and $[\eta]^\gamma = [b_1(\gamma), b_2(\gamma)]$. According to$[\lambda_1\xi + \lambda_2\eta]^\gamma = \left[\big(\lambda_1 a_1(\gamma)\big) + \lambda_2 b_1(\gamma), \lambda_1 a_2(\gamma) + \lambda_2 b_2(\gamma)\right]$, It follows from Defination 3.1 and theorem 3.2

$$E(\lambda_1\xi + \lambda_2\eta) = \int_0^1 \gamma\left(\lambda_1 a_1(\gamma) + \lambda_2 b_1(\gamma)\right)d\gamma + \int_0^1 \gamma\left(\lambda_1 a_2(\gamma) + \lambda_2 b_2(\gamma)\right)d\gamma$$

$$= \lambda_1 \int_0^1 \gamma \left(a_1(\gamma) + a_2(\gamma)\right) d\gamma + \lambda_2 \int_0^1 \gamma \left(b_1(\gamma) + b_2(\gamma)\right) d\gamma$$

$$= \lambda_1 E(\xi) + \lambda_2 E(\eta)$$

The proof is completes

The linearity is an important property for mean value as an extension of Theorem 3.3 for any fuzzy numbers $\xi_1, \xi_2, \xi_3, \dots \dots .. \xi_n$ and $\lambda_1, \lambda_2, \lambda_3, \dots \dots .. \lambda_n \geq 0$ we have $E(\lambda_1 \xi_1 + \lambda_1 \xi_2 + \cdots \dots \lambda_1 \xi_2) = \lambda_1 E(\xi_1) + \lambda_2 E(\xi_2) + \lambda_3 E(\xi_3) \dots \dots .. \lambda_n E(\xi_n)$

Defination III.2: Let ξ be a fuzzy numbers with differential e membership function $\mu(x)$ and finite mean value $E(\xi)$. Then its variance is defined as

$$V(\xi) = \int_{-\infty}^{+\infty} \left(x - E(\xi)\right)^2 \mu(x)\, \mu'(x) dx$$

If ξ is afuzzy number with mean $E(\xi)$,then its variance is used to measure the spread of its distribution about $E(\xi)$

Example III.2: Let $\eta = (s_1, s_2, s_3, s_4)$ be a trapezoidal fuzzy number .According to Defination 3.2ita variance is

$$\int_{s_1}^{s_2} \left(x - E(\eta)\right)^2 . \frac{x - s_1}{s_2 - s_1} . \frac{1}{s_2 - s_1} dx + \int_{s_3}^{s_4} \left(x - E(\eta)\right)^2 . \frac{s_4 - x}{s_4 - s_3} . \frac{1}{s_4 - s_3} dx$$

$$= \frac{s_2 - s_1 - 2\left(s_2 - E(\eta)\right)^2 + s_4 - s_3 + 2\left(s_3 - E(\eta)\right)^2}{12} + \frac{2\left(s_2 - E(\eta)\right)^2 + 2\left(s_3 - E(\eta)\right)^2}{12}$$

If η is symmetric with $s_2 - s_1 = s_4 - s_3$,we have $V(\eta) = \frac{[2(s_2 - s_1)^2 + 3(s_3 - s_2)^2 + 4(s_3 - s_2)(s_2 - s_1)]}{12}$ If η is a triangular fuzzy number (r_1, r_2, r_3) we have $V(\eta) = \frac{[2(r_2 - r_1)^2 + 2(r_3 - r_2)^2 - (r_3 - r_2)(r_2 - r_1)]}{18}$

Theorem III.4: Let ξ be a fuzzy number with $\gamma - level\ set [\xi]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and mean value $E(\xi)$.Then we have

$$V(\xi) = \int_0^1 \gamma \left[\left(a_1(\gamma) - E(\xi)\right)^2 + \left(a_2(\gamma) - E(\xi)\right)^2\right] d\gamma$$

Proof: without loss of generality, we assume $\mu(x_0) = 1$ then .according to Definition 3.2 we have

$$V(\xi) = \int_{-\infty}^{+\infty} \left(x - E(\xi)\right)^2 \mu(x) \mu'(x) dx$$

$$= \int_{-\infty}^{x_0} \left(x - E(\xi)\right)^2 \mu(x) \mu'(x) dx + \int_{x_0}^{+\infty} \left(x - E(\xi)\right)^2 \mu(x) \mu'(x) dx$$

Taking $x = a_1(\gamma)$ it follows from the integration by substitution that

$$\int_{-\infty}^{x_0} \left(x - E(\xi)\right)^2 \mu(x) \mu'(x) dx = \int_{-\infty}^{x_0} \left(x - E(\xi)\right)^2 \mu(x) d\, \mu(x)$$

$$= \int_0^1 \gamma \left(a_1(\gamma) - E(\xi)\right)^2 d\gamma$$

similarly taking $x = a_2(\gamma)$ it follows from the integration by substitution that

$$\int_{x_0}^{+\infty} \left(x - E(\xi)\right)^2 \mu(x) \mu'(x) dx = \int_{x_0}^{+\infty} \left(x - E(\xi)\right)^2 \mu(x) d\mu(x)$$

$$= -\int_0^1 \gamma \left(a_2(\gamma) - E(\xi)\right)^2 d\gamma$$

The proof is complete

RemarkIII.2: based on the above theorem we have $V = \frac{(V^{+}+V^{-})}{2} + \frac{(E^{+}-E^{-})^{2}}{4}$, Which implies that Defination 3.2 is closely related to the lower and upper possibilistic variance based on the credibility measures and Choquet integral, which has no relation with the lower and upper possibilistic variances

DefinationIII.3 : Suppose that ξ is a fuzzy number with $\gamma -$ level set $[a_1(\gamma), a_2(\gamma)]$ and finite mean value E_1. $\eta$is another fuzzy number with $\gamma -$ level set $[b_1(\gamma), b_2(\gamma)]$ and finite mean value E_2.The covariance between ξ and η is defined as

$$cov(\xi,\eta) = \int_0^1 \gamma[(a_1(\gamma) - E_1)(b_1(\gamma) - E_2) + (a_2(\gamma) - E_1)(b_2(\gamma) - E_2)]$$

TheoremIII.5: Let ξ and η be two fuzzy numbers with finite mean values. Then for any nonnegative real numbers λ_1 and λ_2 we have

$$V(\lambda_1\xi + \lambda_2\eta) = \lambda_1^2 V(\xi) + \lambda_2^2 V(\eta) + 2\lambda_1\lambda_2 Cov(\xi,\eta)$$

Proof: Assume that $[\xi]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$and $[\eta]^{\gamma} = [b_1(\gamma), b_2(\gamma)]$. According to$[\lambda_1\xi + \lambda_2\eta]^{\gamma} = [(\lambda_1 a_1(\gamma)) + \lambda_2 b_1(\gamma), \lambda_1 a_2(\gamma) + \lambda_2 b_2(\gamma)]$, It follows from theorem 3.4

$$V(\lambda_1\xi + \lambda_2\eta) = \int_0^1 \gamma\Big[\big((\lambda_1 a_1(\gamma) + \lambda_2 b_1(\gamma)) - (\lambda_1 E_1 + \lambda_2 E_2)\big)^2 + \big((\lambda_1 a_2(\gamma) + \lambda_2 b_2(\gamma)) - (\lambda_1 E_1 + \lambda_2 E_2)\big)^2\Big] d\gamma$$

$$= \int_0^1 \gamma\left[\left((\lambda_1 a_1(\gamma) - \lambda_1 E_1) + (\lambda_2 b_1(\gamma) - \lambda_2 E_2)\right)\right]^2 d\gamma + \int_0^1 \gamma\left[\left((\lambda_1 a_2(\gamma) - \lambda_1 E_1) + (\lambda_2 b_2(\gamma) - \lambda_2 E_2)\right)\right]^2 d\gamma$$

$= \lambda_1^2 \int_0^1 \gamma(a_1(\gamma) - E_1)^2\, d\gamma +$
$\lambda_2^2 \int_0^1 \gamma(b_1(\gamma) - E_2)^2\, d\gamma + 2\lambda_1\lambda_2 \int_0^1 \gamma(a_1(\gamma) - E_1)(b_1(\gamma) - E_2)\, d\gamma + \lambda_1^2 \int_0^1 \gamma(a_2(\gamma) - E_1)^2\, d\gamma +$

$$\lambda_2^2 \int_0^1 \gamma(b_2(\gamma) - E_2)^2\, d\gamma + 2\lambda_1\lambda_2 \int_0^1 \gamma(a_2(\gamma) - E_1)(b_2(\gamma) - E_2)\, d\gamma$$

$= \lambda_1^2 V(\xi) + \lambda_2^2(\eta) + 2\lambda_1\lambda_2 cov(\xi,\eta)$

The proof is complete

DefinationIII.4 : let ξ be a fuzzy number with differentiable membership function $\mu(x)$ and finite mean value $E(\xi)$ .Then ,its Skewness is defined as

$S(\xi) = \int_{-\infty}^{+\infty} \big(x - E(\xi)\big)^3 \mu(x)|\mu'(x)|dx$………………(7)

Example III.4: Assume that η is a trapezoidal fuzzy number (s_1, s_2, s_3, s_4) with finite mean value $E(\eta)$ Then we have

$$S(\eta) = \int_{s_1}^{s_2} \big(x - E(\eta)\big)^3 \cdot \frac{x - s_1}{s_2 - s_1} \cdot \frac{1}{s_2 - s_1} dx + \int_{s_3}^{s_4} \big(x - E(\eta)\big)^3 \cdot \frac{s_4 - x}{s_4 - s_3} \cdot \frac{1}{s_4 - s_3} dx$$

$$= \frac{\big(s_2 - E(\eta)\big)^4}{4(s_2 - s_1)} - \frac{\left[\big(s_2 - E(\eta)\big)^5 - \big(s_1 - E(\eta)\big)^5\right]}{20(s_2 - s_1)^2} - \frac{\big(s_3 - E(\eta)\big)^4}{4(s_4 - s_3)} + \frac{\left[\big(s_4 - E(\eta)\big)^5 - \big(s_3 - E(\eta)\big)^5\right]}{20(s_4 - s_3)^2}$$

Ifη is symmetric with $s_4 - s_3 = s_2 - s_1$ we have $E(\xi) = \frac{s_2+s_3}{2}$ and $S(\eta) = 0$ If η is a triangular fuzzy number (r_1, r_2, r_3) we have $S(\eta) = \frac{[19(r_3-r_2)^3 - 19(r_2-r_1)^3 + 15(r_2-r_1)(r_3-r_2)^2 - 15(r_2-r_1)^2(r_3-r_2)]}{1080}$

Theorem III.6: For any fuzzy number ξ with $\gamma -$ level set $[\xi]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$

$$S(\xi) = \int_0^1 \gamma \left[\left(a_1(\gamma) - E(\xi)\right)^3 + \left(a_2(\gamma) - E(\xi)\right)^3 \right] d\gamma$$

Proof: Suppose that x_0 is the point withe $\mu(x_0) = 1$ then .according to Definition 3.4 we have

$$S(\xi) = \int_{-\infty}^{x_0} \left(x - E(\xi)\right)^3 \mu(x)\mu'(x)dx + \int_{x_0}^{+\infty} \left(x - E(\xi)\right)^3 \mu(x)\mu'(x)dx$$

Taking $x = a_1(\gamma)$ it follows from the integration by substitution that

$$\int_{-\infty}^{x_0} \left(x - E(\xi)\right)^3 \mu(x)\mu'(x)dx = \int_{-\infty}^{x_0} \left(x - E(\xi)\right)^3 \mu(x)d\,\mu(x)$$

$$= \int_0^1 \gamma\left(a_1(\gamma) - E(\xi)\right)^3 d\gamma$$

Similarly taking $x = a_2(\gamma)$ it follows from the integration by substitution that

$$\int_{x_0}^{+\infty} \left(x - E(\xi)\right)^3 \mu(x)\mu'(x)dx = \int_{x_0}^{+\infty} \left(x - E(\xi)\right)^3 \mu(x)d\mu(x)$$

$$= -\int_0^1 \gamma\left(a_2(\gamma) - E(\xi)\right)^2 d\gamma$$

The proof is complete

Theorem III.7: Suppose that ξ is a fuzzy number with finite mean value .For any real numbers $\lambda \geq 0$ and b we have

$S(\lambda\xi + b) = \lambda^3 S(\xi)$………..(8)

Proof : Assume that ξ has $\gamma -$ level set $[a_1(\gamma), a_2(\gamma)]$ and mean value $E(\xi)$.Then fuzzy number $\lambda\xi + b$ has mean value $\lambda E(\xi) + b$ and $\gamma -$ level set $[\lambda a_1(\gamma) + b, \lambda a_2(\gamma) + b]$. According to Theorem 3.6,we have

$$S(\lambda\xi + b) = \int_0^1 \gamma \left[\left(\lambda a_1(\gamma) + b - (\lambda E(\xi) + b)\right)^3 + \left(\lambda a_2(\gamma) + b - (\lambda E(\xi) + b)\right)^3 \right] d\gamma$$

$$= \lambda^3 \int_0^1 \gamma \left[\left(a_1(\gamma) - E(\xi)\right)^3 + \left(a_2(\gamma) - E(\xi)\right)^3 \right] d\gamma$$

$$= \lambda^3 S(\xi).$$

The proof is complete.

TABLE I
FUZZY RETURNS FOR RISKY ASSETS IN EXAMPLE V.I

Asset	Fuzzy return	Mean	variance	Skewness
1	$(-0.26,0.10,0.36)$	8.67×10^{-2}	1.54×10^{-2}	-4.80×10^{-4}
2	$(-0.10,0.20,0.45)$	1.92×10^{-1}	1.28×10^{-2}	-2.52×10^{-4}
3	$(-012,0.14,0.30)$	1.23×10^{-1}	8.00×10^{-3}	-2.95×10^{-4}
4	$(-0.05,0.05,0.10)$	4.17×10^{-2}	1.10×10^{-3}	-1.89×10^{-5}
5	$(-0.30,0.10,0.20)$	5.00×10^{-2}	1.67×10^{-2}	-1.30×10^{-3}

TABLE II
OPTIMAL PORTFOLIO IN EXAMPLE V.I

Asset	1	2	3	4	5
Allocation(%)	13.88	55.42	18.11	12.59	0.00

IV-MEAN-VARIANCE-SKEWNESS PORTFOLIO SELECTION MODEL

Suppose that there are n risky assets .Let ξ be the return rate of asset i,and let x_i be the proportion of wealth invested in this asset$(i = 1,2,3 \dots . n)$.
If $\xi_1, \xi_2, \xi_3, \dots \dots \xi_n$ are regarded as fuzzy numbers,the total return of portfolio $(x_1, x_2, x_3, \dots \dots x_n)$ is also a fuzzy number $\xi = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n$.We use mean value $E(\xi)$ to denote the expected return of the total portfolio, and use the variance$V(\xi)$ to denote the risk of the total portfolio.For a rational investor ,when minimal expected return level and maximal risk level are given, he/she prefers a portfolio with higher skewness. Therefore we propose the following mean-variance-skewness model.

Max $S[\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n]$
s.t $E[\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n]$
$V[\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n]$……………………(9)

$$x_1 + x_2 + x_3 \dots \dots \dots . +x_n = 1$$
$$0 \le x_i \le 1, \qquad i = 1,2,3, \dots . . n$$

The first constraints ensures that the expected return is no less than α ,and the second one ensures that the total risk does not exceed β.The last two constraints mean that there are n risky assets and no short selling is allowed
The first variation of mean –variance –skewness model (9) is as follows:
Min $V[\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n]$
s.t $E[\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n]$
$S[\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n]$……………………(10)

$$x_1 + x_2 + x_3 \dots \dots \dots . +x_n = 1$$
$$0 \le x_i \le 1, \qquad i = 1,2,3, \dots . . n$$

It means that when the expected return is lower and α and the skewness is no less than γ,the investor tries to minimize the total risk. The second variation of a mean- variance-skewness.

TABLE III

FUZZY RETURNS FOR RISKY ASSETS IN EXAMPLE V.2

Asset	Fuzzy return
1	$(-0.15,0.15,0.30)$
2	$(-0.10,0.20,0.30)$
3	$(-0.06,0.10,0.18)$
4	$(-0.12,0.20,0.24)$
5	$(-0.10,0.08,0.18)$
6	$(-0.45,0.20,0.60)$
7	$(-0.20,0.30,0.50)$
8	$(-0.07,0.08,0.17)$
9	$(-0.30,0.40,0.50)$
10	$(-0.10,0.20,0.50)$

Model (9) is
Max $E[\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n]$
s.t $V[\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n]$
$S[\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \dots \dots \dots . +\xi_n x_n]$……………………(11)

$$x_1 + x_2 + x_3 \dots \dots \dots . +x_n = 1$$
$$0 \le x_i \le 1, \qquad i = 1,2,3, \dots . . n$$

The objective is no maximize return when the risk is lower than β and the skewness is no less than γ
Now ,we analyze the crisp expressions for mean variance and skewness of total return ξ.Denote $[\xi]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $[\xi_i]^\gamma = [a_{i1}(\gamma), a_{i2}(\gamma)]$ and $E(\xi_i) = e_i \; for \; i = 1,2,3 \dots . n$ .It is readily to prove that

$$a_1(\gamma) = a_{11}(\gamma)x_1 + a_{21}(\gamma)x_2 + a_{31}(\gamma)x_3 \dots \dots \dots + a_{n1}(\gamma)x_n$$
$$a_2(\gamma) = a_{12}(\gamma)x_1 + a_{22}(\gamma)x_2 + a_{32}(\gamma)x_3 \dots \dots \dots + a_{n2}(\gamma)x_n$$

First according to the linearity theorem of mean value ,we have
$E(\xi) = e_1 x_1 + e_2 x_2 + e_3 x_3 \dots \dots \dots . +e_n x_n$. Second ,according to Theorem III.4 the variance for fuzzy number ξ is

$$V(\xi) = \int_0^1 \gamma \left[\left(a_1(\gamma) - E(\xi)\right)^2 + \left(a_2(\gamma) - E(\xi)\right)^2 \right] d\gamma$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij}\, x_i x_j$$

Where $v_{ij} = \int_0^1 \gamma\left[(a_{i1}(\gamma) - e_i)(a_{j1}(\gamma) - e_j) + (a_{i2}(\gamma) - e_i)(a_{j2}(\gamma) - e_j)\right] d\gamma$ for $i, j = 1,2,3 \ldots\ldots n$ Finally, according to Theorem III.6.The skewness for a fuzzy number ξ is

$$S(\xi) = \int_0^1 \gamma \left[\left(a_1(\gamma) - E(\xi)\right)^3 + \left(a_2(\gamma) - E(\xi)\right)^3 \right] d\gamma$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} s_{ijk}\, x_i x_j x_k$$

whereWhere$s_{ijk} = \int_0^1 \gamma\left[(a_{i1}(\gamma) - e_i)(a_{j1}(\gamma) - e_j)(a_{k1}(\gamma) - e_k) + (a_{i2}(\gamma) - e_i)(a_{j2}(\gamma) - e_j)(a_{k2}(\gamma) - \right.$ $ekd\gamma$ for $i,j,k=1,2,3.....n$.

TABLE IV

OPTIMAL PORTFOLIO IN EXAMPLE 5.2

	Asset	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Credibilistic model this work	**Allocation%**	**0**	**0.00**	**41.67**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**58.33**
	Allocation%	**9.50**	**10.24**	**8.89**	**9.99**	**8.60**	**10.09**	**12.01**	**8.65**	**11.12**	**10.91**

Based on the above analysis, the mean-variance- skewnessmodel(9) has the following crisp equivalent:

Max $\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} s_{ijk}\, x_i x_j x_k$

s.t $\sum_{i=1}^{n} e_i x_i \geq \alpha$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij}\, x_i x_j \leq \beta$$

$$x_1 + x_2 + x_3 \ldots\ldots\ldots. + x_n = 1$$

$$0 \leq x_i \leq 1, \qquad i = 1,2,3, \ldots\ldots n$$

The crisp equivalent for model (10 and model (11) can be obtained similarly.Since this model has polynomial objective and constraint functions.It can be well solved by using analytical methods.In 2010,Li et al [26] proposed a fuzzy mean variance skewness model with in the framework of credibility theory, in which a genetic algorithm integrated with fuzzy simulation was used to solve the suboptimal solution.Compared with the credibilistic approach this study significantly reduces the computation time and improves the performance on optimality

Example IV.I: Suppose that $\xi = (r_{i1}, r_{i2}, r_{i3})(i = 1,2,3 \ldots. n)$
are triangular fuzzy numbers. Then, model (9) has the following equivalent.

max$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n}\left[19(r_{i3} - r_{i2})(r_{j3} - r_{j2})(r_{k3} - r_{k2}) - 19(r_{i2} - r_{i1})(r_{j2} - r_{j1})(r_{k2} - r_{k1}) + \right.$ $15ri2-ri1rj3-rj2rk3-rk2-$ $15ri3-ri2rj2-rj1rk2-rk1xixjxk$

s.t $\sum_{i=1}^{n}(r_{i1} + 4r_{i2} + r_{i3})\, x_i \geq 6\alpha$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[2(r_{i2}-r_{i1})(r_{j2}-r_{j1})+2(r_{i3}-r_{i2})(r_{j3}-r_{j2}) - (r_{i2}-r_{i1})(r_{j3}-r_{j2})\right]x_i x_j \le 18\beta$$

$$x_1+x_2+x_3\ldots\ldots\ldots.+x_n=1$$

$0 \le x_i \le 1,\ i=1,2,3,\ldots..n$……………………… (13)

Example IV.2: suppose that $\eta_i=(s_{i1},s_{i2},s_{i3},s_{i4})(i=1,2,3,\ldots n)$ are symmetric trapezoidal fuzzy numbers. Then model (10) has the following crisp equivalent

$\min\sum_{i=1}^{n}\sum_{j=1}^{n}\left[2(s_{i2}-s_{i1})(s_{j2}-s_{j1})+3(s_{i3}-s_{i2})(s_{j3}-s_{j2})+\right.$ $4(s_{i3}-$
ri2sj2−sj1xixj

$$\sum_{i=1}^{n}(s_{i2}+s_{i3})\,x_i \ge 2\alpha$$

$$x_1+x_2+x_3\ldots\ldots\ldots.+x_n=1$$

$0 \le x_i \le 1,\ i=1,2,3,\ldots..n$……………..(14)

V. NUMERICAL EXAMPLES

In this section we present some numerical examples to illustrate the efficiency of the proposed models.

Example V.I: In this example ,we consider a portfolio selection problem with five risky assets. Suppose that the returns of these risky assets are all triangular fuzzy numbers (see table 1) According to model (13) ,if the investor wants to get a higher skewness under the given risk level $\beta=0.01$ and return level $\alpha=0.12$ we have

$\max\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[19(r_{i3}-r_{i2})(r_{j3}-r_{j2})(r_{k3}-r_{k2})-19(r_{i2}-r_{i1})(r_{j2}-\right.$ $\left.r_{j1})(r_{k2}-r_{k1})+\right.$
15ri2−ri1rj3−rj2rk3−rk2− *15ri3−ri2rj2−rj1rk2−rk1xixjxk*

s.t $\sum_{i=1}^{n}(r_{i1}+4r_{i2}+r_{i3})\,x_i \ge 0.72$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[2(r_{i2}-r_{i1})(r_{j2}-r_{j1})+2(r_{i3}-r_{i2})(r_{j3}-r_{j2}) - (r_{i2}-r_{i1})(r_{j3}-r_{j2})\right]x_i x_j \le 0.18$$

$$x_1+x_2+x_3\ldots\ldots\ldots.+x_n=1$$

$$0 \le x_i \le 1, \qquad i=1,2,3,\ldots..n$$

By using the nonlinear optimization software lingo 11,we obtain the optimal solution. Table II lists the optimal allocations to assets.It is shown that the optimal portfolio invests in assets 1,2,3 and4.Assets 5 is excluded since it has lower mean and higher variance than assets 1,2 and 3.For asset2, since it has the highest mean and better variance and skewness, the optimal portfolio invests in it with the maximum allocation 55.42%

Example V.2. In this example ,we compare this study with the credibilistic mean-variance-skewness model Li et al.[26],Suppose that there are ten risky assets with fuzzy returns (see Table III) ,the minimum return level is $\alpha=0.15$,

and the maximum risk level is $\beta=0.02$ .The optimal portfolios are listed by TableIV .It is shown that a credibilistic model provides a concentrated investment solution, While our study leads to a distributive investment strategy, which satisfies the risk diversification theory.

VI.CONCLUSION

In this paper ,we redefined the mean and variance for fuzzy numbers based on membership functions .Most importantly. We proposed the concept of skewness and proved some desirable properties. As applications,we considered the multiassets portfolio selection problem and formulated a mean –variance skewness model in fuzzy circumstance.These results can be used to help investors to make the optimal investments decision under complex market situations

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