International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 3 Issue 5 || July. 2015 || PP-12-21

Some Fixed Point Theorems of Expansion Mapping In G-Metric Spaces

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ABSTRACT: *Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variation and linear inequalities, optimization and approximation theory. Therefore, different Authors proved many fixed points results for self mapping defined on complete G-Metric space. The objectives of this study are to prove fixed point results for mapping satisfying expansion conditions.*

KEY WORDS: *Metric spaces, Generalized metric space, D -metric spaces, 2 -metric spaces.*

I. INTRODUCTION

Now a days, the study of metric spaces is considers fascinating and highly useful because of its increasing role in mathematics and applied sciences.

In the past two decades metric spaces have gained much attention due to the advancement of metric fixed point theory. Fixed point theorem plays a major role in many applications, such as variation and linear inequalities, optimization and applications in the field of approximation theory. Thus the study of the fixed point theory has been researched extensively.

During the sixties, the notion of 2 -metric spaces was introduced by [2] as a generalization of usual notion of metric spaces (X, D). But other authors proved that there is no relation between these two functions. For instance [3] showed that 2 -metric spaces needs not be a continuous function on its variable, where as the ordinary metric is. These considerations led [1] in 1992 to introduce a new class of generalized metric spaces called D-metric space as a generalization of ordinary metric spaces(X, D). However Z. Mustafa and B. Sims have demonstrated [4] that most of the claims concerning the fundamental topological structure of D-metric space are incorrect. Alternatively, they have introduced [6] more appropriate notion of generalized metric space which called G-metric space.

II. PRELIMINARY NOTES:

Definition 2.1[6]-Let X be a nonempty set and let $G: X \times X \times X \to R^+$ be a function satisfying
(G1) $G(x, y, z) = 0 \; if \; x = y = z$
(G2) $0 < G(x, x, y\,); \forall \; x, y \in X \; with \; x \neq y$
(G3) $G(x, x, y\,) < G(x, y, z\,)$, for all $x, y, z \in X$with $z \neq y$
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (Symmetry in all three variables)
(G5) $G(x, y, z) < G(x, a, a) + G(a, y, z)$for all $x, y, z \in X$(rectangle inequality)
Then the function is called a generalized metric space or, more specifically a G- Metric on X and the pair $(X, G)$is G -Metric space.
Clearly these properties are satisfied when $G(x, y, z)$is the perimeter of the triangle with vertices at x, y, z inR^2, Moreover taking a in the interior of the triangle shows that $(G5)$is the best possible.

If (X, D) is an ordinary metric space then (X, D)can define G-metrics on X by:

$$(E_s)\;\; D_s(d)(x, y, z) = d(x, y) + \; d(y, z) = d(x, z)$$

$$(E_m)\;\; D_m(d)(x, y, z) = \max\{d(x, y) + \; d(y, z) + d(x, z)\}$$

Proposition 2.2[6]: Let is (X, G)a G metric space. Then for any x, y, z and $a \in X$ it follows that:

- $G(x, y, z) = 0 \Rightarrow x = y = z$
- $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$

- $G(x,y,y) \leq 2G(y,x,x)$
- $G(x,y,z) \leq G(x,a,z)\, G(a,y,z)$
- $G(x,y,z) \leq \frac{2}{3}\{G(x,y,a)G(x,a,z)G(a,y,z)\}$

Proposition 2.3: Let $(X,G)$is a G-metric space. Then for any $x,y,z$and $a\, \epsilon X$ it follows

$$G(x,y,z) \leq 2[G(x,x,y) + G(y,y,a) + G(z,z,a)]$$

Proof- $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$

$$G(x,y,z) \leq G(x,a,a)\, G(y,z,a)$$
$$G(x,y,z) \leq G(x,x,a) + G(y,a,a) + G(a,z,a)$$
$$G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a)$$
$$G(x,y,z) \leq 2G(x,x,a) + 2G(y,y,a) + 2G(z,z,a)$$
$$G(x,y,z) \leq 2[G(x,x,a) + G(y,y,a) + G(z,z,a)$$

Proposition 2.4[6]: Every G-Metric space (X,G) will define a metric space $(X,\, d_G), d_G(x,y) = G(x,y,y) + G(y,x,x)$, for all $x,y,z\,\epsilon\, X$

Definition 2.5[6]: Let (X,G) be a G-Metric space. Then for $x_0 \epsilon\, X, r > 0$ the G-ball with centre x_0 and a radius r is $B_G(x_0\,, r) = \{y\epsilon\, X\colon G(x_0,y,y) < r\}$

Proposition 2.6[6]: Let (X,G) be a G-metric space. Then for any $x_0 \epsilon\, X$ and,we $r > 0$ have $G(x_0,y,z) < r \Rightarrow x,y\,\epsilon\, B_G(x_0,r), y \in B_G(x_0,r)$Then there exist a $\delta > 0$ such that $B_G(y,\delta) \subseteq B_G(x_0,r)$

Proof-(1) Follows directly from (G3), while (2) follows from (G5) with . $\delta = r - G(x_0 y,y)$
It follows from (2) the above proposition that the family of all G-balls $B = \{B_G(x,r)\colon x \in X, r > 0\}$. Is the base of a topology $\tau(G)$ on X , the G-Metric topology.

Definition 2.7 [6]: Let (X,G)be a metric space. The sequence $(x_n) \subseteq X$ is G-convergent to x if it converges to the G-Metric topology, $\tau(G)$.

Definition 2.8 [6]: Let be a G-metric space. Then for a sequence $(x_n) \subseteq X$ and a point $x \in X$ the following are equivalent:

- (x_n)
- $G(x_{n,}x_{n,}\,, x) \to 0, as\ n \to \infty$
- $G(x_{n,}x\,, x) \to 0, as\ n \to \infty$
- $G(x_{n,}x_{m,}\,, x) \to 0, as\ n, m \to \infty$

Proposition 2.9[6]: Let $(X,G)(X',G')$,be two G-metric spaces. Then a function $f\colon X \to X'$is G-continuous at a point $x \in X$ iff it is s sequentially continuous at x ; that is, whenever (x_n)
is G-convergent to x we have $(f(x_n))$ is G-convergent to $f(x)$.

Proposition 2.10[6]: Let (X,G) be a G-metric space. Then the function $G(x,y,z)$is jointly continuous in all three of its variables.

Definition 2.11 [6]: Let (X,G) be a G-metric space. Then the sequence $(x_n) \subseteq X$ is said to be G-Cauchy sequence if for every $\in > 0$, there exist $n_0 \in N$such that $G(x_{n,}x_{n,}\,, x_m) < \in$ for all $n, m \geq n_0$.

Definition 2.12 [6]: A G-metric space $(X,G)$is said to be a G-complete if every G-Cauchy sequence in (X,G) is G-convergent in (X,G), and called uncomplete G-metric space , if it is not complete.
In [5] the fixed point theory of contractive mapping satisfied variety of contractive type condition on complete G-metric space (X,G) has been discussed. In this study we introduce the existence of fixed points of expansion mappings in complete G-metric space.

Lemma2.13- Let (X, G) is a G-Metric space. If sequence (x_n) in X converges to x and (y_n)converges to y, then $\lim_{n\to\infty} G(x_n, y_n, y_n) = G(x, y, y)$

Proof- Since $\lim x_n = x$, $\lim y_n = y$. For $\in > 0$ there exist for every

- $n \geq n_1 \Rightarrow G(x, x, x_n) < \frac{\in}{3}$
- $n \geq n_2 \Rightarrow G(y, y, y_n) < \frac{\in}{3}$

Let $n_0 = \max\{n_1, n_2\}$ then for every by $n \geq n_0$ rectangular inequality we have
$G(x_n, y_n, y_n) \leq G(x_n, x, x) + G(x, y_n, y_n)$

Now by proposition (2.2)

- $G(x_n, y_n, y_n) \leq G(x_n, x, x) + G(x, y, y) + G(y_n, y, y) + G(y_n, y, y)$
- $G(x_n, y_n, y_n) < \frac{\in}{3} + G(x, y, y) + \frac{\in}{3} + \frac{\in}{3}$
- $G(x_n, y_n, y_n) < G(x, y, y) + \in$ …(1)
  Again $G(x, y, y) \leq G(x, x_n, x_n) + G(x_n, y, y)$
  By preposition (2.2)
- $G(x, y, y) \leq 2G(x, x, x_n) + G(x_n, y_n, y_n) + G(y_n, y, y)$
- $G(x, y, y) < 2\left(\frac{\in}{3}\right) + G(x_n, y_n, y_n) + \frac{\in}{3}$
- $G(x, y, y) < G(x_n, y_n, y_n) + \in$ …(2)
  $\therefore$ *by* (1) & (2) $|G(x_n, y_n, y_n) - G(x, y, y)| < \in$
  or $\lim G(x_n, y_n, y_n) = G(x, y, y)$ *for all* $n \geq n_0$

III MAIN RESULTS

Theorem3.1- Let $F$and T be self maps of complete G-metric space with

(a) $F(X) \subset T(X)$
(b) $G(Fx, Fy, Fy) \geq \alpha G(Ty, Ty, Tx) + \beta \min\{G(Tx, Fy, Fy), G(Tx, Fy, Ty)\}$
For all $x, y \in X$ and also $\alpha > 1, \beta > 2, \alpha + \beta > 1$
(c) Either F or T is continuous.
(d) Pair $(F, T)$is semi compatible.
Then F, T have a unique common fixed point in X .

Proof-Let $x_0 \epsilon X$ be an arbitrary point. Since $F(X) \subset T(X)$ then there exist a point $x_1$such that $Fx_1 = Tx_0 = y_0$.Inductively we can define a sequence $Fx_{n+1} = Tx_n = y_n$.
Now using (b) with $x = x_n, y = x_{n+1}$

$$G(Fx_n, Fx_{n+1}, Fx_{n+1}) \geq \alpha[(Tx_{n+1}, Tx_{n+1}, Tx_n) + \beta \min\{G(Tx_n, Fx_{n+1}, Fx_{n+1})\, G(Tx_n, Fx_{n+1}, Tx_{n+1})$$

$$G(y_{n-1}, y_n, y_n) \geq \alpha G(y_{n+1}, y_{n+1}, y_n) + \beta \min\{G(y_n, y_n, y_n)\ G(y_n, y_n, y_{n+1})$$

$$G(y_{n-1}, y_n, y_n) \geq \alpha G(y_{n+1}, y_{n+1}, y_n)$$

$$G(y_{n+1}, y_{n+1}, y_n) \leq \frac{1}{\alpha} G(y_{n-1}, y_n, y_n)$$

Let $\frac{1}{\alpha} = k$ then $G(y_{n+1}, y_{n+1}, y_n) \leq k\, G(y_{n-1}, y_n, y_n)$ ... (1)

Similarly it can be found that

$$G(y_{n-1}, y_n, y_n) \leq k\, G(y_{n-2}, y_{n-1}, y_{n-1})$$

By (1) we can write $G(y_{n+1}, y_{n+1}, y_n) \leq k^2\, G(y_{n-2}, y_{n-1}, y_{n-1})$
inductively we have

$$G(y_{n+1}, y_{n+1}, y_n) \leq k^n\, G(y_0, y_1, y_n)$$

Now we will prove that sequence $\{y_n\}$ is Cauchy sequence
By using (G5) of definition (2.1), for $n > m$

$$G(y_m, y_n, y_n) \leq G(y_m, y_{m+1}, y_{m+1}) + G(y_{m+1}, y_n, y_n)$$

$$\leq G(y_m, y_{m+1}, y_{m+1}) + G(y_{m+1}, y_{m+2}, y_{m+2}) + G(y_{m+2}, y_n, y_n)$$

:

:

$$\leq G(y_m, y_{m+1}, y_{m+1}) + G(y_{m+1}, y_{m+2}, y_{m+2}) + \cdots + G(y_{n-1}, y_n, y_n) + G(y_n, y_n, y_n)$$

$$\leq k^m G(y_0, y_1, y_1) + k^{m+1} G(y_0, y_1, y_1) + \cdots + k^{n-1} G(y_0, y_1, y_1)$$

since $m < n$ let $m = n + p$ then

$$G(y_m, y_n, y_n) \leq k^m G(y_0, y_1, y_1) + k^{m+1} G(y_0, y_1, y_1) + \cdots + k^{m+p-1} G(y_0, y_1, y_1)$$

$$\leq k^m (1 + k + k^2 + \cdots + k^{p-1}) G(y_0, y_1, y_1)$$

$$\leq k^m (\tfrac{1-k^p}{1-k}) G(y_0, y_1, y_1)$$

$$\leq k^m (\tfrac{1}{1-k}) G(y_0, y_1, y_1)$$

Since $\alpha > 1$ or $k < 1$ then limiting $n \to \infty$, $G(y_m, y_n, y_n) \to 0$, therefore$\{y_n\}$ is Cauchy sequence. Since $(X, G)$is complete then

$\lim(y_n) = Tz$ or its all subsequences also converges to z.Then it can be written $\lim(Fx_{n+1}) = z$ & $\lim(Tx_n) = z$.

Case 1$F$is continuous map. Since$\lim Tx_n = \lim Fx_n = z$, then

$$\lim FFx_n = Fz \ \& \ \lim FTx_n = Fz$$

Since (F, T) is semi compatible. Since $\lim Tx_n = z$ then $\lim \mathrm{T} Fx_n = Fz$

Now using (b) with $x = x_n, y = Fx_n$

$$G(Fx_n, FFx_{n+1}, FFx_{n+1}) \geq \alpha\{(TFx_n, TFx_{n+1}, Tx_n) + \beta \min\{G(Tx_n, FFx_n, FFx_n), G(Tx_n, FFx_n, TFx_n)\}$$

Limiting $n \to \infty$ $G(z, Fz, Fz) \geq \alpha G(Fz, Fz, z) + \beta$ min $\{G(z, Fz, Fz), G(z, Fz, Fz)\}$

$$G(z, Fz, Fz) \geq (\alpha + \beta) G(Fz, Fz, z)$$

$$G(Fz, Fz, z)(\alpha + \beta - 1) \leq 0$$

$since (\alpha + \beta - 1) > 0 \ or \ (\alpha + \beta) > 1$

$$G(Fz, Fz, z) \leq 0 \ or \ Fz = z$$

Now by using (b) with $x = x_n$ & $y = z$

$$G(Fx_n, Fz, Fz) \geq \alpha G(Tz, Tz, z) + \beta \min\{G(Tx_n, Fz, Fz), G(Tx_n, Fz, Tz)\}$$

$Limiting \ n \to \infty$

$$G(z, z, z) \geq \alpha G(Tz, Tz, z) + \beta \min\{G(z, z, z), G(z, z, Tz)\}$$

$$0 \geq \alpha G(Tz, Tz, z)$$

$since \ \alpha > 0 \ \therefore \ G(Tz, Tz, z) \leq 0 \Rightarrow Tz = z$

$\therefore Fz = Tz = z$

Therefore z is common fixed point of T & F

T

Case 2-T is continuous map. Since $\lim Tx_n = \lim Fx_n = z$, then $\lim TTx_n = Tz$ & $\lim TFx_n = Tz$

Since (F, T) is semi compatible. Since $\lim Fx_n = z$ then $\lim FTx_n = z$.

By using (b) with $x = x_n, y = Tx_n$

$$G(Fx_n, FTx_n, FTx_n) \geq \alpha G(TTx_n, TTx_n, Tx_n) + \beta \min\{G(Tx_n, FTx_n, FTx_n), G(Tx_n, FTx_n, TTx_n)\}$$

$Limiting \ n \to \infty$

$$G(z, Tz, Tz) \geq \alpha G(Tz, Tz, z) + \beta \min\{G(z, Tz, Tz), G(z, Tz, Tz)\}$$

$$G(z, Tz, Tz) \geq (\alpha + \beta) G(z, Tz, Tz)$$

$$G(z, Tz, Tz)(\alpha + \beta - 1) \leq 0$$

$since (\alpha + \beta - 1) > 0$

$\therefore \ G(z, Tz, Tz) \leq 0 \ or \ Tz = z$

Again using (b) with $x = x_n$, $y = z$

$$G(Fx_n, Fz, Fz) \geq \alpha[G(Tz, Tz, Tx_n) + \beta \min\{G(Tx_n, Fz, Fz), G(Tx_n, Fz, Fz)\}$$

$Limiting \ n \to \infty$

$$G(z, Fz, Fz) \geq \alpha G(z, z, z) + \beta \min\{G(z, Fz, Fz), G(z, Fz, z)\}$$
$$G(z, Fz, Fz) \geq \beta \min\{ G(z, Fz, Fz), G(Fz, z, z)$$

$Since\ By\ proposition$ (2.2)

$$G(z, Fz, Fz) \leq 2G(z, z, Fz) or$$
$$G(z, z, Fz) \geq \frac{1}{2} G(z, Fz, Fz)\ then$$
$$G(z, Fz, Fz) \geq \beta \min\{ G(z, Fz, Fz), \frac{1}{2} G(z, Fz, Fz)\}$$
$$G(z, Fz, Fz) \geq \frac{\beta}{2} G(z, Fz, Fz)$$
$$2G(z, Fz, Fz) \geq \beta G(z, Fz, Fz)$$
$$G(z, Fz, Fz)(\beta - 2) \leq 0$$

Since $\beta > 2 \therefore G(z, Fz, Fz) = 0 \Rightarrow Fz = z$

$\therefore Fz = Tz = z$

Uniqueness –Let u be another fixed point of F & T then $Fu = Tu = u$

By using (b) with $x = z$, $y = u$

$$G(Fz, Fu, Fu) \geq \alpha G(Tu, Tu, Tz) + \beta \min\{G(Tz, Fu, Fu), G(Tz, Fu, Tu)$$
$$G(z, u, u) \geq \alpha G(u, u, z) + \beta \min\{G(z, u, u), G(z, u, u)\}$$
$$G(z, u, u) \geq \alpha G(z, u, u) + \beta\ G(z, u, u)$$
$$G(z, u, u)(\alpha + \beta - 1) \leq 0$$

$since\ (\alpha + \beta - 1) > 0\ or\ \alpha + \beta > 1$

$$G(u, u, z) \leq 0 \Rightarrow u = z$$

Therefore z is unique common fixed point of F & T .

Theorem3.2- Let $F$and T be self maps of complete G-metric space with

(a) $F(X) \subset T(X$

(b) $G(Fx, Ty, Ty) \quad \geq \alpha G(Ty, Ty, Fy) + \beta\ G(Fx, Tx, Tx) + \gamma \min\{G(Tx, Fy, Fy), G(Tx, Fy, Ty)\}$
For all $\alpha > 1, 0 < \beta < 1$,$\alpha + \beta > 1$ and for all$\gamma > 1$,$\alpha + \gamma > 2$

(c) Either F or T is continuous.

(d) Pair $(F, T)$is semi compatible..

Then F& T have a unique common fixed point in X.

Proof-Let be an $x_0 \in X$ $a$rbitrary point. Since $F(X) \subset T(X)$,then there exist a point $x_1$such that $Fx_1 = Tx_0 = y_0$.Inductively we can define a sequence$Fx_{n+1} = Tx_n = y_n$.

Now using (b) with $x = x_n, y = x_{n+1}$

$$G(Fx_n, Fx_{n+1}, Fx_{n+1})$$
$$\geq \alpha[G(Tx_{n+1}, Tx_{n+1}, Fx_{n+1}) + \beta G(Fx_n, Tx_n, Tx_n)$$
$$+ \gamma \min\{G(Tx_n, Fx_{n+1}, Fx_{n+1})\ G(Tx_n, Fx_{n+1}, Tx_{n+1})$$
$$G(y_{n-1}, y_n, y_n) \geq \alpha G(y_{n+1}, y_{n+1}, y_n) + \beta\ G(y_{n-1}, y_n, y_n) +$$
$$\gamma \min G(y_n, y_n, y_n)\ \ G(y_n, y_n, y_{n+1})$$
$$G(y_{n-1}, y_n, y_n) \geq \alpha G(y_{n+1}, y_{n+1}, y_n) + \beta G(y_{n-1}, y_n, y_n)$$
$$G(y_{n-1}, y_n, y_n)(1 - \beta) \geq \alpha G(y_{n+1}, y_{n+1}, y_n)$$
$$G(y_{n+1}, y_{n+1}, y_n) \leq \frac{1-\beta}{\alpha} G(y_{n-1}, y_n, y_n)$$

$\because \frac{1-\beta}{\alpha} < 1\ or\ \alpha + \beta > 1$

$Let\ \frac{1-\beta}{\alpha} = k\ then$

$$G(y_{n+1}, y_{n+1}, y_n) \leq k\ G(y_{n-1}, y_n, y_n)\ \ldots (1)$$

Similarly it can be found that

$$G(y_{n-1}, y_n, y_n) \leq k\ G(y_{n-2}, y_{n-1}, y_{n-1})$$

By (1)

$$G(y_{n+1}, y_{n+1}, y_n) \leq k^2\ G(y_{n-2}, y_{n-1}, y_{n-1})$$

Inductively we can define

$$G(y_{n+1}, y_{n+1}, y_n) \leq k^n\ G(y_0, y_1, y_1) \ldots (2)$$

Now we can show by the manner of theorem (3.1) that sequence $\{y_n\}$ is Cauchy sequence. Since $(X, G)$is complete then $\lim(y_n) = z$ or its all subsequences also converges to z .Then it can be written $\lim\ Fx_n = z, \lim\ Tx_n = z$,.

Case 1- F is continuous map.
Since $\lim(Fx_n) = \lim(Tx_n) = z$, Then
$\lim(FFx_n) = Fz$ & $\lim(FTx_n) = Fz$
Since (F,T) is semi compatible, $\lim Tx_n = z$ then $\lim TFx_n = Fz$
Now using (b) with $x = Fx_n, y = x_{n+1}$.

$$G(FFx_n, Fx_{n+1}, Fx_{n+1}) \geq \alpha[G(Tx_{n+1}, Tx_{n+1}, Fx_{n+1}) + \beta G(FFx_n, TFx_n, TFx_n) + \gamma \min\{G(TFx_n, Fx_{n+1}, Fx_{n+1}) G(TFx_n, Fx_{n+1}, Tx_{n+1})$$

Limiting $n \to \infty$

$$G(Fz,z,z) \geq \alpha G(z,z,z) + \beta G(Fz,Fz,Fz) + \gamma \min\{G(Fz,z,z), G(Fz,z,z)\}$$
$$G(Fz,z,z) \geq \gamma G(z,z,Fz)$$
$$G(Fz,z,z)(\gamma - 1) \geq 0$$
$$since\ \gamma > 1\ implies\ Fz = z$$

Now using (b) with $x = x_n$ & $y = z$

$$G(Fx_n, Fz, Fz) \geq \alpha[G(Tz,Tz,Tz) + \beta G(Fx_n, Tx_n, Tx_n) + \gamma \min\{G(Tx_n, Fz, Fz) G(Tx_n, Fz, Tz)$$

$limiting\ n \to \infty$

$$G(z,z,z) \geq \alpha G(Tz,Tz,z) + \beta G(z,z,z) + \gamma \min\{G(z,z,z), G(z,z,Tz)$$
$$0 \geq \alpha G(Tz,Tz,z)$$

$since\ \alpha > 0\ therefore\ G(Tz,Tz,z) \leq 0 \Longrightarrow Tz = z, So\ Tz = Fz = z$

Case 2- T is continuous map. Since $\lim(Fx_n) = \lim(Tx_n) = z$, Then

$$\lim(TTx_n) = Tz\ \&\ \lim(TFx_n) = Tz$$

Since (F,T) is semi compatible. Since $\lim(Fx_n) = z$ then $\lim(FTx_n) = Tz$.
By using (b) with $x = Tx_n, y = x_{n+1}$

$$G(FTx_{n+1}Fx_{n+1}, Fx_{n+1}) \geq \alpha[G(Tx_{n+1}, Tx_{n+1}, Fx_{n+1}) + \beta G(FTx_n, TTx_n, TTx_n) + \gamma \min\{G(TTx_n, Fx_{n+1}, Fx_{n+1}) G(TTx_n, Fx_{n+1}, Tx_{n+1})$$

$n \to \infty$

$$G(Tz,z,z) \geq \alpha G(z,z,z) + \beta G(Tz,Tz,Tz) + \gamma \min\{G(Tz,z,z), G(Tz,z,z)$$
$$G(Tz,z,z) \geq \gamma G(Tz,z,z)$$
$$(\gamma - 1) G(Tz,z,z) \leq 0$$

$Since\ \gamma - 1 \geq 0\ or\ \gamma \geq 1\ therefore\ G(Tz,z,z) \leq 0 or\ Tz = z$

Again using (b) with $x = x_n, y = z$

$$G(Fx_n, Fz, Fz) \geq \alpha[G(Tz,Tz,Fz) + \beta G(Fx_n, Tx_n, Tx_n) + \gamma \min\{G(Tx_n, Fz, Fz) G(Tx_n, Fz, Tz)$$

$limiting\ n \to \infty$

$$G(z,Fz,Fz) \geq \alpha G(z,z,Fz) + \beta G(z,z,z) + \gamma \min\{G(z,Fz,Fz), G(z,Fz,z)$$

By proposition (2.2) we have $G(z,z,Fz \geq \frac{1}{2} G(z,Fz,Fz)$ so

$$G(z,z,Fz \geq \frac{\alpha}{2} G(z,Fz,Fz) + \frac{\gamma}{2} G(z,Fz,Fz)$$
$$G(z,z,Fz)(\frac{\alpha+\gamma}{2} - 1) \leq 0$$

Since $\frac{\alpha+\gamma}{2} - 1 > 0\ or\ \alpha + \gamma > 2$ then $G(z,Fz,Fz) \leq 0$implies $Fz = z$

Uniqueness –Let u be another fixed point of F & T then $Fu = Tu = u$
By using (b) with $x = z$, $y = u$

$$G(Fz,Fu,Fu) \geq \alpha G(Tu,Tu,Fu) + \beta G(Fz,Tz,Tz) + \gamma \min\{G(Tz,Fu,Fu), G(Tz,Fu,Fu\})$$
$$G(z,u,u) \geq \alpha G(u,u,u) + \beta G(z,z,z) + \gamma \min\{G(z,u,u), G(z,u,u)\}$$
$$G(z,u,u) \geq \gamma G(z,u,u)$$
$$(\gamma - 1)G(z,u,u) \leq 0$$

$since\ \gamma - 1 > 0, \gamma > 1$

$$G(u,u,z) \leq 0 or\ u = z$$

Therefore z is unique common fixed point of F & T .

Example- Let $x, y \in X$and $X = [3,0]$, $F(x) = x^2,\ T(x) = \frac{x}{2},\ x_n = \left\{\frac{1}{n}\right\}$ And $G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$

$\lim F(x_n) = \lim {x_n}^2 = \lim \frac{1}{n^2} = 0$

$\lim T(x_n) = \frac{x_n}{2} = \lim \frac{1}{2n} = 0\ \therefore\ \lim F(x_n) = \lim T(x_n) = 0$

$\lim \{F(Tx_n)\} = \lim (Tx_n)^2 = \lim \left(\frac{x_n}{2}\right)^2 = \lim \frac{{x_n}^2}{4} = 0$ and

$T(z) = T(0) = 0 \therefore \lim \{F(Tx_n)\} = T(z)$

So that $(F, T)$is semi compatible.

Taking $x = 3, y = 0$ and $\alpha > 1, \beta = \frac{1}{2}, \gamma = 2$, it satisfy greater than condition of (b) and if $\alpha > 1, \beta = \frac{1}{3}, \gamma = 5$ it satisfy equal to condition of (b) and 0 is the fixed point of F & T.

Theorem3.3- Let E, F, T & H be self of complete G -metric space with

(a) $F(X) \subset T(X)\ and\ \ E(X) \subset H(X)$

(b) $G(Fx, Ey, Ey) \geq \alpha G(Ty, Hy, Ey) + \beta G(Fx, Tx, Tx) + \gamma \min\{G(Tx, Ey, Ey), G(Tx, Fy, Hy)\}$

For all $\alpha, \gamma > 1, \alpha + \beta > 1$ & $\beta + 2\gamma > 0$

(c)Either F or T is continuous

(d) Pair $(F, T)$is semi compatible.

(e) $TE = ET, FE = EF, TF = FT.$

If F^2 is an identity map then E, F, T & H have a unique common fixed point in X

Proof- Let $x_0 \in X$ be an arbitrary point. Since) $F(X) \subset T(X)\ and\ \ E(X) \subset H(X)$then there exist a point x_1, x_2such that $Fx_1 = Tx_0 = y_0$ & $Ex_2 = Hx_1 = y_1$.Inductively we can define a sequence $Fx_{n+1} = Tx_n = y_n$ & $Ex_{n+2} = Hx_{n+1} = y_{n+1}$.

Now using (b) with $x = x_n, y = x_{n+1}$

$$G(Fx_n, Ex_{n+1}, Fx_{n+1}) \geq \alpha[G(Tx_{n+1}, Hx_{n+1}, Ex_{n+1}) + \beta G(Fx_n, Tx_n, Tx_n) + \gamma \min\{G(Tx_n, Ex_{n+1}, Ex_{n+1})\ G(Tx_n, Fx_{n+1}, Hx_{n+1})\}$$

$$G(y_{n-1}, y_n, y_n) \geq \alpha G(y_{n+1}, y_{n+1}, y_n) + \beta\ G(y_{n-1}, y_n, y_n) + \gamma \min G(y_n, y_n, y_n)\ \ G(y_n, y_n, y_{n+1})$$

$$G(y_{n-1}, y_n, y_n) \geq \alpha G(y_{n+1}, y_{n+1}, y_n) + \beta G(y_{n-1}, y_n, y_n)$$
$$\geq \alpha G(y_{n+1}, y_{n+1}, y_n) + \beta G(y_{n-1}, y_n, y_n)$$

$$G(y_{n-1}, y_n, y_n)(1-\beta) \geq\ \alpha G(y_{n+1}, y_{n+1}, y_n)$$

$$G(y_{n+1}, y_{n+1}, y_n) \leq \frac{1-\beta}{\alpha}\ G(y_{n-1}, y_n, y_n)$$

Since $\frac{1-\beta}{\alpha} < 1\ or\ \alpha + \beta > 1$

Let $\frac{1-\beta}{\alpha} = k$ then

$$G(y_{n+1}, y_{n+1}, y_n) \leq k\ G(y_{n-1}, y_n, y_n)\ \ \dots (1)$$

Similarly $\quad G(y_n, y_n, y_{n-1}) \leq kG(y_{n-2}, y_{n-1}, y_{n-1})$

By (1) $\quad G(y_{n+1}, y_{n+1}, y_n) \leq k^2\ G(y_{n-2}, y_{n-1}, y_{n-1})$

Inductively we can define

$$G(y_{n+1}, y_{n+1}, y_n) \leq k^n\ G(y_0, y_1, y_1) \dots (2)$$

By theorem 3.1it can be considered that sequence $\{y_n\}$is Cauchy sequence.

$\lim Fx_{n+1} = z$ & $\lim\ Tx_n = z$, $\lim Ex_{n+2} = z$ & $\lim\ Hx_{n+1} = z$

Case 1-is continuous map. Since $\lim(Fx_n) = z$ & $\lim\ (Tx_n) = z$ then

$\lim(\mathrm{F} Fx_n) = Fz$ & $\lim\ (FTx_n) = Fz$

Since $(F, T)$is semi compatible. Since $\lim(Tx_n) = z$ then $\lim\ (TFx_n) = Fz$

Now using (b) with $x = Fx_n, y = x_{n+1}$

$$G(FFx_n, Ex_{n+1}, Ex_{n+1}) \geq \alpha[G(Tx_{n+1}, Hx_{n+1}, Ex_{n+1}) + \beta G(FFx_n, TFx_n, TFx_n) + \gamma \min\{G(TFx_n, Ex_{n+1}, Ex_{n+1})\, G(TFx_n, Fx_{n+1}, Hx_{n+1})$$

Limiting $n \to \infty$

$$G(Fz,z,z) \geq \alpha G(z,z,z) + \beta G(Fz,Fz,Fz) + \gamma \min\{G(Fz,z,z), G(Fz,z,z)\}$$
$$G(Fz,z,z) \geq \gamma\, G(z,z,Fz)$$
$$(\gamma - 1)\, G(Fz,z,z) \leq 0$$

$Since\ \gamma > 1\ implies\ Fz = z$

Now using (b) with $x = z$ & $y = x_{n+1}$

$G(Fz, Ex_{n+1}, Ex_{n+1}) \geq$
$\alpha[G(Tx_{n+1}, Hx_{n+1}, Ex_{n+1}) + \beta G(Fz, Tz, Tz) + \gamma \min\{G(Tz, Ex_{n+1}, Ex_{n+1})\, G(Tz, Fx_{n+1}, Hx_{n+1})\}$
$\lim\ n \to \infty$
$G(z,z,z) \geq \alpha G(z,z,z) + \beta\, G(z,Tz,Tz) + \gamma \min\{G(Tz,z,z)\ , G(Tz,z,z)\}$
$0 \geq\ \beta\, G(z,Tz,Tz) + \gamma\, G(Tz,z,z)$

By proposition (2.2) that $G(z,Tz,Tz) \geq \frac{1}{2} G(z,z,Tz)$

$$0 \geq \frac{\beta}{2}\, G(z,Tz,Tz) + \gamma\, G(Tz,z,z)$$
$$G(Tz,z,z)\left(\frac{\beta}{2} + \gamma\right) \leq 0$$

Since $\frac{\beta}{2} + \gamma\ > 0\ or >\ \beta + 2\gamma > 0$

$$\therefore G(Tz,z,z) \leq 0 \Longrightarrow Tz = z$$

By using (b) with $x = Ez$ & $y = x_{n+1}$

$$G(FEz, Ex_{n+1}, Ex_{n+1}) \geq \alpha[G(Tx_{n+1}, Hx_{n+1}, Ex_{n+1}) + \beta G(FEz, TEz, TEz) + \gamma \min\{G(TEz, Ex_{n+1}, Ex_{n+1})\, G(TEz, Fx_{n+1}, Hx_{n+1})$$

Since $EF = EF$ & $TE = ET\ also$ $\lim\ n \to \infty$

$G(Ez,z,z) \geq \alpha G(z,z,z) + \beta G(Ez,Ez,Ez) + \gamma \min\{G(Ez,z,z), G(Ez,z,z\})$
$G(Ez,z,z) \geq \gamma\, G(Ez,z,z)$
$(\gamma - 1) G(Ez,z,z) \leq 0$
$since\ \gamma > 1\ \ or\ \gamma - 1 > 0$
$G(Ez,z,z) \leq 0 or\ Ez = z$

By using (b) with $x = x_n$, $y = z$

$G(Fx_n, Ez, Ez) \geq \alpha[G(Tz, Hz, Ez) + \beta G(Fx_n, Tx_n, Tx_n) + \gamma \min\{G(Tx_n, Ez, Ez)\, G(Tx_n, Fz, Hz)$
$Taking$ $\lim\ n \to \infty$

$$G(z,z,z) \geq \alpha G(z,Hz,z) + \beta G(z,z,z) + \gamma \min\{G(z,z,z), G(z,z,Hz)$$
$$0 \geq\ \alpha G(z,Hz,z)$$

$since\ \alpha > 0\ therefore\ G(z,Hz,z) \leq 0 \Longrightarrow Hz = z$
$so\ Tz = Fz = Ez = Hz = z$
Or z is acommon fixed point of all four maps.

Case 2-When T is continuous map. Since $\lim Fx_n = z$ & $\lim\ Tx_n = z$ then

$$\lim T\, Fx_n = Tz\ \ \&\ \ \lim\ TTx_n = Tz$$

Since(F,T) is semi compatible map. Since $\lim Fx_n = z$ then $\lim FTx_n = Tz$
By using (b) with $x = x_n$ & $y = x_{n+1}$

$$G(FTx_n, Ex_{n+1}, Ex_{n+1}) \geq \alpha[G(Tx_{n+1}, Tx_{n+1}, Ex_{n+1}) + \beta G(FTx_n, TTx_n, TTx_n) + \gamma \min\{G(TTx_n, Ex_{n+1}, Ex_{n+1})\, G(TTx_{n+1}, Fx_{n+1}, Tx_{n+1})\}$$

$\lim\ n \to \infty$

$$G(Tz,z,z) \geq \alpha G(z,z,z) + \beta\, G(Tz,Tz,Tz) + \gamma \min\{G(Tz,z,z)\ , G(Tz,z,z)\}$$
$$G(Tz,z,z) \geq\ \gamma\, G(Tz,z,z)$$
$$G(Tz,z,z)(\gamma - 1) \leq 0$$

$Since\ \gamma - 1 > 0\ or\ \gamma > 1$

$$G(Tz, z, z) \leq 0 \Longrightarrow Tz = z$$

By using (b) with $x = Fz$ & $y = x_{n+1}$

$$G(F^2z, Ex_{n+1}, Ex_{n+1}) \geq \alpha[G(Tx_{n+1}, Hx_{n+1}, Ex_{n+1}) + \beta G(F^2z, TFz, TFz) + \gamma \min\{G(TFz, Ex_{n+1}, Ex_{n+1}), G(TFz, Fx_{n+1}, Hx_{n+1})\}$$

$\lim n \to \infty$ with $FT = TF$ & $F^2 = I$

$$G(z, z, z) \geq \alpha G(z, z, z) + \beta\, G(z, Fz, Fz) + \gamma \min\{G(Fz, z, z)\ , G(Fz, z, z)\}$$

$$0 \geq \beta G(z, Fz, Fz) + \gamma\, G(Fz, z, z)$$

Then by proposition (2.2)

$$0 \geq \frac{\beta}{2}\, G(z, z, Fz) + \gamma\, G(z, z, Fz)$$

$$G(z, z, Fz)(\tfrac{\beta+2\gamma}{2} - 1) \leq 0$$

Since $\frac{\beta+2\gamma}{2} > 0\ or\ \beta + 2\gamma > 0$, therefore $G(z, z, Fz) = 0$ implies $Fz = z$.

Again by using (b) with $x = Ez$ & $y = x_{n+1}$

$$G(FEz, Ex_{n+1}, Ex_{n+1}) \geq \alpha[G(Tx_{n+1}, Hx_{n+1}, Ex_{n+1}) + \beta G(FEz, TEz, TEz) + \gamma \min\{G(TEz, Ex_{n+1}, Ex_{n+1}), G(TEz, Fx_{n+1}, Hx_{n+1})\}$$

$since\ FE = EF\ , TE = ET\ and \lim n \to \infty$

$$G(Ez, z, z) \geq \alpha G(z, z, z) + \beta\, G(Ez, Ez, Ez) + \gamma \min\{G(Ez, z, z)\ , G(Ez, z, z)\}$$

$$G(Ez, z, z) \geq \gamma\, G(Ez, z, z)$$

$$(\gamma - 1)G(Fz, z, z) \leq 0 since$$

$$\gamma - 1 > 0\ or\ \gamma > 1$$

Therefore $G(Ez, z, z) \leq 0 \Rightarrow Ez = z$

Now by (b) with $x = x_n$ & $y = z$

$$G(Fx_n, Ez, Ez) \geq \alpha[G(Tz, Hz, Ez) + \beta G(Fx_n, Tx_n, Tx_n) + \gamma \min\{G(Tx_n, Ez, Ez)\, G(Tx_n, Fz, Hz)\}$$

$Taking \lim n \to \infty$

$$G(z, z, z) \geq \alpha G(z, Hz, z) + \beta G(z, z, z) + \gamma \min \{G(z, z, z), G(z, z, Hz)$$

$$0 \geq \alpha G(z, Hz, z)$$

$since\ \alpha > 0\ \therefore\ Hz = z$

Therefore z is a common fixed point of.E, F, T & H

Uniqueness - Let u be another fixed point of F & T then $Eu = Fu = Tu = Hu = u$

By using (b) with $x = z, y = u.$

$$G(Fz, Eu, Eu) \geq \alpha G(Tu, Hu, Eu) + \beta G(Fz, Tz, Tz) + \gamma \min \{G(Tz, Eu, Eu), G(Tz, Fu, Hu)\}$$

$$G(z, u, u) \geq \alpha G(u, u, u) + \beta\, G(z, z, z) + \gamma \min\{G(z, u, u)\ , G(z, u, u)\}$$

$$G(z, u, u) \geq \gamma\, G(z, u, u)$$

$$(\gamma - 1)G(z, u, u) \leq 0 \quad since\ \gamma - 1 > 0\ or\ \gamma > 1$$

$$G(z, u, u) \leq 0 \Longrightarrow u = z$$

Therefore z is a unique common fixed point of $E\ , F, T$ & H.

Theorem3.4- Let $E, F,$ & T be self maps of complete G-metric space with

(a) $F(X) \subset T(X)$ and $E(X) \subset T(X)$

$(b) G(Fx, Ey, Ey) \geq \alpha G(Ty, Ty, Ey) + \beta\, G(Fx, Tx, Tx) + \gamma \min \{G(Tx, Ey, Ey), G(Tx, Fy, Ty)\}$

For all $\alpha, \gamma > 1, 0 < \beta < 1, \alpha + \beta > 1$ & $\beta + 2\gamma > 0$

(c)Either F or T is continuous

(d) Pair $(F, T)$is semi compatible.

(e) $TE = ET, FE = EF\ , TF = FT.$

If F^2 is an identity map then $E\ , F,$ & T have a unique common fixed point in X.

Proof- theorem can be easily proved by manner of theorem (3.3).

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