International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 3 Issue 5 || July. 2015 || PP-06-11

Ties Adjusted Nonparametric Statististical Method For The Analysis Of Ordered C Repeated Measures

[1]Matthew M. C.; [2]Adigwe, R. U.; [3]Oyeka, I. C. A. and [4]Ajibade Bright
[1-2] Department of Mathematics and Statistics, Delta State Polytechnic, Ogwashi-Uku.
[3]Department of Statistics, Faculty of Physical Sciences, Nnamdi Azikiwe University, PMB 5025, Awka, Anambra State.
[4]Department of General Studies, Petroleum Training Institute, P.M.B 20, Effurun, Warri. Delta State. Nigeria.

ABSTRACT : *This paper proposes a nonparametric statistical method for the analysis of repeated measures that adjusts for the possibility of tied observation in the data which may be observations in time, space or condition. The proposed method develops a test statistic for testing whether subjects are progressively improving (or getting worse) in their experience over time, space or condition. The method is illustrated with some data and shown to be at least as powerful as the Friedman's two way analysis of variance test by ranks even in the absence of any in-built ordering in the variable under study.*

I. INTRODUCTION

If one has repeated measures randomly drawn from a number of related populations that are dependent on some demographic factors or conditions or that are ordered in time or space which do not satisfy the necessary assumptions for the use of parametric tests, then use of nonparametric methods is indicated and preferable. These types of data include subjects' or candidates' scores in examinations or job placement interviews at various points in time; diagnostic test results repeated a certain number of times; commodity prizes at various times, location or market places. Statistical analysis of these types of data often require the use of nonparametric methods such as the Friedman's two way analysis of variance test by ranks or the Cochran's Q-test (Gibbons 1971, Oyeka 2010). However a problem with these two statistical methods is that the Friedman's test often tries to adjust for ties that occur in blocks or batches of sample observations by assigning these tied observations their mean ranks, the Cochran's Q-test requires the observations to be dichotomous, assuming only two possible values. Furthermore, if the null hypothesis to be tested is that subjects are increasingly performing better or worse with time or space; then these two statistical procedures may not be readily applicable. In this case the methods developed by Bartholomew and others (Bartholomew 1959 and 1963) may then be available for use. However, some of these methods are rather difficult to apply in practice and the resolution of any ties that may occur within blocks of observations is not often easy. Authors who have worked on these areas include: Oyeka et al (2010), Oyeka (2010), Krauth (2003), Miller (1996), Vargha et al (1996), Cohen (1983), Gart (1963) and Cochran (1950). In this paper we propose a nonparametric statistical method for the analysis of ordered repeated measures that are related in time, space or condition that takes account of all possible pair-wise combinations of treatment levels.

II. THE PROPOSED METHOD

Let $(x_{i1} x_{i2} \dots x_{ic})$ be the i^{th} batch or block in a random sample of n observations drawn from some c related populations $X_1, X_2, \dots, X_c$ for $i = 1,2, \dots, n$. where c may be indexed in time, space or condition. Population $X_1, X_2, \dots, X_c$ may be measures on as low as the ordinal scale and need not be continuous.

The problem of research interest here is to determine whether subjects are on the average progressively increasing, experiencing no change or worsening in their score or performance overtime, space or remission of condition. It is quite possible that within any specified time interval say some subjects' scores at some time in the interval may be higher than their scores earlier in the interval which are themselves higher than their scores later in the interval.

To adjust for this possibility we develop ties adjusted extended sign test for this purpose that structurally corrects for ties and also considers all possible pair-wise combinations of treatment levels, we may first take pair-wise combinations of the c observations for each of the n subjects. Thus suppose for the i^{th} subject the $c' = \binom{c}{2} = \frac{c(c-1)}{2}$ pairs of the observations on the c treatment levels or population is $(x_{i1}, x_{i2}), (x_{i1}, x_{i3}), \dots, (x_{i1}, x_{ic}), (x_{i2}, x_{i3}), (x_{i2}, x_{i4}), \dots, (x_{ic-1}, x_{ic})$ which are increasingly (decreasingly) indexed in time or space, for $i = 1,2, \dots, n$.

Let

$$U_{ij} = \begin{cases} 1\ if\ the\ first\ observation\ is\ higher\ (better, more) than\ the\ second\ observation\ in\ the\ j^{th} pair\ of\ observations\ for\ the\ i^{th}\ subject \\ 0, if\ the\ first\ and\ second\ observatios\ in\ the\ j^{th}\ pair\ of\ observations\ for\ the\ i^{th}\ subject\ are\ the\ same \\ -1, if\ the\ first\ observation\ is\ lower, (worse, smaller) than\ the\ second\ observation\ in\ the\ j^{th}\ pair\ of\ observations\ for\ the\ j^{th} subject \end{cases}$$
$---(1)$

For $i = 1,2, \dots n; j = 1,2, \dots, c' = \frac{c(c-1)}{2}$

Note that by its specification, U_{ij} for $i = 1,2, \dots n; j = 1,2, \dots, c' = \frac{c(c-1)}{2}$ spans all the $c' = \binom{c}{2}$ pairwise combinations of the c treatment levels.

Now let

$$\pi_j^{\,+} = P(U_{ij} = 1); \pi_j^{\,0} = P(U_{ij} = 0); \pi_j^{\,-} = P(U_{ij} = -1) ---------- (2)$$

Where $\pi_j^{\,+} + \pi_j^{\,0} + \pi_j^{\,-} = 1 ------------------ (3)$

Note that $\pi_j^{\,0}$ provides an adjustment for any possible tied scores or observations by subjects for the j^{th} pair of treatment combinations.

Let

$$W_j = \sum_{i=1}^{n} U_{ij} \qquad --------(4)$$

and

$$W = \sum_{j=1}^{c'} W_j = \sum_{i=1}^{c'} \sum_{i=1}^{n} U_{ij} \qquad ------------ (5)$$

Now

$$E(U_{ij}) = \pi_j^{\,+} - \pi_j^{\,-}; Var(U_{ij}) = \pi_j^{\,+} + \pi_j^{\,-} - (\pi_j^{\,+} - \pi_j^{\,-})^2 ----(6)$$

Also

$$E(W_j) = \sum_{i=1}^{n} E(U_{ij}) = n(\pi_j^{\,+} - \pi_j^{\,-}) -----------(7)$$

And

$$Var(W_j) = \sum_{i=1}^{n} Var(U_{ij}) = n\left(\pi_j^{\,+} + \pi_j^{\,-} - (\pi_j^{\,+} - \pi_j^{\,-})^2\right) ----(8)$$

Similarly,

$$E(W) = \sum_{j=1}^{c'} E(W_j) = n \sum_{j=1}^{c'} (\pi_j^{\,+} - \pi_j^{\,-}) ----(9)$$

And

$$Var(W) = n \sum_{j=1}^{c'} \left(\pi_j^{\,+} + \pi_j^{\,-} - (\pi_j^{\,+} - \pi_j^{\,-})^2\right) ------- (10)$$

Note from equation 9 that

$$\frac{E(W)}{n} = \sum_{j=1}^{c'} (\pi_j^{\,+} - \pi_j^{\,-})$$

measures the difference between the proportion of subjects who on the average successively or progressively earn higher scores and the proportion of subjects who on the average successively earn lower scores over time, space or condition.

Research interest is often in determining whether a population of subjects are on the average progressively experiencing increases (or decreases) in their scores or performance. For example in the case of disease diagnosis research interest may be on whether the proportion of subjects or patients progressively experiencing remission of disease are progressively increasing (or decreasing) over time or condition. In other words, null hypothesis of interest may be

$$H_0: \pi_1^{\,+} - \pi_1^{\,-} = \pi_2^{\,+} - \pi_2^{\,-} = \cdots \pi_{c'}^{\,+} - \pi_{c'}^{\,-} = \pi^+ - \pi^- = 0 \ \ vs \ \ H_1: \pi_j^{\,+} - \pi_j^{\,+} > 0\ say - (11)$$

For some $j = 1,2, \dots, c'$

The null hypothesis of eqn 11 may be tested based on W and its variance. We however here adopt an alternative approach based on the chi-square test for independence. Now $\pi_j^{\,+}, \pi_j^{\,0}\ and\ \pi_j^{\,-}$ are respectively the probabilities that in the j^{th} pair of observations or scores, the first score by a randomly selected subject is on the average higher (better, more), the same as (equal to) or lower (worse, smaller) than the second score for $j = 1,2,\dots,c' = \frac{c(c-1)}{2}$. Their sample estimates are respectively

$$\hat{\pi}_j^{\,+} = p_j^{\,+} = \frac{f_j^{\,+}}{n};\hat{\pi}_j^{\,0} = p_j^{\,0} = \frac{f_j^{\,0}}{n};\hat{\pi}_j^{\,-} = p_j^{\,-} = \frac{f_j^{\,-}}{n}\dots.(12)$$

where $f_j^{\,+}, f_j^{\,0}$ and $f_j^{\,-}$ are respectively the total number of times the first observation or score is higher, equal to, or lower than the second observation or score in the j^{th} pair of observations. That is $f_j^{\,+}, f_j^{\,0}$ and $f_j^{\,-}$ are respectively the number of 1's, 0's and -1's in the frequency distribution of the n values of these numbers in $U_{ij}, i = 1,2,\dots n$ for each j. Note $f_j^{\,0} = n - f_j^{\,+} - f_j^{\,-}$; $p_j^{\,0} = 1 - p_j^{\,+} - p_j^{\,-} ------(13)$

Also let f^+, f^- and f^0 be respectively the total number of 1's, -1's and 0's in all the c' pair wise treatment combinations. That is

$$f^+ = \sum_{i=1}^{c'} f_j^{\,+}; f^- = \sum_{i=1}^{c'} f_j^{\,-}; f^0 = \sum_{i=1}^{c'} f_j^{\,0} = nc' - f^+ - f^- = \frac{nc(c-1)}{2} - f^+ - f^- \quad (14)$$

Note that f^+, f^- and f^0 are respectively the total number of times subjects successively score higher, lower or experience non change in scores for all the pair-wise treatment combinations. The overall sample proportions for all the c' pair-wise treatment combinations are respectively

$$\hat{\pi}^+ = P^+ = \frac{f^+}{nc'} = \frac{2f^+}{nc(c-1)}; \hat{\pi}^- = P^- = \frac{f^-}{nc'} = \frac{2f^-}{nc(c-1)}; \hat{\pi}^0 = P^0 = \frac{f^0}{nc'} = \frac{2f^0}{nc(c-1)} = 1 - P^+ - P^- ----(15)$$

Now in terms of contingency tables, the observed frequencies for these three situations for the j^{th} pair of treatment combinations are respectively

$$O_{1j} = f_j^{\,+};\ O_{2j} = f_j^{\,-};\ O_{3j} = f_j^{\,0} = n - f_j^{\,+} - f_j^{\,-} \ ----(16)$$

The corresponding proportions are given by equation 12.

Now under the null hypothesis of equation 11 that subjects are on the average as likely to progressively earn higher as lower score at all treatment levels, then the expected number of 1's (higher scores), -1's (lower scores) or 0's (no change in scores) at the j^{th} pair of treatment combinations are respectively

$$E_{1j} = \frac{nf^+}{nc'} = \frac{nf^+}{\frac{nc(c-1)}{2}};\ E_{2j} = \frac{nf^-}{nc'} = \frac{f^-}{c'};\ E_{3j} = \frac{nf^0}{nc'} = \frac{f^0}{c'} = \frac{nc' - f^+ - f^0}{c'} \quad (17)$$

Under the null hypothesis of equation 11 of no difference in response calls, that is of equal population medians, the test statistic

$$\chi^2 = \sum_{i=1}^{3}\sum_{j=1}^{c'} \frac{\left(O_{ij} - E_{ij}\right)^2}{E_{ij}} \ -------- (18)$$

has approximately the chi-square distribution with $(3-1)(c'-1) = 2(c'-1) = c(c-1) - 2$ degrees of freedom for sufficiently large n.

Using equations 16 and 17 in equation 18 we have that

$$\chi^2 = \sum_{j=1}^{c'} \frac{\left[f_j^{\,+} - \frac{f^+}{c'}\right]^2}{\frac{f^+}{c'}} + \sum_{j=1}^{c'} \frac{\left[f_j^{\,-} - \frac{f^-}{c'}\right]^2}{\frac{f^-}{c'}} + \sum_{j=1}^{c'} \frac{\left[n - f^+ - f^- - n\left[\frac{nc' - f^+ - f^-}{c'}\right]\right]^2}{\frac{n(nc' - f^+ - f^-)}{nc'}}$$

$$= \frac{c'}{f^+f^-(nc - f^+ - f^-)}\left[f^-(nc' - f^+ - f^-)\sum_{j=1}^{c'}\left(f^+ - \frac{f^+}{c'}\right)^2 + f^+(nc' - f^+ - f^-)\sum_{j=1}^{c'}\left(f_j^{\,-} - \frac{f^-}{c'}\right)^2 + f^+f^-\sum_{j=1}^{c'}\left[\left[f_j^{\,+} - \frac{f^+}{c'}\right] + \left[f_j^{\,-} - \frac{f^-}{c'}\right]\right]^2\right]$$

which when further simplified reduces to

$$\chi^2 = \frac{c'}{f^+f^-(nc' - f^+ - f^-)}\left[f^-(nc' - f^-)\sum_{j=1}^{c'}\left(f_j^{\,+} - \frac{f^+}{c'}\right)^2 + f^+(nc' - f^+)\sum_{j=1}^{c'}\left(f_j^{\,-} - \frac{f^-}{c'}\right)^2 + 2f^+f^-\sum_{j=1}^{c'}\left(f_j^{\,+} - \frac{f^+}{c'}\right)\left(f_j^{\,-} - \frac{f^-}{c'}\right)\right] \quad 19$$

which has a chi-square distribution with $2(c^{'} - 1) = c(c - 1) - 2$ degrees of freedom; and may be used to test the null hypothesis of equation (11)
H_0 is rejected at the α −level of significance if

$$\chi^2 \geq \chi^2{}_{1-\alpha,c(c-1)-2} \quad - - - - - (20)$$

otherwise H_0 is accepted.
Now an alternative expression for the statistic of equation (19) in terms of the sample proportions of equations $(12) - (15)$ when simplified becomes

$$\chi^2 = \frac{n}{p^+p^-(1-p^+-p^-)}\left[p^-(1-p^-)\sum_{j=1}^{c'}\left(p_j{}^+ - p^+\right)^2 + p^+(1-p^+)\sum_{j=1}^{c'}\left(p_j{}^- - p^-\right)^2 + 2p^+p^-\sum_{j=1}^{c'}\left(p_j{}^+ -\right.\right.$$

p+p−−p−−−(21)

An easier to use computational form of equation (21) is

$$\chi^2 = \frac{n}{p^+p^-(1-p^+-p^-)}\left[p^-(1-p^-)\left(\sum_{j=1}^{c'} p_j{}^{+2} - c'p^{+2}\right) + p^+(1-p^+)\left(\sum_{j=1}^{c'} p_j{}^{-2} - c'p^{-2}\right) + 2p^+p^-\left(\sum_{j=1}^{c'} p_j{}^{+}p^- - c'p^+p^-\right)\right] \quad (22)$$

which also has a chi-square distribution with $2(c^{'} - 1) = c(c - 1) - 2$ degrees of freedom, H_0 is rejected at the α −level of significance if equation 20 is satisfied otherwise H_0 is accepted.

V. ILLUSTRATIVE EXAMPLE

We here use the data of table 1 on the grade point average (GPA) of a random sample of 17 students during each of their four years of study for a degree in a programme of a certain University to illustrate the proposed method. The data are presented in table 1 which also shows the pair-wise difference d_{ij} between the GPA's for 4 years for $i = 1,2,\dots,6$. These six pair-wise differences, d_{ij}, are taken and used here for simplicity of presentation because the data being analyzed are numeric. However the analysis could still be done by comparing the GPA's in pairs for each subject over the four years and applying equation 1.

TABLE 1: DATA ON GRADE POINT AVERAGE (GPA) FOR FOUR YEARS FOR A RANDOM SAMPLE OF STUDENTS

S/No	Year 1	Year 2	Year 3	Year 4	d_{i12}	d_{i13}	d_{i14}	d_{i23}	d_{i24}	d_{i34}
1	3.7	1.7	2.2	4.0	2.0	1.5	-0.3	-0.5	-2.3	-1.8
2	3.8	3.3	4.4	4.6	0.5	-0.6	-0.8	-1.1	-1.3	-0.2
3	4.1	4.0	4.4	4.3	0.1	-0.3	-0.2	-0.4	-0.3	0.1
4	4.2	3.1	2.5	3.8	1.1	1.7	0.4	0.6	-0.7	-1.3
5	3.7	3.3	4.3	4.3	0.4	-0.6	-0.6	-0.1	-1.0	0.0
6	3.7	2.9	4.1	3.6	0.8	-0.4	0.1	-1.2	-0.7	0.5
7	2.8	2.1	3.1	3.3	0.7	-0.3	-0.5	-1.0	-1.2	-0.2
8	3.7	2.9	2.8	4.0	0.8	0.9	-0.3	0.1	-1.1	-1.2
9	4.1	2.7	4.0	3.9	1.4	0.1	0.2	-1.3	-1.2	0.1
10	3.0	2.8	2.6	4.0	0.2	0.4	-1.0	0.2	-1.2	-1.4
11	3.5	2.5	3.7	3.7	1.0	-0.2	-0.2	-1.2	-1.2	0.0
12	3.5	3.1	4.0	3.9	0.4	-0.5	-0.4	-0.9	-0.8	0.1
13	4.5	4.4	4.6	4.7	0.1	-0.1	-0.2	-0.2	-0.3	-0.1
14	4.0	3.4	4.3	4.2	0.6	-0.3	-0.2	-0.9	-0.8	0.1
15	3.8	3.5	3.9	4.0	0.3	-0.1	-0.2	-0.4	-0.5	-0.1
16	3.4	3.0	4.0	4.6	0.4	-0.6	-1.2	-1.0	-1.6	-0.6
17	3.9	4.0	4.4	4.7	-0.1	-0.5	-0.8	-0.4	-0.7	-0.3

Now to illustrate the proposed method we apply equation 2 to the differences d_{ij} of table 1 to obtain values of U_{ij} shown in table 2.

TABLE 2: VALUES OF U_{ij} (EQN 1) FOR THE DIFFERENCES d_{ij} IN TABLE 1 AND OTHER STATISTICS

S/No	U_{i1}	U_{i2}	U_{i3}	U_{i4}	U_{i5}	U_{i6}	Total
1	1	1	-1	-1	-1	-1	
2	1	-1	-1	-1	-1	-1	
3	1	-1	-1	-1	-1	1	
4	1	1	1	1	-1	-1	
5	1	-1	-1	-1	-1	0	
6	1	-1	1	-1	-1	1	
7	1	-1	-1	-1	-1	-1	
8	1	1	-1	1	-1	-1	
9	1	1	1	-1	-1	1	
10	1	1	-1	1	-1	-1	
11	1	-1	-1	-1	-1	0	
12	1	-1	-1	-1	-1	1	
13	1	-1	-1	-1	-1	-1	
14	1	-1	-1	-1	-1	1	
15	1	-1	-1	-1	-1	-1	
16	1	-1	-1	-1	-1	-1	
17	-1	-1	-1	-1	-1	-1	
n	17	17	17	17	17	17	$102\left(\frac{nc(c-1)}{2}\right)$
f_j^+	16	5	3	3	0	5	32 (f^+)
f_j^-	1	12	14	14	17	10	68 $(= f^+)$
f_j^0	0	0	0	0	0	2	$2(= f^0)$
p_j^+	0.941	0.294	0.176	0.176	0.00	0.294	0.314 $(= P^+)$
p_j^-	0.059	0.706	0.824	0.824	1.00	0.588	0.667 $(= P^-)$
p_j^0	0.00	0.00	0.00	0.00	0.00	0.118	$0.02= P^0$

Using the sample proportion shown at the bottom of table 2 in the computational formular of equation 22 we have

$$\chi^2 = \frac{(17)\left((0.667)(0.333)(1.119-(6)(0.314)^2)+(0.314)(0.686)(3.205-6(0.667)^2)\right)}{(0.314)(0.667)(0.020)}$$

$$= \frac{(17)\left((0.222)(0.527)+(0.215)(0.536)+(0.419)(-0.530)\right)}{0.004}$$

$= \frac{(17)(0.117+0.115-0.222)}{0.004} = \frac{(17)(0.010)}{0.004} = 42.50\ (p-value = 0.000)$ which with $c(c-1)-2 = 4(3)-2 = 12-2 = 10$ degrees of freedom is highly statistically significant, indicating that students did not progressively earn higher GPAs during their years of study.

One may wish to compare the present results with what could have been obtained if the Friedman's Two-Way Analysis of variance Test by ranks had been used to analyze the data. To do this we may rank the GPAs of each student in table 1 from the lowest assigned the rank 1 to the highest assigned the rank 4. Tied GPAs for each student are assigned their mean ranks. The results are presented in table 3.

TABLE 3: Ranks assigned to students GPAs of Table 1 for use with the Friedman's Test.

S/No	Year 1	Year 2	Year 3	Year 4
1	3	1	2	4
2	2	1	3	4
3	2	1	4	3
4	4	2	1	3
5	2	1	3.5	3.5
6	3	1	4	2
7	2	1	3	4
8	3	2	1	4
9	4	1	3	2
10	3	2	1	4
11	2	1	3.5	3.5
12	2	1	4	3
13	2	1	3	4
14	2	1	4	3
15	2	1	3	4
16	2	1	3	4
17	1	2	3	4
$R_{.j}$	41	21	49	59

Now using the sum of ranks given at the bottom of table 3 with the Friedman's Two-Way Analysis of Variance by Ranks test statistic given as

$$\chi^2 = \frac{12}{nc(c+1)} \sum_{j=1}^{4} R_{.j}{}^2 - 3n(c+1) \quad --------(23)$$

we have

$$\chi^2 = \frac{12(41^2 + 21^2 + 49^2 + 59^2)}{17(4)(5)} - 3(17)(5)$$

$$= \frac{12(1681 + 441 + 2401 + 3481)}{340} - 255 = 280.14 - 255$$

$$= 25.14 \; (p - value = 0.000)$$

which with 3 degrees of freedom is also statistically significant showing that students median GPAs during their four years of study were probably different.

The present conclusion using the Friedman's Two-Way Analysis of Variance by ranks does not however adjust for and reflect any possible gradient in the proportions of students experiencing increase (decrease) in their GPAs during the four years of study. There are observable progressive increases and decreases in these proportions for the present data, making the application of the Friedman's Two-Way Analysis of Variance test by rank rather inappropriate here. In fact the calculated chi-square values of 42.50 for the proposed method and only 25.14 for the Friedman's Two-Way Analysis of Variance test by ranks indicate at least for the present data that it is likely to accept a false null hypothesis (Type II error) more frequently than the proposed method even if there exist no inbuilt ordering in the variables under study.

VI. SUMMARY AND CONCLUSION

This paper has presented and discussed a nonparametric statistical method for the analysis of related measures that are ordered in time, space or condition. The proposed method may be used to check for the existence of any gradient in proportions of responses and whether subjects are progressively experiencing increase (or decrease) in their performance levels. The method is illustrated with some data and shown to be more powerful than the Friedman's Two-Way Analysis of Variance test by ranks especially when the data being analyzed have inbuilt order.

REFERENCES

[1]. Bartholomew, D. J. (1959): A test of homogeneity for ordered alternatives. Biometrika, 46: 36-48

[2]. Bartholomew, D. J. (1963): On Chassan's test for order. Biometrika, 19, 188-191

[3]. Cochran, W. G. (1950): "The Comparison of Percentages in Matched Samples." Biometrika, 37, 256-266

[4]. Cohen, J. (1983): The Cost of Dichotomization, Appl. Pschol. Measure, 7: 249-253

[5]. Gart, J. J. (1963): A Median test with Sequential Application. Biometrika, 50: 55-62

[6]. Gibbons, J. D. (1971): Nonparametric Statistical Inference, McGraw Hill, New York

[7]. Miller, J (1996): Statistics for Advanced level (2nd Edition) Cambridge University Press

[8]. Oyeka, C. A. (2010): An Introduction to Applied Statistical Methods (8th Edition). Nobern Avocation Publishing Company, Enugu, Nigeria

[9]. Oyeka, C. A.; Utazi C. E.; Nwosu, C. R.; Ebuh, G. U.; Ikpegbu, P. A.; Obiora-Ilouno H. O.; and Nwankwo, C. C. (2010): Non-Homogenous Sample Data. International Journal of Mathematics Research Vol. 2, No 1, Pp 93-99