International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 2 Issue 4 || April. 2014 || PP-58-66

Controllability of Neutral Integrodifferential Equations with Infinite Delay

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ABSTRACT: In this paper sufficient conditions for the controllability of nonlinear neutral integrodifferential equations with infinite delay where established using the fixed point theorem due to Schaefer.

KEYWORDS: Controllability, Infinite delay, Neutral Integrodifferential equation, Schaefer fixed point.

I. INTRODUCTION

In this paper, we establish a controllability result to the following nonlinear neutral integrodifferential equations with infinite delay:

$$\frac{d}{dt}\left[x(t)-h(t,x_t)\right]=A(t)x(t)+Bu(t)+\int_{-\infty}^{t}g(t,s,x_s)ds+f(t,x_t),\quad t\geq 0$$
$$x_0=\phi\in\Re \qquad (1.1)$$

where the state $x(.)$ takes values in the Banach space X endowed with the norm $|.|$, the control function $u(.)$ is given in the Banach space of admissible control function $L^2(J,U)$ with U as a Banach space. $A(t):D(A)\rightarrow X$ is an infinitesimal generator of a strongly continuous semigroup of bounded linear operator $T(t),\ t\geq 0$ in X. $\Re$ is a bounded linear operator from U into X, where $g:J\times J\times\Re\rightarrow X$, $f:J\times\Re\rightarrow X$, and $h:J\times\Re\rightarrow X$ are given functions. The delay $x_t:(-\infty,0]\rightarrow X$ defined by $x_t(\theta)=x(t+\theta)$ belongs to some abstract phase space $\Re$, which will be a linear space of functions mapping $(-\infty,0]$ into X endowed with the seminorm $\|.\|_{\Re}$ in $\Re$. Controllability problems of linear and nonlinear systems represented by Ordinary differential Equations in finite dimensional space has been studies extensively [6, 15]. Several authors extended the controllability concept to infinite dimensional systems in abstract spaces with unbounded operators [1, 2, 8]. There are many systems that can be written as abstract neutral functional equations with infinite delays. In recent years, the theory of neutral functional differential equations with infinite delay in infinite dimension has received much attention [4, 7, 10]. Meanwhile, the controllability problem of such systems was also discussed by several scholars. Meili et al. [13] studied the controllability of neutral functional integrodifferential systems in abstract spaces using fractional power of operators and Sadovski fixed point theorem. Balachandran and Nandha [5] established sufficient conditions for the controllability of neutral functional integrodifferential systems with infinite delay in Banach spaces by means of Schaefer fixed point theorem. Li et al. [12] used Hausdoff measure of noncompactness and Kakutani's fixed point theorem to establish sufficient conditions for the controllability of nonlinear integrodifferential systems with nonlocal condition in separable Banach space. The purpose of this paper is to establish a set of sufficient conditions for the controllability of neutral integrodifferential equations with infinite delay using Schaefer fixed point theorem.

II. PRELIMINARIES

In this section, we introduce notations, definitions and theorems which are used throughout this paper. In the study of equations with infinite delay such as Eq. (1.1), we need to introduce the phase space. In this paper we employ an axiomatic definition of the phase space first introduced by Hale and Kato [9] and widely discussed by Hino et al [11]. The phase space $\Re$ is a linear space of functions mapping $(-\infty,0]$ into X endowed with the seminorm $|.|_{\Re}$ and assume that $\Re$ satisfies the following axioms:

(A1) If $x:(-\infty,b)\to X, b>0,$ is continuous on $[0,b]$ and $x_0\in\mathfrak{R}$, then for every $t\in[0,b]$ the Following conditions hold:

(i) x_t is in $\mathfrak{R}$

(ii) $\|x(t)\|\le H\|x_t\|_{\mathfrak{R}}$

(iii) $\|x_t\|_{\mathfrak{R}}\le K(t)\sup\{|x(s)|:0\le s\le t\}+M(t)\|x_0\|_{\mathfrak{R}}$

where $H\ge 0$ is a constant; $K,M:[0,\infty)\to[0,\infty)$, K is continuous and M is locally bounded, and H,K,M are independent of $x(.)$.

(A2) For the function $x(.)$ in (A1), x_t is a $\mathfrak{R}$-valued continuous function on $[0,b]$.

(A3) The space $\mathfrak{R}$ is complete.

We need the fixed point theorem as stated below,

Schafer's theorem [14]: Let S be a convex subset of a normed linear space E and $0\in S$. Let $F:S\to S$ be completely continuous operator and let

$$\varsigma(F)=\{x\in S:\lambda Fx=x \ \ for \ \ some \ \ \lambda\in(0,1)\}$$

Then either $\varsigma(F)$ is unbounded or F has a fixed point.

We assume the following hypothesis:

(H1) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t),\ t>0$ on X and their exists $M\le 1$ and $M_1>0$ such that

$$|T(t)|\le M \ \text{ and } \ |AT(t)|\le M_1.$$

(H2) For $h:J\times\mathfrak{R}\to X$ is completely continuous and there exists constants c_1,c_2 such that $\tilde{K}c_1<1$ and

$$\|h(t,\phi)\|\le c_1\|\phi\|+c_2$$

$t\in[0,b],\phi\in\mathfrak{R}$, where $\tilde{K}=\max\{K(t):t\in J\}$.

(H3) The function $f:J\times\mathfrak{R}\to X$ satisfies the following conditions

(i). For each $t\in J$, the function $f:J\times\mathfrak{R}\to X$ is continuous.

(ii) For each $\psi\in\mathfrak{R}$, the function $f(;,\psi):J\to X$ is strongly measurable.

(iii). For each positive integer k, there exists $\alpha_k(.)\in L^1[(0,b)]$ such that

$$\sup\{\|f(t,x_t)\|:\|x\|\le k\}\le\alpha_k(t),\ \ t\in J \quad a.e.$$

and

$$|f(t,x)|\le\Omega\|x\|,\ \ t\in J,\ \ x\in\mathfrak{R}$$

where $\Omega:[0,\infty)\to[0,\infty$ is a continuous nondecreasing function.

(H4) $g : J \times J \times \Re \to X$ is continuous and there exists $N > 0$, such that

$$\|g(t,s,\eta)\| \le N$$

For every $0 \le s \le t \le b.$

(H5) for each $\phi \in \Re$, $q(t) = \lim_{a\to\infty} \int_{-a}^{0} g(t,s,\phi(s))ds$ exists and it is continuous. Further there exists

$$N_1 > 0 \text{ such that } \|q(t)\| \le N_1.$$

(H6) there exists a compact set $V \subseteq X$, such that $T(t)f(s,\psi)$, $T(t)Bu(s)$, $T(t)g(s,\tau,\eta)$ and

$$T(t)q(s) \in V \text{ for all } \psi,\eta \in \Re, \text{ and } 0 \le \tau \le s \le t \le b.$$

(H7) The linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \int_0^b T(b-s)Bu(s)ds$$

has an inverse operator W^{-1}, which takes values in $L^2(J,U)/\ker W$ and there exist positive constants $M_3, M_4 > 0$ such that $|B| \le M_3$ *and* $|W^{-1}| \le M_4$

(H8) $\displaystyle\int_s^b \tilde{m}(s)ds < \int_0^\infty \frac{ds}{s+\Omega(s)}$

where

$$N_3 = M_3M_4\Big\{|x_1| - M\big(|\phi(0)| - c_1\|\phi\| + c_2\big) - c_1\|x_t\| + c_2 - M_1\int_0^b \big(c_1\|x_s\| + c_2\big)ds$$
$$- MN_1b - MNb - M\int_0^b \Omega\|x_s\|ds\Big\}$$

$$c = \frac{1}{1-\tilde{K}c_1}\left\{\tilde{M}\|\phi\| + \tilde{K}\big[M|\phi(0)| + c_1\|\phi\| + c_2\big] + c_1 + M_1c_2b + MN_3 + MN_1 + MNb\right\}$$

$$\tilde{M} = \max\{M(t) : t \in J\} \qquad \tilde{m}(t) = \max\left\{\frac{\tilde{K}M_1c_1}{1-\tilde{K}c_1}, \frac{\tilde{K}M}{1-\tilde{K}c_1}\right\}$$

Definition 2.1: The system (1.1) is said to be controllable on the interval J iff, for every $x_0, x_b \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(t)$ of (1.1) satisfies $x(0) = x_0$ and $x(b) = x_b$.

Definition 2.2: A function $x : (-\infty, b] \to X$ is called a mild solution of the integrodifferential equation (1.1) on [0, b] iff ,

$$x(t) = T(t)[\phi(0) - h(0,\phi)] + h(t,x_t) + \int_0^t AT(t-s)h(s,x_t)ds$$
$$+ \int_0^t T(t-s)\left[Bu(s) + q(s) + \int_0^t g(s,\tau,x_s)d\tau + f(s,x_s)\right]ds \quad t \in [0,b] \qquad (2.1)$$

is satisfied. See [3,10]

III. MAIN RESULT

Theorem 3.1: Assume that (H1)-(H8) holds. Then the system (1.1) is controllable.

Proof: Consider the map defined by

$$\Gamma x(t) = \Gamma(t)[\phi(0) - h(0,\phi)] + h(t, x_t) + \int_0^t AT(t-s)h(s, x_s)ds$$
$$+ \int_0^t T(t-s)\left[(Bu)(s) + q(s) + \int_0^t g(s,\tau,x_s)d\tau + f(s,x_s)\right]ds \qquad t \in J \tag{3.1}$$

and define the control function $u(t)$ as

$$u(t) = W^{-1}\{x_b - T(b)[\phi(0) - h(0,\phi)] - h(b, x_b) - \int_0^b AT(b-s)h(s,x_s)ds$$
$$- \int_0^b T(b-s)q(s)ds - \int_0^b \int_0^s T(b-s)g(s,\tau,x_\tau)d\tau ds - \int_0^b T(b-s)f(s,x_s)ds\Big\}(t) \tag{3.2}$$

First we show that there is a priori bound $K > 0$ such that $\|x_t\| \le K,\ t \in J$, where K depends only on b and on the function $\Omega(.)$. Then we show that the operator has a fixed point, which is then a solution to the system (1.1). Obviously, $(\Gamma x)(b) = x_1$, which means that the control u steers the system from the initial function ϕ to x_1 in time b, provided that the nonlinear operator Γ has a fixed point.

Substituting (3.2) into (2.1) we obtain

$$x(t) = T(t)[\phi(0) - h(0,\phi)] + h(t, x_t) + \int_0^t AT(t-s)h(s,x_s)ds$$
$$+ \int_0^t T(t-\eta)BW^{-1}\Big\{x_1 - T(b)[\phi(0) - h(0,\phi)] - h(b,x_b) - \int_0^b AT(b-s)h(s,x_s)ds$$
$$- \int_0^b T(b-s)q(s)ds - \int_0^b \int_0^s T(b-s)g(s,\tau,x_\tau)d\tau ds - \int_0^b T(b-s)f(s,x_s)ds\Big\}d\eta$$
$$+ \int_0^t T(t-s)\left[q(s) + \int_0^s g(s,\tau,x_\tau)d\tau + f(s,x_s)\right]ds$$

Then, we have

$$|x(t)| \le M\big[|\phi(0)| + c_1\|\phi\| + c_2\big] + c_1\|x_t\| + c_2 + M_1\int_0^t (c_1\|x_s\| + c_2)ds + MN_3 + MN_1 + MNb$$
$$+ M\int_0^t \Omega\|x_s\|ds$$

From which using Axiom (A1)(iii), it follows that

$$\|x_t\| \le K(t)\sup\{|x(s)| : 0 \le s \le t\} + M(t)\|\phi\|$$
$$\le \tilde{K}\sup\{|x(s)| : 0 \le s \le t\} + \tilde{M}\|\phi\|$$
$$\le \tilde{M}\|\phi\| + \tilde{K}\{M\big[|\phi(0)| + c_1\|\phi\| + c_2\big] + c_2 + M_1c_2b + MN_3 + MN_1 + MNb\}$$
$$+ \tilde{K}c_1\sup\|x_s\| + \tilde{K}M_1c_1\int_0^t \|x_s\|ds + \tilde{K}M\left(\int_0^t \Omega\|x_s\|\right)ds$$

Let $\mu(t)=\sup\{\|x_s\|:0\le s\le t\}$, then the function $\mu(t)$ is continuous and nondecreasing and from Axiom (A2) we have

$$\mu(t)\le \widetilde{M}\|\phi\|+\widetilde{K}\{[M|\phi(0)|+c_1\|\phi\|+c_2]+c_1+M_1c_2b+MN_3+MN_1+MNb\}$$

$$+\widetilde{K}c_1\mu(t)+\widetilde{K}M_1c_1\int_0^t\mu(s)ds+\widetilde{K}M\int_0^t\Omega\mu(\tau)ds$$

From which it follows that

$$\mu(t)\le c+\frac{\widetilde{K}M_1c_1}{1-\widetilde{K}c_1}\int_0^t\mu(s)ds+\frac{\widetilde{K}M}{1-Kc_1}\int_0^t\Omega\mu(s)ds$$

Denoting the right hand side of the above inequality as $\gamma(t)$, we have $\gamma(0)=c$, $\mu(t)\le\gamma(t),\ t\in J$ and

$$\gamma'(t)=\frac{\widetilde{K}M_1c_1}{1-\widetilde{K}c_1}\mu(s)+\frac{\widetilde{K}M}{1-\widetilde{K}c_1}\Omega\mu(s)$$

$$\le\frac{\widetilde{K}M_1c_1}{1-\widetilde{K}c_1}\gamma(s)+\frac{\widetilde{K}M}{1-\widetilde{K}c_1}\Omega\gamma(s)$$

$$\le\frac{1}{1-\widetilde{K}c_1}\widetilde{K}M_1c_1\left[\gamma(s)+\frac{M}{M_1c_1}\Omega\gamma(s)\right]\quad t\in J$$

$$\le\vec{m}(t)(\gamma(s)+\Omega\gamma(s))$$

which implies that

$$\int_{\gamma(0)}^{\gamma(t)}\frac{ds}{s+\Omega(s)}<\int_0^t\widetilde{m}(s)ds<\int_0^\infty\frac{ds}{s+\Omega(s)}\qquad t\in J$$

This inequality implies that there is a constant K such that $\gamma(t)\le K,\ t\in J$ and hence,

$\|x_t\|\le\mu(t)\le\gamma(t)\le K,\ t\in K$, where K only depends on b and on the function $\Omega(.)$.

We now rewrite the initial value problem (1.1) as follows:

For $\phi\in\Re$, define $\hat{\phi}\in\Re$ by

$$\hat{\phi}(t)=\begin{cases}\phi(t)-h(t,\phi) & if\ -\infty\le t\le 0\\ T(t)[\phi(0)-h(0,\phi)] & if\ 0\le t\le a\end{cases}$$

If $y\in\Re\ and\ x(t)=y(t)+\hat{\phi}(t),\ t\in[-\infty,a]$, then it is easy to see that x satisfies (2.1) if and only if y satisfies,

$$y(t)=y_0=0,\quad -\infty\le t\le 0$$

$$y(t)=T(t)h(0,\phi)+h(t,y_t+\hat{\phi}_t)+\int_0^t AT(t-s)h(s,y_s+\hat{\phi}_s)ds$$

$$+\int_0^t T(t-\eta)BW^{-1}\{x_1-T(b)[\phi(0)-h(0,\phi)]-h(b,y_b+\hat{\phi}_b)-\int_0^b AT(b-s)h(s,y_s+\hat{\phi}_s)ds$$

$$-\int_0^b T(b-s)q(s)ds-\int_0^b\int_0^s T(b-s)g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau ds-\int_0^b T(b-s)f\left(s,y_s+\hat{\phi}_s\right)ds\Big\}d\eta$$

$$+\int_0^t T(t-s)\left[q(s)+\int_0^s g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau+f\left(s,y_s+\hat{\phi}_s\right)\right]ds,\quad t\in J$$

We define the operator $\Gamma:\Re_0\to\Re_0$, $\Re_0=\{y\in\Re: y_0=0\}$ by

$$\begin{aligned}(\Gamma y)(t)=&T(t)h(0,\phi)+h\left(t,y_t+\hat{\phi}_t\right)+\int_0^t AT(t-s)h\left(s,y_s+\hat{\phi}_s\right)ds\\&+\int_0^t T(t-\eta)BW^{-1}\Big\{x_1-T(b)[\phi(0)-h(0,\phi)]-h\left(b,y_b+\hat{\phi}_b\right)-\int_0^b AT(b-s)h\left(s,y_s+\hat{\phi}_s\right)ds\\&-\int_0^b T(b-s)q(s)ds-\int_0^b\int_0^\tau T(b-s)g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau ds-\int_0^b T(b-s)f\left(s,y_s+\hat{\phi}_s\right)ds\Big\}d\eta\\&+\int_0^t T(t-s)\left[q(s)+\int_0^s g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau+f\left(s,y_s+\hat{\phi}_s\right)\right]ds,\ t\in J\end{aligned}$$

From the definition of an operator Γ defined on equation (3.1), it can be noted that the equation (2.1) can be written as

$$y(t)=\lambda\Gamma y(t),\quad 0<\lambda<1 \qquad (3.3)$$

Now, we prove that Γ is completely continuous. For any $y\in B_k$, let $0<t_1<t_2<b$, then

$$\begin{aligned}&|(\Gamma y)(t_1)-(\Gamma y)(t_2)|\\&\le|T(t_1)-T(t_2)||h(0,\phi)|+\left|h\left(t_1,y_{t_1}+\hat{\phi}_{t_1}\right)-h\left(t_2,y_{t_2}+\hat{\phi}_{t_2}\right)\right|\\&+\left|\int_0^{t_1}A[T(t_1-s)-T(t_2-s)]h\left(s,y_s+\hat{\phi}_s\right)ds\right|+\left|\int_{t_1}^{t_2}AT(t_2-s)h\left(s,y_s+\hat{\phi}_s\right)ds\right|\\&+\Big|\int_0^{t_1}[T(t_1-\eta)-T(t_2-\eta)]BW^{-1}\Big\{x_1-T(b)[\phi(0)-h(0,\phi)]-h\left(b,y_b+\hat{\phi}_b\right)\\&-\int_0^b AT(b-s)h\left(s,y_s+\hat{\phi}_s\right)-\int_0^b T(b-s)q(s)ds\\&-\int_0^b\int_0^\tau T(b-s)g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau ds-\int_0^b T(b-s)f\left(s,y_s+\hat{\phi}_s\right)ds\Big\}d\eta\Big|\\&+\Big|\int_{t_1}^{t_2}T(t_2-\eta)BW^{-1}\Big\{x_1-T(b)[\phi(0)-h(0,\phi)]-h\left(b,y_b+\hat{\phi}_b\right)\\&-\int_0^b AT(b-s)h\left(s,y_s+\hat{\phi}_s\right)ds-\int_0^b T(b-s)q(s)ds\\&-\int_0^b\int_0^\tau T(b-s)g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau ds-\int_0^b T(b-s)f\left(s,y_s+\hat{\phi}_s\right)ds\Big\}d\eta\Big|\\&+\left|\int_0^{t_1}[T(t_1-s)-T(t_2-s)]\left[q(s)+\int_0^s g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau+f\left(s,y_s+\hat{\phi}_s\right)\right]ds\right|\\&+\left|\int_{t_1}^{t_2}T(t_2-s)\left[q(s)+\int_0^s g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau+f\left(s,y_s+\hat{\phi}_s\right)\right]ds\right|\end{aligned}$$

$$\leq \left|T(t_1)-T(t_2)\right|\left|h(0,\phi)\right| + \left|h\left(t_1, y_{t_1}+\hat{\phi}_{t_1}\right)-h\left(t_2, y_{t_2}+\hat{\phi}_{t_2}\right)\right|$$

$$+\int_0^{t_1}\left|A[T(t_1-s)-T(t_2-s)]\right|\left(c_1\left\|y_s+\hat{\phi}_s\right\|+c_2\right)ds+\int_{t_1}^{t_2}\left|AT(t_2-s)\right|\left(c_1\left\|y_s+\hat{\phi}_s\right\|+c_2\right)ds \quad (3.4)$$

$$+\int_0^{t_1}\left|T(t_1-s)-T(t_2-s)\right|M_3M_4\left\{\left|x_1\right|+M\left[\left|\phi(0)-h(0,\phi)\right|\right]+c_1\left\|y_b+\hat{\phi}_b\right\|+c_2+MN_1b\right.$$

$$\left.+M\int_0^b N\tau ds+M\int_0^b \Omega\left\|y_s+\hat{\phi}_s\right\|ds\right\}ds+\int_{t_1}^{t_2}\left|T(t_2-s)\right|M_3M_4\left\{\left|x_1\right|+M\left[\left|\phi(0)-h(0,\phi)\right|\right]\right.$$

$$\left.+c_1\left\|y_b+\hat{\phi}_b\right\|+c_2+M_1\int_0^b\left(c_1\left\|y_b+\hat{\phi}_b\right\|+c_2\right)ds+MN_1b+M\int_0^b\int_0^\tau Nd\tau ds+M\int_0^b\Omega(s)ds\right\}ds$$

$$+\int_0^{t_1}\left|T(t_1-s)-T(t_2-s)\right|\left[N_1+N\tau+\Omega(s)\right]ds+\int_{t_1}^{t_2}T(t_2-s)\left[N_1+N\tau+\Omega(s)\right]ds$$

The right-hand side of equation (3.4) is independent of $y \in \Re_k$ and tends to zero as $t_2-t_1 \to 0$, since g is completely continuous and the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology. Thus Γ maps $\Re_k$ into an equicontinuous family of functions.

$$(\Gamma_\varepsilon y)(t)=T(t)h(0,\phi)+h\left(t-\varepsilon, y_{t-\varepsilon}+\hat{\phi}_{t-\varepsilon}\right)+\int_0^{t-\varepsilon}AT(t-s)h\left(s,y_s+\hat{\phi}_s\right)ds$$

$$+\int_0^{t-\varepsilon}T(t-s)BW^{-1}\left\{x_1-T(b)[\phi(0)-h(0,\phi)]-h\left(b,y_b+\hat{\phi}_b\right)\right.$$

$$-\int_0^b AT(b-s)h\left(s,y_s+\hat{\phi}_b\right)ds-\int_0^b T(b-s)q(s)ds-\int_0^b T(b-s)f\left(s,y_s+\hat{\phi}_s\right)ds$$

$$\left.-\int_0^b\int_0^s T(b-s)g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau ds\right\}ds+\int_0^t T(t-s)\left[q(s)+f\left(s,y_s+\hat{\phi}_s\right)+\int_0^s g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau\right]ds$$

$$=T(t)h(0,\phi)+h\left(t-\varepsilon,y_{t-\varepsilon}+\hat{\phi}_{t-\varepsilon}\right)+T(\varepsilon)\int_0^{t-\varepsilon}AT(t-\varepsilon)h\left(s,y_s+\hat{\phi}_s\right)$$

$$+T(\varepsilon)\int_0^{t-\varepsilon}T(t-s-\varepsilon)BW^{-1}\left\{x_1-T(b)[\phi(0)-h(0,\phi)]-h\left(b,y_b+\hat{\phi}_b\right)\right.$$

$$-\int_0^b AT(b-s)h\left(s,y_b+\hat{\phi}_b\right)ds-\int_0^b T(b-s)q(s)ds-\int_0^b T(b-s)f\left(s,y_s+\hat{\phi}_s\right)ds$$

$$\left.-\int_0^b\int_0^s T(b-s)g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau ds\right\}ds$$

$$+T(\varepsilon)\int_0^{t-\varepsilon}T(t-s-\varepsilon)\left[q(s)+f\left(s,y_s+\hat{\phi}_s\right)+\int_0^s g\left(s,\tau,y_\tau+\hat{\phi}_\tau\right)d\tau\right]ds$$

Since $T(t)$ is a compact operator, the set $Y_\varepsilon(t)=\{(\Gamma_\varepsilon y)(t): y\in\Re_k\}$ is precompact in X for every ε,

$0 < \varepsilon < t$. Moreover, for every $y \in \Re_k$ we have

$$\left|(\Gamma y)(t) - (\Gamma_\varepsilon y)(t)\right| \le \left|h\left(t, y_t + \hat{\phi}_t\right) - h\left(t-\varepsilon, y_{t-\varepsilon} + \hat{\phi}_{t-\varepsilon}\right)\right| + \int_{t-\varepsilon}^{t} \left|AT(t-s)h\left(s, y_s + \hat{\phi}_s\right)\right| ds$$
$$+ \int_{t-\varepsilon}^{t} |T(t-s)| M_3 M_4 \left\{ |x_1| + M|\phi(0) - h(0,\phi)| + \left|h\left(b, y_b + \hat{\phi}_b\right)\right| \right.$$
$$+ \int_0^b \left|AT(t-s)h\left(s, y_s + \hat{\phi}_s\right)\right| ds + \int_0^b |T(t-s)q(s)| ds + \int_0^b \left|T(t-s) f\left(s, y_s + \hat{\phi}_s\right)\right| ds$$
$$\left. + \int_0^b \int_0^s \left|T(t-s) g\left(s, \tau, y_\tau + \hat{\phi}_\tau\right)\right| d\tau ds \right\} ds$$
$$+ \int_{t-\varepsilon}^{t} |T(t-s)| \left[q(s) + f\left(s, y_s + \hat{\phi}_s\right) + \int_0^s g\left(s, \tau, y_\tau + \hat{\phi}_\tau\right) ds \right] ds$$

Clearly $\left|(\Gamma y)(t) - (\Gamma_\varepsilon y)(t)\right| \to 0$ as $\varepsilon \to 0^+$. Therefore, there is a family of precompact sets which are arbitrarily close to the set $\{(\Gamma y)(t) : y \in \Re_k\}$. Hence, the set $\{(\Gamma y)(t) : y \in \Re_k\}$ is precompact in X .

Next we prove that the set $\Gamma : \Re_b^0 \to \Re_b^0$ is continuous. Let $\{y_n\}_{n\ge 1} \subset \Re_b^0$ with $y_n \to y$ in $\Re_b^0$. We have

$$\left\|(\Gamma y_n)(t) - (\Gamma y)(t)\right\| \le \left|h\left(t, y_{n_t} + \hat{\phi}_t\right) - h\left(t, y_t + \hat{\phi}_t\right)\right| + \int_0^t |AT(t-s)| \left|h\left(s, y_{n_s} + \hat{\phi}_s\right) - h\left(s, y_s + \hat{\phi}_s\right)\right| ds$$
$$+ \int_0^t |T(t-s)| M_3 M_4 \left\{ M_1 \int_0^t \left|h\left(s. y_{n_s} + \hat{\phi}_s\right) - h\left(s, y_s + \hat{\phi}_s\right)\right| ds \right.$$
$$\left. + M \int_0^b \int_0^s \left|g\left(s, \tau, y_{n_\tau} + \hat{\phi}_\tau\right) - g\left(s, \tau, y_\tau + \hat{\phi}_\tau\right)\right| ds + M \int_0^b \left| f\left(s, y_{n_s} + \hat{\phi}_s\right) - f\left(s, y_s + \hat{\phi}_s\right)\right| ds \right\} ds$$
$$+ \int_0^t |T(t-s)| \left[\left| f\left(s, y_{n_s} + \hat{\phi}_s\right) - f\left(s, y_s + \hat{\phi}_s\right)\right| + \int_0^s \left|g\left(s, \tau, y_{n_\tau} + \hat{\phi}_\tau\right) - g\left(s, \tau, y_\tau + \hat{\phi}_\tau\right)\right| ds \right]$$

From the assumption (H1) and the Lebesgue Dominated Convergence theorem we conclude that $\Gamma y_n \to \Gamma y$ as $n \to \infty$ in $\Re_b^0$. Thus $\Gamma(.)$ is continuous, which completes the proof that $\Gamma(.)$ is completely continuous. Hence there exists a unique fixed point $x(t)$ for Γ *on* $\Re_b$. Obviously $x(.)$ is a mild solution of the system (1.1) satisfying $x(b) = x_1$.

IV. CONCLUTION

Sufficient condition for the controllability of the nonlinear neutral integrodiffential equation was established using Schaefer's fixed point theorem. First we show that there is a priori bound $K > 0$ such that $\|x_t\| \le K,\ t \in J$. Then we show that the operator has a fixed point, which is then a solution to the system (1.1). Obviously, $(\Gamma x)(b) = x_1$, which means that the control u steers the system from the initial function ϕ to x_1 in time b, provided that the nonlinear operator Γ has a fixed point.

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