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Some Accepts Of Banach Summability of a Factored Conjugate Series

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ABSTRACT: An elementary proposition states that an absolutely convergent series is convergent, *i.e.* that if $/s_0-s_{1/+/}s_1-s_{2/}+\ldots+/s_n-s_{n-1}/<\infty$

This is the analogue for series of the theorem on functions that if a function f*(x)* is of bounded variation in an interval, the limits exist at every point. Consider the function $f(x) = \sum a_n x_n$, then the series being supposed convergent in (0 < x < 1) .
Summability theory has historically been concerned with the notion of assigning a limit to a linear space-valued sequences, especially if the sequence is divergent. In this paper we have been proved a theorem on Banach summability of a factored conjugate series.

KEY WORDS: *Summability theory, Absolute Banach summability, Conjugate series, infinite series.*

I. INTRODUCTION

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive numbers such that

(1.1) $$P_n = \sum_{r=0}^{n} p_v \to \infty,\ as\ n \to \infty\ \left(P_{-i} = p_{-i} = 0,\ i \geq 1\right)$$

The sequence to sequence transformation

(1.2) $$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v}\ s_v$$

defines the sequence of the $(N,\ p_n)$-mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$.

The series $\sum a_n$ is said to be summable $\left|N,\ p_n\right|_k,\ k \geq 1$, if

(1.3) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty$$

In the case when $p_n = 1$, for all $n\ and\ k = 1, \left|N, p_n\right|_k$ summability is same as $\left|C,1\right|$ summability . For $k = 1, \left|N, p_n\right|_k$ summability is same an $\left|N, p_n\right|$ -summability.

Now , Let $\sum_{n=1}^{\infty} B_n(x)$ be the conjugate Fourier series of a 2π -periodic function $f(t)$ and L-integrable on $(-\pi, \pi)$. Then

(1.4) $$B_n(x) = \frac{2}{\pi} \int_0^{\pi} \psi(t) \sin\, dt,\ n = 1,2,3,\ldots$$

where

(1.5) $$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

Dealing with $|B|$ -summability of a conjugate Fourier Series, We have the following results:

Known Results:

Theorem-2.1:

Let $f(t)$ be a 2π -periodic, L-integrable function on $(-\pi,\pi)$. Then the conjugate Fourier Series $\sum B_n(x)$ of $f(t)$ is $|B|$ -integrable if

(i) $\psi(t) \in BV(0,\pi)$

and

(ii) $\int_0^{\pi} \frac{\psi(t)}{t}\, dt < \infty$

Theorem-2.2: If

$$\int_0^{\pi} \frac{\psi(t)}{t}\, dt < \infty,$$

then the factored Fourier series $\sum \lambda_n B_n(x)$ is $|B|$-summable for $\{\lambda_n\}$ to be a non-negative convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$.

we wish to generalize the above two results to absolute Riesz-Banach summability. We prove

Theorem-2.3:

Let $\{p_n\}$ be a positive non-decreasing sequence of numbers such that $P_n = \sum_{v=1}^{n} p_v \to \infty,\ on\ n \to \infty$. Let

(i) $\psi(t) \in BV(0,\pi)$

(ii) $\int_0^{\pi} \frac{\psi(t)}{t}\, dt < \infty,$

and

(iii) $np_n = O(P_n),\ as\ n \to \infty$.

Then the conjugate Fourier Series $\sum B_n(x)$ is absolutely Riesz-Banach summable i.e. $|(\overline{N}, p_n) - B| -$summable.

Theorem-2.4:

Let $\{p_n\}$ be a sequence of positive numbers such that $P_n = \sum_{v=1}^{n} p_v \to \infty$, as $n \to \infty$. Let

(i) $\int_0^\pi \frac{\psi(t)}{t}\, dt < \infty$,

and

(ii) $np_n = 0(P_n),\ as\ n \to \infty$. Then the factored conjugate Fourier series $\sum \lambda_n B_n(x)$ is absolutely Riesz-Banach summable i.e. $\left|(\overline{N},\ p_n) - B\right|$ -summable for $\{\lambda_n\}$ to be a non-negative convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$.

We need the following Lemmas for the proof of the above theorems.

Lemma-2.3.1

Let $\{p_n\}$ be a positive non-decreasing sequence of numbers.

Let $\tau = \left[\frac{1}{t}\right]$, then $\{t_n\}$ is a monotonically decreasing sequence.

Lemma-2.3.2

Let $\{p_n\}$ be a sequence of positive non-decreasing, then $\left\{\frac{P_k}{n+k}\right\}$ is monotonically increasing in k.

Proof. We have

$$\frac{P_k}{n+k} - \frac{P_{k-1}}{n+k-1} = \frac{(n+k-1)\,P_k - (n+k)P_{k-1}}{(n+k)(n+k-1)}$$

$$= \frac{(n+k)p_k - P_k}{(n+k)(n+k-1)} = \frac{np_k + (p_k - p_1) + (p_k - p_2) + \cdots + (p_k - p_{k-1})}{(n+k)(n+k-1)}$$

$> 0,\ m\{p_n\}$ is non-decreasing.

This proves the lemma.

Lemma-2.3.3

If $\{\lambda_n\}$ is a positive convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$, then $\{\lambda_n\}$ is a monotonically decreasing sequence.

Proof of the theorem - 2.3

If $T_k(n)$ is the k-th element of the Riesz-Banach transformation of the conjugate Fourier Series $\sum B_n(x)$, then

$$T_k(n) = \frac{1}{P_k}\sum_{v=1}^{k} p_v\, s_{n+v-1}$$

$$= \frac{1}{P_k}\sum_{v=1}^{k} p_v \sum_{r=1}^{n+v-1} B_i(x)$$

$$= \sum_{i=1}^{n} B_i(x) + \frac{1}{P_k}\sum_{i=n}^{k+n-1}(P_k - P_{i-n})\, B_i(x).$$

Now

$$T_k(n) - T_{k+1}(n) = -\frac{p_{k+1}}{P_k P_{k+1}}\sum_{v=1}^{k} P_v\ B_{n+v}(x)$$

For the series $\sum B_n(x)$, we have

$$\sum_{k=1}^{\infty}\left|T_k(n) - T_{k+1}(n)\right| = \sum_{k=1}^{\infty}\frac{p_{k+1}}{P_k P_{k+1}}\left|\sum_{v=1}^{k} P_v\ B_{n+v}(x)\right|$$

$$= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{p_{k+1}}{P_k\ P_{k+1}}\left|\sum_{v=1}^{k} P_v \int_0^{\pi}\psi(t)\sin\,(n+v)\ t\ dt\right|$$

$$= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{p_{k+1}}{P_k\ P_{k+1}}\left|\int_0^{\pi}\sum_{v=1}^{k} P_v\ \psi(t)\sin\,(n+v)\ t\ dt\right|$$

$$= \frac{2}{\pi}\left[\sum_{k=1}^{\tau} + \sum_{k>\tau}\ \right],\ where\ \tau = \left[\frac{1}{t}\right] = \frac{2}{\pi}\ \left[\Sigma_1 + \Sigma_2\right], \text{say}$$

We have

$$\sum_1 = \sum_{k=1}^{\tau}\frac{p_{k+1}}{P_k\ P_{k+1}}\left|\int_0^{\pi}\sum_{v=1}^{k} P_v\ \psi(t)\sin(n+v)t\ dt\right| = \sum_{k=1}^{\tau}\frac{p_{k+1}}{P_k P_{k+1}}\left|\int_0^{\pi} t\sum_{v=1}^{k} P_v\ \sin\,(n+v)\,t\ \frac{\psi(t)}{t}\ dt\right|$$

Since

$$\int_0^{\pi}\frac{\psi(t)}{t}\ dt < \infty,\ \sum_1 = \sum_{k=1}^{\tau} < \infty, \text{ if}$$

$$\sum_{k=1}^{\tau}\frac{p_{k+1}}{P_k\ P_{k+1}}\ t\left|\sum_{v=1}^{k} P_v\ \sin(n+v)\,t\right| < \infty, \text{ uniformly for } 0 < t < \pi$$

Now

$$\sum_{12} \le t\sum_{k=1}^{\tau}\frac{p_{k+1}}{P_k\ P_{k+1}}\sum_{v=1}^{k} P_v\ \left|\sin(n+k)\ t\right|$$

$$\le 0(t)\sum_{k=1}^{\tau}\frac{p_{k+1}}{P_k P_{k+1}}\sum_{v=1}^{k} P_v$$

$$= 0(t)\sum_{k=1}^{\tau}\frac{P_{k+1}}{P_k P_{k+1}}\ (k+1)\ P_k\ = 0(t)\sum_{k=1}^{\tau}\frac{(k+1)p_{k+1}}{P_{k+1}} = 0(t)\cdot 0(\tau) = 0(1),\ as\ np_n\ = 0(P_n)$$

Next,

$$\sum_2 = \sum_{k>\tau} \frac{p_{k+1}}{P_k P_{k+1}} \left| \int_0^{\pi} \sum_{v=1}^{k} p_v \ \sin\ (n+v)\ t\ \psi(t)\ dt \ \right|$$

Since $\psi(0) = \psi(\pi) = 0$, we have

$$\int_0^{\pi} \psi(t) \sin(n+v)\ t\ dt = \int_0^{\pi} \frac{\cos(n+v)t}{n+v}\ d\psi(t)$$

Then,

$$\sum_2 = \sum_{k>\tau} \frac{p_{k+1}}{P_k P_{k+1}} \left| \int_0^{\pi} \sum_{v=1}^{k} \frac{P_v}{n+v} \cos(n+v)\ t\, d\psi(t) \right|$$

$$= 0(1) \sum_{k>\tau} \frac{p_{n+1}}{P_k P_{k+1}} \left| \sum_{v=1}^{k} \frac{P_v}{n+v} \cos(n+v)\ t \right|, \ as\ \psi(t) \in BV(0,\pi) \text{ by results, } \int_0^{\pi} |d\psi(t)| < \infty$$

$$= 0(1) \sum_{k>\tau} \frac{p_{k+1}}{P_k P_{k+1}} \cdot \frac{P_k}{n+k} \left| \sum_{n+k}^{k} \cos\ (n+v)\ t \right|,$$

$$= 0(\tau) \sum_{k>\tau} \frac{p_{k+1}}{(n+k) P_{k+1}} \quad \text{, by Lemma-2.3.1,}$$

$$= 0(\tau) \sum_{k>\tau} \frac{1}{(n+k)(n+1)}\ ,\ as\ np_n = 0(P_n)$$

$$= 0(\tau) \cdot 0(\tau^{-1}) = 0(1).$$

Then

$$\sum_{k=1}^{\infty} |T_k(n) - T_{k+1}(n)| < \infty \text{, uniformly for } n \in N.$$

Hence $\sum B_n(x)$ is absolutely Riesz-Banach summable.

This completes the proof of the theorem.

Proof of the Theorem - 2.4

Let $T_k(n)$ be the k-th element of the Riesz – Banach transformation of the factored conjugate Fourier series $\sum \lambda_n\ B_n(x)$. Then

$$T_k(n) = \sum_{i=1}^{n} \lambda_i B_i(x) + \frac{1}{P_k} \sum_{i=n}^{k+n-1} (P_k - P_{i-n})\ \lambda_i\ \ B_i(n)$$

Then

$$T_k(n) - T_{k+1}(n) = -\frac{p_{k+1}}{P_k P_{k+1}} \sum_{v=1}^{k} P_v\ \lambda_{n+v}\ \ B_{n+v}(x)$$

…series $\sum \lambda_n B_n(x)$, we have

$$\sum_{k=1}^{\infty}\left|T_k(n)-T_{n+1}(n)\right| = \sum_{k=1}^{\infty}\frac{p_{n+1}}{P_k P_{n+1}}\left|\sum_{v=1}^{k}P_v\ \lambda_{n+v}\ B_{n+v}(x)\right|$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{p_{n+1}}{P_k P_{n+1}}\left|\sum_{v=1}^{k}P_v\ \lambda_{n+v}\int_0^{\pi} t\frac{\psi(t)}{t}\sin(n+v)+dt\right|$$

Since

$$\int_0^{\pi}\frac{\psi(t)}{t}\,dt<\infty,\qquad \sum_{k=1}^{\infty}\left|T_k(n)-T_{n+1}(n)\right|<\infty$$

if

$$\sum = \sum_{k=1}^{\infty}\frac{p_{k+1}}{P_k P_{n+1}}\left|t\sum_{v=1}^{k}P_v\,\lambda_{n+v}\ \sin(n+v)t\right|<\infty,$$

uniformly for $0<t<\pi$.

Now

$$\sum=\left[\sum_{k=1}^{\tau}+\sum_{k>\tau}\ \right]\frac{p_k}{P_k\ P_{k+1}}\left|t\sum_{v=1}^{k}P_v\,\lambda_{n+v}\ \sin(n+v)\,t\ \right|$$

$$=\sum_1+\sum_2\quad\text{, say.}$$

We have

$$\sum_1=\sum_{k=1}^{\tau}\frac{p_{k+1}}{P_k\ P_{k+1}}\left|t\sum_{v=1}^{k}P_v\ \lambda_{n+v}\ \sin(n+v)\,t\ \right|$$

$$=0(t)\sum_{k=1}^{\tau}\frac{p_{k+1}}{P_k P_{k+1}}\sum_{v=1}^{k}\ P_v\left|\lambda_{n+v}\sin(n+v)\,t\ \right|$$

$$=0(t)\sum_{k=1}^{\tau}\ \frac{p_{k+1}}{P_k P_{k+1}}\sum_{v=1}^{k}P_v$$

$$=0(t)\sum_{k+1}^{\tau}\ \frac{p_{k+1}}{P_k\ \ P_{k+1}}\,(k+1)\,P_k$$

$$=0(t)\sum_{k=1}^{\tau}\ \frac{(k+1)\,p_{k+1}}{P_{k+1}}=0(t)\cdot 0(\tau)=0(1),\ on\ np_n\ =0(P_n).$$

Next

$$\sum_2=t\sum_{k>\tau}\ \frac{p_{n+1}}{P_k P_{k+1}}\left|\sum_{v=1}^{k}P_v\ \lambda_{n+v}\ \sin(n+v)\,t\ \right|.$$

By Abel's partial summation formula

$$\sum_{v=1}^{k}(P_v\ \lambda_{n+v})\sin(n+v)\,t=\sum_{v=1}^{k-1}\Delta(P_v\ \lambda_{n+v})\sum_{p=1}^{v}\sin\ (n+p)\,t\ +\ P_k\ \lambda_{n+k}\ \sum_{p=1}^{k}\sin(n+p)\,t$$

$$= 0(\tau)\left[\sum_{v=1}^{k-1}\left(-p_{v+1}\ \lambda_{n+v}\ +P_{v+1}\,\Delta\lambda_{n+v}\right)+P_k\ \lambda_{n+k}\right]$$

Thus

$$\sum_2 = 0(1)\sum_{k>\tau}\frac{p_{k+1}}{P_k P_{k+1}}\left|\sum_{v=1}^{k-1}\left(-p_{v+1}\ \lambda_{n+v}\ +P_{v+1}\ \Delta\lambda_{n+v}\right)+P_k\ \lambda_{n+k}\right|$$

$$= 0(1)\left[\sum_{k>\tau}\frac{p_{k+1}}{P_k\ P_{k+1}}\left(p_{v+1}\ \lambda_{n+v}\ +\ p_{v+1}\ \Delta\lambda_{n+v}\right)+\sum_{k>\tau}\frac{P_k\ \lambda_{n+k}}{P_n\ P_{n+1}}\right]$$

$$= 0(1)\left[\sum_{v=1}^{\tau}\left(p_{v+1}\ \lambda_{n+v}\ +P_{v+1}\ \Delta\lambda_{n+1}\right)\sum_{k=\tau}^{\infty}\frac{p_{k+1}}{P_k P_{k+1}}+\right.$$

$$\left.+\sum_{v=\tau}^{\infty}\left(p_{v+1}\ \lambda_{n+v}+P_{v+1}\ \Delta\lambda_{n+v}\right)\sum_{k=v}^{\infty}\frac{p_{n+1}}{P_n P_{n+1}}+\sum_{n>2}\frac{p_{k+1}}{P_{k+1}}\lambda_{n+k}\right]$$

$$= 0(1)\left[\frac{1}{P_\tau}\sum_{v=1}^{\tau}\left(p_{v+1}\ \lambda_{n+v}+P_{v+1}\ \Delta\lambda_{n+v}\right)+\sum_{v=\tau}^{\infty}\frac{p_{v+1}\,\lambda_{n+v}+P_{v+1}\ \Delta\lambda_{n+v}}{P_v}+\sum_{k>\tau}\frac{\lambda_{n+k}}{k+1}\right]$$

$$= 0(1)\left[\left(\sum_{v=1}^{\tau}\frac{p_{v+1}\,\lambda_{n+v}}{P_\tau}+\sum_{v=\tau}^{\infty}\frac{p_{v+1}\ \lambda_{n+v}}{P_\tau}\right)+\left(\sum_{v=1}^{\tau}\frac{P_{v+1}\ \Delta\lambda_{n+v}}{P_\tau}+\sum_{v=\tau}^{\infty}\frac{P_{v+1}\ \Delta\lambda_{n+v}}{P_v}\right)+\sum_{k>\tau}\frac{\lambda_{k-1}}{k+1}\right]$$

$$= 0(1)\left[\sum_{v=1}^{\infty}\frac{p_{v+1}\ \lambda_{n+v}}{P_{v+1}}+\sum_{v=1}^{\infty}\Delta\lambda_{n+v}+\sum_{k>\tau}\frac{\lambda_{n+1}}{k+1}\right]$$

$$= 0(1)\left[\sum_{v=1}^{\infty}\frac{\lambda_{n+v}}{v+1}+\sum_{v=1}^{\infty}\Delta\lambda_{n+v}\right]$$

$$= 0(1)\left(\sum_{v=1}^{\infty}\frac{\lambda_{n+v}}{v+1}+\sum_{v=1}^{\infty}\Delta\lambda_{n+v}\right)$$

$= 0(1)$, on λ_n is decreasing and $\sum\frac{\lambda_n}{n}<\infty$,

Hence $\sum<\infty$, uniformly for $n\in N$.

Then $\sum\lambda_n\ B_n(n)\ is\ \left|\left(\overline{N},\ p_n\right)-B\right|$-summable.

This proves the theorem.

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