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Relationships among determining matrices, partials of indices of control systems matrices and systems coefficients for single–delay linear neutral control systems.

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ABSTRACT: *This paper obtained various relationships among determining matrices, partial derivatives of indices of control systems matrices of all orders, as well as their relationships with systems coefficients for single – delay autonomous neutral linear differential systems through a sequence of lemmas, theorems and corollaries and the exploitation of key facts about permutations.*

The proofs were achieved using appropriate combinations of summation notations, multinomial distribution, change of variables techniques and deft deployment of skills in the differentiation of matrix functions of several variables.

KEYWORDS- *Determining, Feasible, Generalized, Index, Systems.*

I. INTRODUCTION

The importance of the relationships among determining matrices, indices of control systems matrices and systems coefficient stems from the fact that these relationships pave the way for the determination of Euclidean controllability and compactness of cores of Euclidean targets. This paper brings fresh perspectives to bear on such relationships.

The derivation of necessary and sufficient condition for the Euclidean controllability of system (1) below on the interval $[0,\ t_1]$, using determining matrices depends on

(i) Obtaining workable expressions for the determining equations of the $n\times n$ matrices $Q_k(jh)$, for $j: t_1 - jh > 0, k = 0, 1, \cdots$

(ii) Showing that $\Delta X^{(k)}(t_1 - jh, t_1) = (-1)^k Q_k(jh)$, for j: $t_1 - jh > 0, k = 0, 1, \ldots$
where $\Delta X^{(k)}(t_1 - jh, t_1) = X^{(k)}((t_1 - jh)^-, t_1) - X^{(k)}((t_1 - jh)^+, t_1)$

(iii) showing that $Q_\infty(t_1)$ is a linear combination of $Q_0(s), Q_1(s), \cdots, Q_{n-1}(s); s = 0, h, \cdots (n-1)h.$

Our objective is to prosecute task (ii) and (iii). Tasks (i) has been prosecuted in Ukwu [1].

II. IDENTIFICATION OF WORK-BASED DOUBLE-DELAY AUTONOMOUS CONTROL SYSTEM

We consider the single-delay autonomous neutral control system:

$$\dot{x}(t) = A_{-1}\dot{x}(t-h) + A_0 x(t) + A_1 x(t-h) + B\,u(t); t \geq 0 \tag{1}$$

$$x(t) = \phi(t), t \in [-h, 0], h > 0 \tag{2}$$

where A_{-1}, A_0, A_1 are $n\times n$ constant matrices with real entries, B is an $n\times m$ constant matrix with real entries. The initial function ϕ is in $C\big([-h, 0], \mathbf{R}^n\big)$, the space of continuous functions from $[-h, 0]$ into the real n-dimension Euclidean space, $\mathbf{R}^n$ with norm defined by $\|\phi\| = \sup_{t\in[-h,0]} |\phi(t)|$, (the sup norm). The control *u* is in the space $L_\infty\left([0, t_1], \mathbf{R}^n\right)$, the space of essentially bounded measurable functions taking $[0, t_1]$ into $\mathbf{R}^n$ with norm $\|\phi\| = \underset{t\in[0,t_1]}{ess\sup}|u(t)|$.

Any control $u \in L_\infty\left([0,t_1], \mathbf{R}^n\right)$ will be referred to as an admissible control. For full discussion on the spaces C^{p-1} and L_p (or L^p), $p \in \{1,2,...,\infty\}$, see Chidume [2 and 3] and Royden [4].

1.2 Preliminaries on the partial derivatives $\frac{\partial^k X(\tau, t)}{\partial \tau^k}, k = 0,1,\cdots$

Let $t, \tau \in [0,t_1]$. For fixed *t*, let $\tau \to X(\tau, t)$ satisfy the matrix differential equation:

$$\frac{\partial}{\partial\tau}X(\tau,t) = \frac{\partial}{\partial\tau}X(\tau+h,t)A_{-1} - X(\tau,t)A_0 - X(\tau+h,t)A_1 \tag{3}$$

for $0 < \tau < t, \tau \neq t - kh, k = 0,1,...$ where $X(\tau, t) = \begin{cases} I_n; \tau = t \\ 0; \tau > t \end{cases}$

See Chukwu [5 and 6], Hale [7] and Tadmore [8] for properties of $X(t, \tau)$. Of particular importance is the fact that $\tau \to X(\tau,t)$ is analytic on the intervals $(t_1 - (j+1)h, t_1 - jh)$, $j = 0,1,\ldots, t_1 - (j+1)h > 0$. Any such $\tau \in (t_1 - (j+1)h, t_1 - jh)$ is called a regular point of $\tau \to X(t, \tau)$. Let $X^{(k)}(\tau,t)$ denote $\frac{\partial^k}{\partial\tau^k}X(\tau,t_1)$, the k^{th} partial derivative of $X(\tau,t_1)$ with respect to *τ*, where *τ* is in $(t_1 - (j+1)h, t_1 - jh)$; $j = 0,1,\ldots, r$, for some integer *r* such that $t_1 - (r+1)h > 0$.

Write $X^{(k+1)}(\tau,t_1) = \frac{\delta}{\delta\tau}X^k(\tau,t_1)$.

Define:

$$\Delta X^{(k)}(t_1 - jh, t_1) = X^{(k)}\left(t_1, (t_1 - jh)^-, t_1\right) - X^{(k)}\left((t_1 - jh)^+, t_1\right), \tag{4}$$

for $k = 0,1,...;\ j = 0,1,...;\ t_1 - jh > 0$,

where $X^{(k)}\left((t_1 - jh)^-, t_1\right)$ and $X^{(k)}\left(t_1, (t_1 - jh)^+, t_1\right)$ denote respectively the left and right hand limits of $X^{(k)}(\tau,t_1)$ at $\tau = t_1 - jh$. Hence:

$$X^{(k)}\left((t_1 - jh)^-, t_1\right) = \lim_{\substack{\tau \to t_1 - jh \\ t_1 - (j+1)h < \tau < t_1 - jh}} X^{(k)}(\tau,t_1) \tag{5}$$

$$X^{(k)}\left((t_1 - jh)^+, t_1\right) = \lim_{\substack{\tau \to t_1 - jh \\ t_1 - jh < \tau < t_1 - (j-1)h}} X^{(k)}(\tau,t_1) \tag{6}$$

2.2 Definition, existence and uniqueness of Determining matrices for system (1)

Let $Q_k(s)$ be then $n \times n$ matrix function defined

by: $Q_k(s) = A_{-1}Q_{k-1}(s-h) + A_0Q_{k-1}(s) + A_1Q_{k-1}(s-h)$ (7)

for $k = 1, 2, \cdots;\ s > 0,$ with initial conditions:

$$Q_0(0) = I_n \quad (8)$$

$$Q_k(s) = 0;\ k < 0 \ \text{ or } s < 0 \quad (9)$$

These initial conditions guarantee the unique solvability of (7). Cf.Gabasov and Kirillova [9].

III. MAIN RESULTS

The investigation in this section will be carried out through the following sequence of results.

3.1 Theorem relating $\Delta X^{(k)}((t_1 - jh),\ t_1)$ to $Q_k(jh)$

For all nonnegative integers $j : t_1 - jh > 0,$ and for $k \in \{0, 1, \cdots, \}$:

$$\Delta X^{(k)}(t_1 - jh, t_1) = (-1)^k Q_k(jh) \quad (10)$$

Proof

If $k = 0,$ then $\Delta X^{(k)}(t_1 - jh, t_1) = \Delta X(t_1 - jh, t_1) = A_{-1}^{j} = Q_0(jh) = (-1)^k Q_k(jh),$ with $k = 0.$

By lemma 2.7 of [1], $\Delta X^{(k)}((t_1 - jh), t_1) = -\sum_{r=0}^{j}\left[\sum_{i=0}^{1} \Delta X^{(k-1)}((t_1 - (j - (r+i))h), t_1)A_i\right]A_{-1}^{r},$

$$k = 1 \Rightarrow \Delta X^{(1)}((t_1 - jh), t_1) = -\sum_{r=0}^{j}\left[\sum_{i=0}^{1} \Delta X((t_1 - (j - (r+i))h), t_1)A_i\right]A_{-1}^{r}$$

$$= -\sum_{r=0}^{j}\left[\Delta X((t_1 - (j-r)h), t_1)A_0 + \Delta X((t_1 - (j - (r+1))h), t_1)A_1\right]A_{-1}^{r}$$

$$= -\sum_{r=0}^{j}\left[\Delta X((t_1 - (j-r)h), t_1)A_0\right]A_{-1}^{r} - \sum_{r=0}^{j}\left[\Delta X((t_1 - (j-r-1)h), t_1)A_0\right]A_{-1}^{r}$$

$$= -\sum_{r=0}^{j}\left[\Delta X((t_1 - (j-r)h), t_1)A_0\right]A_{-1}^{r} - \sum_{r=0}^{j-1}\left[\Delta X((t_1 - (j-r-1)h), t_1)A_1\right]A_{-1}^{r}$$

(since $\Delta X((t_1 - (j-r-1)h, t_1) = 0$, for $r = j$)

$$= -\sum_{r=0}^{j}\left[A_{-1}^{j-r}A_0\right]A_{-1}^{r} - \sum_{r=0}^{j-1}\left[A_{-1}^{j-r-1}A_1\right]A_{-1}^{r} \ \text{(by vii of lemma 2.4 of [1])}$$

$$= -A_{-1}^{j-r}A_0 - \sum_{r=1}^{j}\left[A_{-1}^{j-r}A_0\right]A_{-1}^{r} - \sum_{r=0}^{j-1}\left[A_{-1}^{j-r-1}A_1\right]A_{-1}^{r}$$

$$= -A_{-1}^{j-r}A_0 - \sum_{r=0}^{j-1}\left[A_{-1}^{j-(r+1)}A_0\right]A_{-1}^{r+1} - \sum_{r=0}^{j-1}\left[A_{-1}^{j-r-1}A_1\right]A_{-1}^{r} \ \text{(by change of variables)}$$

$$= -A_{-1}^{j-r}A_0 - \sum_{r=0}^{j-1}\left[A_{-1}^{j-(r+1)}(A_0A_{-1} + A_1)\right]A_{-1}^{r} = -A_{-1}^{j-r}A_0 - \sum_{r=0}^{j-1}A_{-1}^{r}(A_0A_{-1} + A_1)A_{-1}^{j-(r+1)} = Q_1(jh)$$

(by change of variables $\tilde{r} = j - (r+1)$ and then by iv, lemma 2.4 of [1])

Therefore, the theorem is true for $k \in \{0, 1\}$.

The rest of the proof is by induction on

Assume that the theorem is valid for $2 \le k \le p,$ for some integer $p,$ and for all j such that $t_1 - jh > 0.$

k. Then, $\Delta X^{(p+1)}((t_1 - jh), t_1) = -\sum_{r=0}^{j}\left[\sum_{i=0}^{1}\Delta X^{(p)}((t_1-(j-(r+i))h), t_1)A_i\right]A_{-1}^r,$

(by lemma 2.7 of [1]).

$$= -\sum_{r=0}^{j}\left[\sum_{i=0}^{1}(-1)^p Q_p([j-(r+i)]h)A_i\right]A_{-1}^r \quad \text{(by the induction hypothesis)}$$

$$= -\sum_{r=0}^{j}\left[\sum_{i=0}^{1}(-1)^p Q_p([j-(r+i)]h)A_i\right]A_{-1}^r = (-1)^{p+1}\sum_{r=0}^{j}\left[Q_p([j-r]h)A_0 + Q_p([j-r-1]h)A_1\right]A_{-1}^r$$

$$= (-1)^{p+1}\sum_{r=0}^{j}\left[Q_{p+1}([j-r]h) - Q_{p+1}([j-r-1]h)A_{-1}\right]A_{-1}^r \text{ (by lemma 2.7 of [1])}$$

$$= +(-1)^{p+1}\sum_{r=0}^{j}\left[Q_p([j-r]h)A_0 + Q_p([j-r-1]h)A_1\right]A_{-1}^r$$

$$= (-1)^{p+1}\sum_{r=0}^{j}\left[Q_{p+1}([j-r]h) - Q_{p+1}([j-r-1]h)A_{-1}\right]A_{-1}^r \text{ (by lemma 2.7 of [1])}$$

Now we proceed to obtain the above sum by writing out the equivalent exp ressions for each r and then summing the equivalents:

$r = 0 \Rightarrow \; Q_p(jh)A_0 + Q_p([j-1]h)A_1 = Q_{p+1}(jh) - Q_{p+1}([j-1]h)A_{-1}$

$r = 1 \Rightarrow \; (Q_p([j-1]h)A_0 + Q_p([j-2]h)A_1)A_{-1} = Q_{p+1}([j-1]h)A_{-1} - Q_{p+1}([j-2]h)A_{-1}^2\,)$

$r = 2 \Rightarrow \; (Q_p([j-2]h)A_0 + Q_p([j-3]h)A_1)A_{-1}^2 = Q_{p+1}([j-2]h)A_{-1}^2 - Q_{p+1}([j-3]h)A_{-1}^2$

$r = 3 \Rightarrow \; (Q_p([j-3]h)A_0 + Q_p([j-4]h)A_1)A_{-1}^3 = Q_{p+1}([j-3]h)A_{-1}^3 - Q_{p+1}([j-4]h)A_{-1}^3$

$\cdots$

The process continues up to $r = j$, yielding

$r = j-1 \Rightarrow \; (Q_p([j-(j-1)]h)A_0 + Q_p([j-j]h)A_1)A_{-1}^{j-1}$
$\qquad = Q_{p+1}([j-(j-1)]h)A_{-1}^{j-1} - Q_{p+1}([j-j]h)A_{-1}^{j}$

$r = j \; \Rightarrow \; (Q_p([j-j]h)A_0 + Q_p([j-(j+1)]h)A_1)A_{-1}^{j}$
$\qquad = Q_{p+1}([j-j)]h)A_{-1}^{j} - Q_{p+1}([j-(j+1)]h)A_{-1}^{j+1}$

Adding up the terms on the right-hand side for $r = 0, 1, 2, \cdots, j$, it follows that only the first term corresponding to $r = 0$ and the last term corresponding to $r = j$ survive the summation; all other terms cancel out. Therefore:

3.2 Corollary to theorem 3.1

Let $\psi(c,\tau) = c^T X(t,\tau), c \in \mathbf{R}^n$. Let $\psi^{(k)}(c,\tau) = \psi^{(k)}(c,\tau^-) - \psi^{(k)}(c,\tau^+)$ for $\tau \in (0,\infty),$ where $\psi^{(k)}(c,\tau) \; = \; \frac{\partial^k}{\partial \tau^k}\psi(c,\tau).$ Then:

$$\Delta\psi^{(k)}\left(c, t_1 - jh\right) \; = \; \left(-1\right)^k c^{\mathrm{T}} Q_k\left(jh\right)B, \qquad (11)$$

for $j : t_1 - jh \; > \; 0, \; k = 0, 1, \ldots$

Proof

The proof is immediate by noting that

$$\Delta\psi^{(k)}(c,\tau)=c^T\Delta X^{(k)}(\tau,t)B. \text{ Hence } \Delta\psi^{(k)}\left(c,t_1-jh\right)=\left(-1\right)^k c^{\mathrm{T}}Q_k\left(jh\right)B,$$

for $j:t_1-jh\ >\ 0,k=0,1,...$ (by the preceeding theorem).

The following sequence of lemmas is needed in the proof of theorem 3.6

3.3First Corollary to Eq. 9: Expressing the partials of $\left(\sum_{i=0}^{2}\mu_i A_i\right)^k$ in permutation form

Let j,k,r be any nonnegative integers, $j,\ k$ fixed such that $j\ge r,$ and $k\ge r.$ Then

(a) $$\frac{\partial^{k+r}}{\partial\mu_{-1}^{r}\partial\mu_0^{r+k-j}\partial\mu_1^{j-r}}\left[\sum_{i=-1}^{1}\mu_i A_i\right]^{k+r}$$
$$=r!(r+k-j)!(j-r)!\sum_{(v_1,\ldots v_{k+r})\in P_{-1(r),0(r+k-j),1(j-r)}}A_{v_1}\ldots A_{v_{k+r}}\ \text{ if } j\le k,$$

(b) $$\frac{\partial^{j+r}}{\partial\mu_{-1}^{r+j-k}\partial\mu_0^{r}\partial\mu_1^{k-r}}\left[\sum_{i=-1}^{1}\mu_i A_i\right]^{j+r}$$
$$=(r+j-k)!r!(k-r)!\sum_{(v_1,\ldots v_{j+r})\in P_{-1(r+j-k),0(r),1(k-r)}}A_{v_1}\ldots A_{v_{j+r}}\ \text{ if } j\ge k.$$

Proof

(a) And (b) follow from (9) with $i=2$ replaced by $i=-1$ and noting that the superscript triples $r,r+k-j,\ j-r;\ r+j-k,r,k-r$ are all nonnegative and therefore feasible. Moreover they are consistent with (9) as they sum to $k+r$ and $j+r$ in (a) and (b) respectively. This completes the proof.

From above we have the following relations:

$$\frac{1}{r!(r+k-j)!(j-r)!}\frac{\partial^{k+r}}{\partial\mu_{-1}^{r}\partial\mu_0^{r+k-j}\partial\mu_1^{j-r}}\left[\sum_{i=-1}^{1}\mu_i A_i\right]^{k+r}=\sum_{(v_1,\ldots v_{k+r})\in P_{-1(r),0(r+k-j),1(j-r)}}A_{v_1}\ldots A_{v_{k+r}} \quad (12)$$

$$\frac{1}{(r+j-k)!r!(k-r)!}\frac{\partial^{j+r}}{\partial\mu_{-1}^{r+j-k}\partial\mu_0^{r}\partial\mu_1^{k-r}}\left[\sum_{i=-1}^{1}\mu_i A_i\right]^{j+r}=\sum_{(v_1,\ldots v_{j+r})\in P_{-1(r+j-k),0(r),1(k-r)}}A_{v_1}\ldots A_{v_{j+r}} \quad (13)$$

Now sum over $r\in\{1,\cdots,j-1\}$ in (12) to get:

$$\sum_{r=1}^{j-1}\frac{1}{r!(r+k-j)!(j-r)!}\frac{\partial^{k+r}}{\partial\mu_{-1}^{r}\partial\mu_0^{r+k-j}\partial\mu_1^{j-r}}\left[\sum_{i=-1}^{1}\mu_i A_i\right]^{k+r}$$
$$=\sum_{r=1}^{j-1}\sum_{(v_1,\ldots v_{k+r})\in P_{-1(r),0(r+k-j),1(j-r)}}A_{v_1}\ldots A_{v_{k+r}} \quad (14)$$

Now sum over $r\in\{1,\cdots,k-1\},$ in (13) to get:

$$\sum_{r=1}^{k-1}\frac{1}{(r+j-k)!r!(k-r)!}\frac{\partial^{j+r}}{\partial\mu_{-1}^{r+j-k}\partial\mu_0^{r}\partial\mu_1^{k-r}}\left[\sum_{i=1}^{1}\mu_i A_i\right]^{j+r}$$
$$=\sum_{r=1}^{k-1}\sum_{(v_1,\ldots v_{j+r})\in P_{-1(r+j-k),0(r),1(k-r)}}A_{v_1}\ldots A_{v_{j+r}} \quad (15)$$

Therefore, we have proved the following lemma.

3.4 Lemma expressing components of $Q_k(jh)$ as a sum of partial derivatives of $\left(\sum_{i=0}^{2}\mu_i A_i\right)^k$

Let $j,\ k,r$ be any nonnegative integers, j and k fixed such that $j \geq r$ and $k \geq r,\ j+k \neq 0$. Then:

(a) $$\sum_{r=1}^{j-1}\frac{1}{r!(r+k-j)!(j-r)!}\frac{\partial^{k+r}}{\partial\mu_{-1}^{r}\partial\mu_0^{r+k-j}\partial\mu_1^{j-r}}\left[\sum_{i=1}^{1}\mu_i A_i\right]^{k+r}$$
$$= \sum_{r=1}^{j-1}\sum_{(v_1,\ldots v_{k+r})\in P_{-1(r),0(r+k-j),1(j-r)}} A_{v_1}\ldots A_{v_{k+r}}\ ;k\geq j \qquad (16)$$

(b) $$\sum_{r=1}^{k-1}\frac{1}{(r+j-k)!r!(k-r)!}\frac{\partial^{j+r}}{\partial\mu_{-1}^{r+j-k}\partial\mu_0^{r}\partial\mu_1^{k-r}}\left[\sum_{i=1}^{1}\mu_i A_i\right]^{j+r}$$
$$= \sum_{r=1}^{k-1}\sum_{(v_1,\ldots v_{j+r})\in P_{-1(r+j-k),0(r),1(k-r)}} A_{v_1}\ldots A_{v_{j+r}};\ j\geq k \qquad (17)$$

(c) $$\frac{1}{j!k!}\frac{\partial^{j+k}}{\partial\mu_{-1}^{j}\partial\mu_0^{k}}\left[\sum_{i=-1}^{0}\mu_i A_i\right]^{j+k} = \sum_{(v_1,\ldots v_{j+k})\in P_{-1(j),0(k)}} A_{v_1}\ldots A_{v_{j+k}} \qquad (18)$$

(d) $$\frac{1}{(k-j)!j!}\frac{\partial^{k}}{\partial\mu_0^{k-j}\partial\mu_1^{j}}\left[\sum_{i=0}^{1}\mu_i A_i\right]^{k} = \sum_{(v_1,\ldots v_k)\in P_{0(k-j),1(j)}} A_{v_1}\ldots A_{v_k},\quad \text{if } k\geq j \qquad (19)$$

(e) $$\frac{1}{(j-k)!k!}\frac{\partial^{j}}{\partial\mu_{-1}^{j-k}\partial\mu_1^{j}}\left[\sum_{i\in\{-1,1\}}\mu_i A_i\right]^{j} = \sum_{(v_1,\ldots v_k)\in P_{-1(j-k),1(k)}} A_{v_1}\ldots A_{v_j},\ \text{if}\quad j\geq k \qquad (20)$$

Proof
Analogous to the proof of corollary 3.6.9 of [10], the superscripts are all feasible and consistent with (9); consequently the lemma is proved. Note that $\min\{j,k\}\geq 2$ for explicit computational feasibility of (a) and (b). Further, the following result is needed to achieve our objective:

3.5 Lemma relating $\left(\sum_{i\in\{i_1,i_2\}}\mu_i A_i\right)^k$ to $\left(\sum_{i=-1}^{1}\mu_i A_i\right)^k; i_1,i_2\in\{-1,0,1\}, i_1<i_2$ in triple summations

(a) $$\left(\sum_{i=-1}^{0}\mu_i A_i\right)^k = \left(\sum_{i=-1}^{1}\mu_i A_i\right)^k\Bigg|_{\mu_1=0}$$
$$= \sum_{\tilde{j}=0}^{2k}\ \sum_{r=0}^{\left[\left[\frac{2k-\tilde{j}}{2}\right]\right]} \mu_{-1}^{r+\tilde{j}-k}\mu_0^{r}\mu_1^{2k-\tilde{j}-2r}\sum_{(v_1,\cdots v_k)\in P_{-1(r+\tilde{j}-k),0(r),1(2k-\tilde{j}-2r)}} A_{v_1}\cdots A_{v_k}\Bigg|_{\mu_1=0},\ k\geq 1$$

(b) $$\left(\sum_{i=0}^{1}\mu_i A_i\right)^k = \left(\sum_{i=-1}^{1}\mu_i A_i\right)^k\Bigg|_{\mu_{-1}=0}$$
$$= \sum_{\tilde{j}=0}^{2k}\ \sum_{r=0}^{\left[\left[\frac{2k-\tilde{j}}{2}\right]\right]} \mu_{-1}^{r+\tilde{j}-k}\mu_0^{r}\mu_1^{2k-\tilde{j}-2r}\sum_{(v_1,\cdots v_k)\in P_{-1(r+\tilde{j}-k),0(r),1(2k-\tilde{j}-2r)}} A_{v_1}\cdots A_{v_k}\Bigg|_{\mu_{-1}=0},\ k\geq 1$$

(c) $$\left(\sum_{i\in\{-1,1\}}\mu_i A_i\right)^k = \left(\sum_{i=-1}^{1}\mu_i A_i\right)^k\Bigg|_{\mu_0=0}$$

$$= \sum_{\tilde{j}=0}^{2k}\ \sum_{r=0}^{\left[\left[\frac{2k-\tilde{j}}{2}\right]\right]} \mu_{-1}^{r+\tilde{j}-k}\mu_0^{r}\mu_1^{2k-\tilde{j}-2r} \sum_{(v_1,\cdots v_k)\in P_{-1(r+\tilde{j}-k),\,0(r),\,1(2k-\tilde{j}-2r)}} A_{v_1}\cdots A_{v_k}\Bigg|_{\mu_0=0}, \quad k\ge 1$$

Proof

First, note from theorem 3.2 of [11] and theorem 3.6.8 of [10], that $\left(\sum_{i=-1}^{1}\mu_i A_i\right)^k$ equals the right-hand side without the evaluations at $u_i=0$, if $'i=2'$ is replaced by $'i=-1'$ and j is replaced by $\tilde{j}$.

(a) The proof is immediate, observing that $'\mu_1=0'$ annihilates all the terms containing $\mu_1 A_1$, thereby yielding the equivalent expression for $\left(\sum_{i=-1}^{0}\mu_i A_i\right)^k$;

(b) $'\mu_{-1}=0'$ zeroes out all the terms containing $\mu_{-1}A_{-1}$, thereby yielding the equivalent expression for $\left(\sum_{i=0}^{1}\mu_i A_i\right)^k$;

(c) $'\mu_0=0'$ eliminates all the terms containing $\mu_0 A_0$, thereby yielding the equivalent expression for $\left(\sum_{i\in\{-1,1\}}\mu_i A_i\right)^k$.

In the sequel, set $$F=\sum_{i=-1}^{1}\mu_i A_i$$

By the generalized Cayley-Hamilton theorem,

$$F^{n+l}=\sum_{k=0}^{n-1}\alpha_k(\mu)\left(\sum_{i=-1}^{1}\mu_i A_i\right)^k,$$

where the $\alpha_k(\mu)s$ are polynomials of degree $n+l-k$ with respect to the $\mu_i s; \mu=(\mu_{-1},\mu_0,\mu_1)$.

The stage is now set to prove the equality of ranks of some concatenated determining matrices for finite and infinite horizons, using lemmas 3.4, 3.5 and the generalized Cayley-Hamilton theorem.

3.6 Theorem on Rank Equality of some concatenated determining matrices

Let:

$$\hat{Q}_n(t_1)=\left[Q_0(s)B, Q_1(s)B,\cdots,Q_{n-1}(s)B : s\in[0,\, t_1), s=0,h,\cdots,(n-1)h\right], \qquad (21)$$

where $Q_k(s)$ is a determining matrix for the free part of (1) and is defined by (7)

Then:

$$\operatorname{rank}\left[\hat{Q}_\infty(t_1)\right]=\operatorname{rank}\left[\hat{Q}_n(t_1)\right] \qquad (22)$$

In the sequel, set $$F=\sum_{i=-1}^{1}\mu_i A_i$$

Generalized Cayley-Hamilton theorem, Lew (1966, pp. 650-3):

$$F^{n+l}=\sum_{k=0}^{n-1}\alpha_k(\mu)\left(\sum_{i=-1}^{1}\mu_i A_i\right)^k,$$

where the $\alpha_k(\mu)s$ are polynomials of degree $n+l-k$ with respect to the $\mu_i s; \mu=(\mu_{-1},\mu_0,\mu_1)$.
By lemma 3.4:

$$\frac{1}{j!k!}\frac{\partial^{j+k}}{\partial\mu_{-1}^{j}\partial\mu_0^{k}}\left[\sum_{i=-1}^{0}\mu_i A_i\right]^{j+k}+\frac{1}{(j-k)!k!}\frac{\partial^{j}}{\partial\mu_0^{j-k}\partial\mu_1^{k}}\left[\sum_{i\in\{-1,1\}}\mu_i A_i\right]^{j} \qquad (23)$$

$$+\sum_{r=1}^{k-1}\frac{1}{(r+j-k)!r!(k-r)!}\frac{\partial^{j+r}}{\partial\mu_{-1}^{r+j-k}\partial\mu_0^{r}\partial\mu_1^{k-r}}\left[\sum_{i=1}^{1}\mu_i A_i\right]^{j+r}$$

$$=\sum_{(v_1,\ldots v_{j+k})\in P_{-1(j),0(k)}}A_{v_1}\ldots A_{v_{j+k}}+\sum_{(v_1,\ldots v_j)\in P_{-1(j-k),1(k)}}A_{v_1}\ldots A_{v_j}+\sum_{r=1}^{k-1}\sum_{(v_1,\ldots v_{j+r})\in P_{-1(r+j-k),0(r),1(k-r)}}A_{v_1}\ldots A_{v_{j+r}}$$

$$=Q_k(jh);\quad j\geq k \qquad (24)$$

Also:

$$\frac{1}{j!k!}\frac{\partial^{j+k}}{\partial\mu_{-1}^{j}\partial\mu_0^{k}}\left[\sum_{i=-1}^{0}\mu_i A_i\right]^{j+k}+\frac{1}{(k-j)!j!}\frac{\partial^{k}}{\partial\mu_0^{k-j}\partial\mu_1^{j}}\left[\sum_{i=0}^{1}\mu_i A_i\right]^{k} \qquad (25)$$

$$+\sum_{r=1}^{j-1}\frac{1}{r!(r+k-j)!(j-r)!}\frac{\partial^{k+r}}{\partial\mu_{-1}^{r}\partial\mu_0^{r+k-j}\partial\mu_1^{j-r}}\left[\sum_{i=1}^{1}\mu_i A_i\right]^{k+r}$$

$$=\sum_{(v_1,\ldots v_{j+k})\in P_{-1(j),0(k)}}A_{v_1}\ldots A_{v_{j+k}}+\sum_{(v_1,\ldots v_k)\in P_{0(k-j),1(j)}}A_{v_1}\ldots A_{v_k}+\sum_{r=1}^{j-1}\sum_{(v_1,\ldots v_{k+r})\in P_{-1(r),0(r+k-j),1(j-r)}}A_{v_1}\ldots A_{v_{k+r}}$$

$$=Q_k(jh);\quad k\geq j \qquad (26)$$

Hence for every non-negative integer p, we have:

$$Q_{n+p}(jh)=\frac{1}{j!(n+p)!}\frac{\partial^{n+p+j}}{\partial\mu_{-1}^{j}\partial\mu_0^{n+p}}\left[\sum_{i=-1}^{0}\mu_i A_i\right]^{n+p+j} \qquad (27)$$

$$+\frac{1}{(j-[n+p])![n+p]!}\frac{\partial^{j}}{\partial\mu_0^{j-[n+p]}\partial\mu_1^{n+p}}\left[\sum_{i\in\{-1,1\}}\mu_i A_i\right]^{j}$$

$$+\sum_{r=1}^{n+p-1}\frac{1}{(r+j-[n+p])!r!(n+p-r)!}\frac{\partial^{j+r}}{\partial\mu_{-1}^{r+j-[n+p]}\partial\mu_0^{r}\partial\mu_1^{n+p-r}}\left[\sum_{i=1}^{1}\mu_i A_i\right]^{j+r}$$

$$=\sum_{(v_1,\ldots v_{j+n+p})\in P_{-1(j),0(n+p)}}A_{v_1}\ldots A_{v_{j+n+p}}+\sum_{(v_1,\ldots v_j)\in P_{-1(j-[n+p]),1(n+p)}}A_{v_1}\ldots A_{v_j}$$

$$+\sum_{r=1}^{n+p-1}\sum_{(v_1,\ldots v_{j+r})\in P_{-1(r+j-[n+p]),0(r),1(n+p-r)}}A_{v_1}\ldots A_{v_{j+r}},\ \text{if}\quad j\geq n+p, \qquad (28)$$

$$Q_{n+p}(jh)=\frac{1}{j!(n+p)!}\frac{\partial^{n+p+j}}{\partial\mu_{-1}^{j}\partial\mu_{0}^{n+p}}\left[\sum_{i=-1}^{0}\mu_i A_i\right]^{n+p+j}+\frac{1}{(n+p-j)!j!}\frac{\partial^{j}}{\partial\mu_{0}^{n+p-j}\partial\mu_{1}^{j}}\left[\sum_{i=0}^{1}\mu_i A_i\right]^{n+p} \quad (29)$$

$$+\sum_{r=1}^{j-1}\frac{1}{(r!(r+n+p-j)!(j-r)!}\frac{\partial^{n+p-r}}{\partial\mu_{-1}^{r}\partial\mu_{0}^{r+n+p-j}\partial\mu_{1}^{j-r}}\left[\sum_{i=1}^{1}\mu_i A_i\right]^{n+p+r}$$

$$=\sum_{(v_1,\dots v_{j+n+p})\in P_{-1(j),0(n+p)}}A_{v_1}\dots A_{v_{j+n+p}}+\sum_{(v_1,\dots v_j)\in P_{0(n+p-j),1(j)}}A_{v_1}\dots A_{v_k}$$

$$+\sum_{r=1}^{j-1}\sum_{(v_1,\dots v_{k+r})\in P_{-1(r),0(r+n+p-j),1(j-r)}}A_{v_1}\dots A_{v_{k+r}},\quad \text{if } n+p\ge j. \quad (30)$$

Now, using the generalized Cayley-Hamilton theorem in the same spirit as in theorem 3.6, we can prove that $\operatorname{rank}[\hat{Q}_\infty(t_1)]=\operatorname{rank}[\hat{Q}_k(t_1)].$ Indeed, by the generalized Cayley- Hamilton theorem:

$$\left(\sum_{i=-1}^{0}\mu_i A_i\right)^{n+p+j}=\sum_{k=0}^{n-1}\gamma_k(\mu)\left(\sum_{i=-1}^{0}\mu_i A_i\right)^{k},\ \left(\sum_{i=0}^{1}\mu_i A_i\right)^{n+p}=\sum_{k=0}^{n-1}\lambda_k(\mu)\left(\sum_{i=0}^{1}\mu_i A_i\right)^{k} \quad (31)$$

$$\left(\sum_{i=-1}^{1}\mu_i A_i\right)^{n+p+r}=\sum_{k=0}^{n-1}\xi_k(\mu)\left(\sum_{i=-1}^{1}\mu_i A_i\right)^{k} \quad (32)$$

for some polynomials $\gamma_k(\mu),\ \lambda_k(\mu),\ \xi_k(\mu)$ of degrees $n+p+j-k,\ n+p-k,\ n+p+r-k$ respectively, if $n+p\ \ge\ j$.

Hence:

$$Q_{n+p}(jh)$$

$$=\frac{1}{j!(n+p)!}\sum_{k=0}^{n-1}\frac{\partial^{n+p+j}}{\partial\mu_{-1}^{j}\partial\mu_{0}^{n+p}}\gamma_k(\mu)\left[\sum_{i=-1}^{0}\mu_i A_i\right]^{k}$$

$$+\frac{1}{(n+p-j)!\,j!}\sum_{k=0}^{n-1}\frac{\partial^{n+p}}{\partial\mu_{0}^{n+p-j}\partial\mu_{1}^{j}}\lambda_k(\mu)\left[\sum_{i=0}^{1}\mu_i A_i\right]^{k}$$

$$+\sum_{k=0}^{n-1}\sum_{r=1}^{j-1}\frac{1}{(r!(r+n+p-j)!(j-r)!}\frac{\partial^{n+p+r}}{\partial\mu_{-1}^{r}\partial\mu_{0}^{r+n+p-j}\partial\mu_{1}^{j-r}}\xi_k(\mu)\left[\sum_{i=1}^{1}\mu_i A_i\right]^{k} \quad (33)$$

if $n+p\ \ge\ j$.

By lemma3.5,

$Q_{n+p}(jh)$

$$= \frac{1}{j!(n+p)!}\sum_{k=0}^{n-1}\frac{\partial^{n+p+j}}{\partial\mu_{-1}^{j}\partial\mu_0^{n+p}}\gamma_k(\mu)\left(\left[\sum_{\tilde{j}=0}^{2k}\sum_{\tilde{r}=0}^{\left[\left[\frac{2k-\tilde{j}}{2}\right]\right]}\mu_{-1}^{\tilde{r}+\tilde{j}-k}\mu_0^{\tilde{r}}\mu_1^{2k-\tilde{j}-2\tilde{r}} * \sum_{(v_1,\cdots,v_k)\in P_{-1(\tilde{r}+\tilde{j}-k),0(\tilde{r}),1(2k-\tilde{j}-2\tilde{r})}} A_{v_1}\cdots A_{v_k}\right]\Bigg|_{\mu_1=0}\right) \quad (34)$$

$$+\frac{1}{(n+p-j)!\,j!}\sum_{k=0}^{n-1}\frac{\partial^{n+p}}{\partial\mu_0^{n+p-j}\partial\mu_1^{j}}\lambda_k(\mu))\left(\left[\sum_{\tilde{j}=0}^{2k}\sum_{\tilde{r}=0}^{\left[\left[\frac{2k-\tilde{j}}{2}\right]\right]}\mu_{-1}^{\tilde{r}+\tilde{j}-k}\mu_0^{\tilde{r}}\mu_1^{2k-\tilde{j}-2\tilde{r}}\sum_{(v_1,\cdots,v_k)\in P_{-1(\tilde{r}+\tilde{j}-k),0(\tilde{r}),1(2k-\tilde{j}-2\tilde{r})}} A_{v_1}\cdots A_{v_k}\right]\Bigg|_{\mu_{-1}=0}\right)$$

$$+\sum_{k=0}^{n-1}\sum_{r=1}^{j-1}\frac{1}{(r!(r+n+p-j)!(j-r)!}\frac{\partial^{n+p+r}}{\partial\mu_{-1}^{r}\partial\mu_0^{r+n+p-j}\partial\mu_1^{j-r}}\left[\xi_k(\mu)\sum_{\tilde{j}=0}^{2k}\sum_{\tilde{r}=0}^{\left[\left[\frac{2k-\tilde{j}}{2}\right]\right]}\mu_{-1}^{\tilde{r}+\tilde{j}-k}\mu_0^{\tilde{r}}\mu_1^{2k-\tilde{j}-2\tilde{r}} * \sum_{(v_1,\cdots v_k)\in P_{-1(\tilde{r}+\tilde{j}-k),0(\tilde{r}),1(2k-\tilde{j}-2\tilde{r})}} A_{v_1}\cdots A_{v_k}\right]$$

if $n+p \geq j.$

$$Q_{n+p}(jh) = \frac{1}{j!(n+p)!}\frac{\partial^{n+p+j}}{\partial\mu_{-1}^{j}\partial\mu_{0}^{n+p}}\left[\sum_{k=0}^{n-1}\gamma_k(\mu)\sum_{\tilde{j}=0}^{2k}\sum_{\tilde{r}=0}^{\left[\left[\frac{2k-\tilde{j}}{2}\right]\right]}\mu_{-1}^{\tilde{r}+\tilde{j}-k}\mu_0^{\tilde{r}}\mu_1^{2k-\tilde{j}-2\tilde{r}} * \sum_{(v_1,\cdots v_k)\in P_{-1(\tilde{r}+\tilde{j}-k),0(\tilde{r}),1(2k-\tilde{j}-2\tilde{r})}} A_{v_1}\cdots A_{v_k}\right]\Bigg|_{\mu_1=0}$$

$$+\frac{1}{(n+p-j)!\,j!}\frac{\partial^{n+p}}{\partial\mu_0^{n+p-j}\partial\mu_1^{j}}\left[\sum_{k=0}^{n-1}\lambda_k(\mu)\sum_{\tilde{j}=0}^{2k}\sum_{\tilde{r}=0}^{\left[\left[\frac{2k-\tilde{j}}{2}\right]\right]}\mu_{-1}^{\tilde{r}+\tilde{j}-k}\mu_0^{\tilde{r}}\mu_1^{2k-\tilde{j}-2\tilde{r}}\left[\sum_{(v_1,\cdots,v_k)\in P_{-1(\tilde{r}+\tilde{j}-k),0(\tilde{r}),1(2k-\tilde{j}-2\tilde{r})}} A_{v_1}\cdots A_{v_k}\right]\right]\Bigg|_{\mu_{-1}=0} \quad (35)$$

$$+\sum_{r=1}^{j-1}\frac{1}{(r!(r+n+p-j)!(j-r)!}\frac{\partial^{n+p+r}}{\partial\mu_{-1}^{r}\partial\mu_0^{r+n+p-j}\partial\mu_1^{j-r}}\left[\sum_{k=0}^{n-1}\xi_k(\mu)\sum_{\tilde{j}=0}^{2k}\sum_{\tilde{r}=0}^{\left[\left[\frac{2k-\tilde{j}}{2}\right]\right]}\mu_{-1}^{\tilde{r}+\tilde{j}-k}\mu_0^{\tilde{r}}\mu_1^{2k-\tilde{j}-2\tilde{r}} * \sum_{(v_1,\cdots v_k)\in P_{-1(\tilde{r}+\tilde{j}-k),0(\tilde{r}),1(2k-\tilde{j}-2\tilde{r})}} A_{v_1}\cdots A_{v_k}\right]$$

if $n+p \geq j.$

It follows immediately that the sum of the powers of A_0, A_1 and A_2 in every permutation involving A_0, A_1 and A_2 is at most $n-1$. Consequently $\text{rank}[\hat{Q}_{n+p}(t_1)] = \text{rank}[\hat{Q}_n(t_1)], t_1 > 0$ and for every integer $p \geq 0$, leading to the conclusion that $\text{rank}[\hat{Q}_{n+p}(t_1)] = \text{rank}[\hat{Q}_n(t_1)]$, as desired.

IV CONCLUSION

This paper exploited the results in [1] to establish appropriate and relevant relationships among determining matrices, indices of control systems matrices and systems coefficients with respect to single-delay autonomous linear neutral control systems. In the sequel the paper used these relationships in conjunction with the generalized Caley-Hamilton theorem to prove that the associated controllability matrices for finite and infinite horizons have the same rank. The utility of these results can be appreciated in the proof of necessary and sufficient conditions for the Euclidean controllability of system (1) on the intervals $[0, t_1], 0 < t_1 < \infty;$ this will be discussed in a subsequent paper.

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