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A Skewed Generalized Discrete Laplace Distribution

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ABSTRACT: *The family of discrete Laplace distribution is considered. We introduce a skewed distribution called generalized discrete Laplace distribution which arises as the difference of two independently distributed count variables. Some important characteristic properties of the distribution are discussed. The parameter estimation of the proposed model is addressed. The application of the distribution is illustrated using a real data set on the exchange rate of US Dollar to Indian Rupee.*

KEYWORDS: *Discrete Laplace Distribution, Generalized Laplace Distribution,* ***,*** *Generalized Discrete Laplace Distribution,Financial Modelling.*

I. INTRODUCTION

The Laplace distribution provides an important alternative to the dominance of Gaussian-based stochastic models. The growing popularity of the Laplace based models is due to the properties of sharp peak at the mode, heavier than Gaussian tails, existence of all moments, infinite divisibility, and its representation with exponential distribution. A continuous Laplace distribution with scale parameter $\sigma > 0$ and skewness parameter $k > 0$ is given by Kotz [1] with probability density function (pdf),

$$f(x) = \frac{1}{\sigma}\frac{k}{1+k^2}\begin{cases} e^{-\frac{kx}{\sigma}}, & if x \geq 0 \\ e^{\frac{x}{k\sigma}}, & if x \leq 0 \end{cases}$$

Mathai ([2],[3]) introduced generalized Laplace distribution of the continuous type with the characteristic function

$$\varphi(t) = \frac{1}{(1+\beta^2 t^2)^{\alpha}} \quad , \qquad \alpha > 0 \,, \beta > 0$$

and it has applications in various contexts. It is used to model input-output analysis, growth-decay mechanism, growth of melatonin in human body, formation of solar neutrinos etc.A more general form of the generalized Laplacian distribution is obtained by Kotz [1] with characteristic function

$$\varphi(t) = \frac{e^{i\theta t}}{(1+\frac{1}{2}\sigma^2 t^2 - i\mu t)^{\alpha}}, \quad -\infty < t < \infty; \quad \mu, \theta \in \mathbb{R}; \quad \sigma > 0 \,, \alpha \geq 0$$

Usual count data models, such as Poisson distribution and negative binomial distribution can only cover zero and positive integer values. But discrete distributions defined over $\mathbb{Z}$ (including both positive and negative integers) are rare in literature. Recently, there has been much interest to construct discrete versions of continuous distributions. Using procedures outlined by Kemp [4], a discrete normal distribution can be used to study count data supported on the set of integers {0, ±1, ±2, ...}.

Discrete version of skewed Laplace distribution was developed by Kozubowski and Inusah [5]. The probability mass function (pmf) of a skewed discrete Laplace distribution denoted by $DL(p_1, p_2)$ is

$$P(X = k) = \frac{(1-p_1)(1-p_2)}{1-p_1 p_2}\begin{cases} {p_1}^k & \text{when } k = 0,1,2,\ldots \\ {p_2}^{|k|} & \text{when } k = 0,-1,-2,\ldots \end{cases} \qquad (1.1)$$

$$\text{where, } p_1 = e^{\frac{-k}{\sigma}} \text{ and } p_2 = e^{-\frac{1}{k\sigma}}$$

The characteristic function of $DL(p_1, p_2)$ is

$$\varphi_X(t) = \frac{(1-p_1)(1-p_2)}{(1-p_1e^{it})(1-p_2e^{-it})}, \quad t \in \mathbb{R} \qquad (1.2)$$

As a discrete analogue to classical Laplace distribution, $DL(p_1, p_2)$ variable has the same distribution as the difference of two independently but not identically distributed geometric random variables. (Laplace distribution (Kotz [1]) is defined as the difference of two independently but not identically distributed exponential random variables.)

The discrete Laplace distribution is of primary importance in analysis of uncertainty in hydro climatic systems. Hydro climatic episodes, such as droughts, floods, warm spells, cold spells are commonly quantified in terms of their duration and magnitude. The distributions of positive and negative episodes and their differences are of interest for water resources managers, civil engineers and the insurance industry. The durations are frequently modeled by geometric distribution but their differences gives DL model.

In this paper we introduce a distribution that generalizes the discrete Laplace distribution. This new distribution is referred as the generalized discrete Laplace distribution (GDL). The rest of the paper is organized as follows. In section 2, we develop a model leading to the generalized discrete Laplace distribution. Various distributional properties like mean, variance, moment generating function etc. of the proposed distribution are discussed in section 3. In section 4, the parameter estimation of the model is addressed. In section 5, applications of this distribution are illustrated through a real data set on exchange rate of US Dollar to Indian Rupee. Section 6 gives the conclusion.

II. GENERALIZED DISCRETE LAPLACE DISTRIBUTION

Definition 2.1. An integer valued random variable $Y \in \mathbb{Z}$, is said to follow a generalized discrete Laplace distribution denoted by $GDL(\beta, p_1, p_2)$ if it has the characteristic function,

$$\varphi_Y(t) = \left[\frac{(1-p_1)(1-p_2)}{(1-p_1e^{it})(1-p_2e^{-it})}\right]^{\beta} \qquad 0 < p_1 < 1;\ 0 < p_2 < 1;\ \beta > 0 \qquad (2.1)$$

When $\beta = 1$, it reduces to the characteristic function of discrete Laplace distribution given by (1.2).

Theorem 1. $GDL(\beta, p_1, p_2)$ distribution can defined as the difference of two independently distributed negative binomial (NB) random variables with same dispersion parameter.

Proof: Generalized Laplace distribution of the continuous type arises as the distribution of the difference of two independently distributed gamma random variables (Mathai[10]). As a discrete analogue , here we establishes that $GDL(\beta, p_1, p_2)$ distribution can defined as the difference of two independently distributed negative binomial (NB) random variables with same dispersion parameter.

Let X be a NB random variable with parameters (β, q_1). Then pmf of $NB(\beta, q_1)$ is

$$P(X=k) = \begin{cases} \dfrac{\Gamma(\beta+k)}{\Gamma\beta k!} {q_1}^{\beta}(1-q_1)^k, & k = 0,1,2,\dots \\ 0 & Otherwise \end{cases}$$

$$where \quad 0 < q_1 < 1, \quad \beta > 0$$

where q_1 is the probability of success of an event and k is the number of failures before the β^{th} success. On reparameterizing, in terms of mean μ and the shape parameter or the dispersion parameter β, pmf of $NB(\beta, q_1)$ can be written as

$$P(X=k) = \begin{cases} \dfrac{\Gamma(\beta+k)}{\Gamma\beta k!}\left(1+\dfrac{\mu}{\beta}\right)^{-\beta}\left(\dfrac{\mu}{\mu+\beta}\right)^{k}, & k = 0,1,2,\dots \\ 0 & \text{Otherwise} \end{cases} \qquad \mu > 0, \quad \beta > 0 \qquad (2.2)$$

Then, $\mu = \frac{\beta p_1}{1-p_1}$ and $\text{Variance} = \frac{\beta p_1}{1-p_1} + \frac{\beta {p_1}^2}{(1-p_1)^2}, \quad where\ p_1 = 1 - q_1$

The characteristic function of the NB random variable is

$$\varphi_X(t) = \frac{(1-p_1)^{\beta}}{(1-p_1 e^{it})^{\beta}}, \quad where \ \left|p_1 e^{it}\right| < 1 \ and \quad t \in \mathbb{R}$$

Consider two independent NB random variables X_1 and X_2 with parameters (β_1, q_1) and (β_2, q_2)respectively. Define Y= X_1- X_2 .Then, the characteristic function of Y is

$$\varphi_Y(t) = \varphi_{X_1}(t)\varphi_{X_2}(-t)$$

$$\varphi_Y(t) = \frac{(1-p_1)^{\beta_1}(1-p_2)^{\beta_2}}{(1-p_1 e^{it})^{\beta_1}(1-p_2 e^{-it})^{\beta_2}}, \quad p_1 = 1-q_1 \ and \ p_2 = 1-q_2 \qquad (2.3)$$

When $\beta_1 = \beta_2 = \beta$, (2.3) reduces to

$$\varphi_Y(t) = \left[\frac{(1-p_1)(1-p_2)}{(1-p_1 e^{it})(1-p_2 e^{-it})}\right]^{\beta}$$

which is same as the characteristic function of the $GDL(\beta, p_1, p_2)$ distribution given by (2.1). Hence the theorem.

2.1. Probability mass function of $GDL(\beta, p_1, p_2)$

The probability generating function of $GDL(\beta, p_1, p_2)$ distribution is given by

$$P(s) = \left[\frac{(1-p_1)(1-p_2)}{(1-p_1 s)\left(1-\frac{p_2}{s}\right)}\right]^{\beta} \qquad (2.4)$$

The pmf of the $GDL(\beta, p_1, p_2)$ is

$$P(Y=m) = (1-p_1)^{\beta}(1-p_2)^{\beta}\sum_{k=|m|}^{\infty}\binom{\beta+k-1}{k}\binom{\beta+k-|m|-1}{k-|m|}{p_2}^{k}{p_1}^{k-|m|}, \qquad if m < 0 \qquad (2.5)$$

m = 0, ±1, ±2 , … where $\beta > 0$, $0 < p_1 < 1$, $0 < p_2 < 1$,

In terms of hyper geometric function (2.5) can be expressed as

$$P(Y=m) = \begin{cases} \dfrac{(1-p_1)^{\beta}(1-p_2)^{\beta}{p_2}^{|m|}\Gamma(\beta+|m|)\ 2F_1\{(\beta,\beta+|m|),(1+|m|),p_1p_2\}}{\Gamma\beta\Gamma(1+|m|)} & m < 0 \\ \dfrac{(1-p_1)^{\beta}(1-p_2)^{\beta}{p_1}^{|m|}\Gamma(\beta+|m|)\ 2F_1\{(\beta,\beta+|m|),(1+|m|),p_1p_2\}}{\Gamma\beta\Gamma(1+|m|)}, & m \geq 0 \end{cases}$$

where2 F1(a, b; c; z) is the hypergeometric function with series expansion

$$2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \qquad (a)_k = a(a+1)(a+2)\dots(a+k-1)$$

When p_1=p_2=p, (2.5) reduces to the pmf of the symmetric version of generalized discrete Laplace distribution.

From (3.1) and (3.2) , It is clear that variance is greater than mean for GDL(β, p_1, p_2). Hence the distribution is suitable for modeling over dispersed data.

$$\text{Pearson's coefficient of Skewnes,} \quad \beta_1 = \frac{\mu_3^2}{\mu_2^3}.$$

GDL(β,p_1,p_2) distribution is positively skewed when $p_1 > p_2$ and negatively skewed when $p_1 < p_2$ and is symmetric when $p_1 = p_2$.

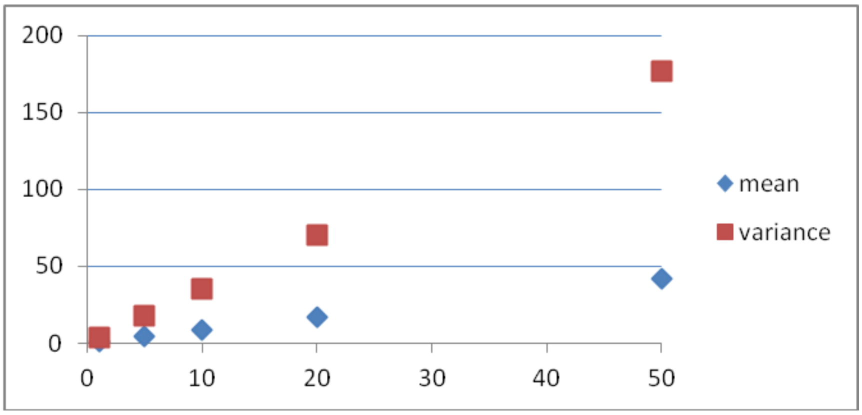

Figure 3.1. Mean and variance graph for different values of β with $p_1 = .4$ and $p_2 = .6$

3.2. Distribution Function

Let Y follows $GDL(\beta, p_1, p_2)$ and $m \leq y < m+1$, where m is an integer Then

$$F(y) = P(Y \leq y) = \sum_{j=-\infty}^{m} P(Y = j)$$

Case I:- when $y < 0$, then m is a negative integer

$$F(y) = \sum_{j=-\infty}^{m} (1-p_1)^{\beta}(1-p_2)^{\beta} \sum_{k=|j|}^{\infty} \binom{\beta+k-1}{k}\binom{\beta+k-|j|-1}{k-|j|} {p_1}^{k-|j|}{p_2}^{k}$$

$$= (1-p_1)^{\beta}(1-p_2)^{\beta} \sum_{j=-\infty}^{m}\sum_{k=|j|}^{\infty} \binom{\beta+k-1}{k}\binom{\beta+k-|j|-1}{k-|j|} {p_1}^{k-|j|}{p_2}^{k}$$

Case II:- When $y \geq 0$ then $m+1$ is a positive integer.

$$F(y) = P(Y \leq y) = 1 - P(Y > y)$$

$$P(y > y) = \sum_{j=m+1}^{\infty} P(y = j)$$

$$F(y) = 1 - \sum_{j=m+1}^{\infty} (1-p_1)^{\beta}(1-p_2)^{\beta} \sum_{k=j}^{\infty} \binom{\beta+k-1}{k}\binom{\beta+k-j-1}{k-j} {p_1}^{k}{p_2}^{k-j}$$

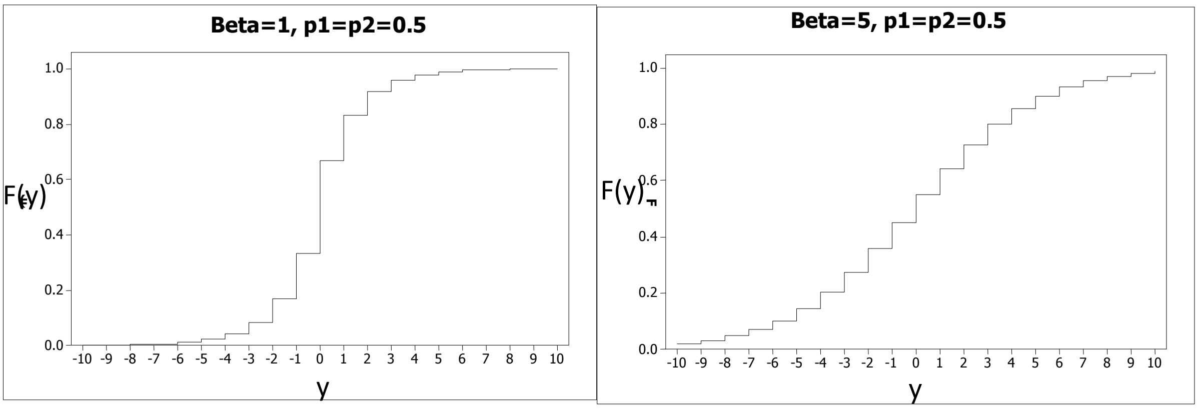

Figure 3.2.Distribution function plot for different values of β and $p_1=p_2=.5$

3.3. Factorial Moments

The first three factorial moments of $GDL(\beta, p_1, p_2)$ is given by

$$F_1 = \beta\vartheta_1 - \beta\vartheta_2 \quad (3.5)$$

$$F_2 = \beta(\beta+1)(\vartheta_1^2 + \vartheta_2^2) - 2\beta^2\vartheta_1\vartheta_2 + 2\beta\vartheta_2 \quad (3.6)$$

$$F_3 = \beta(\beta+1)(\beta+2)(\vartheta_1^3 - \vartheta_2^3) - 3\beta^2(\beta+1)\vartheta_1\vartheta_2(\vartheta_1 - \vartheta_2) + 6\beta^2\vartheta_1\vartheta_2 - 6\beta(\beta+1)\vartheta_2^2 - 6\beta\vartheta_2 \quad (3.7)$$

3.4. Infinite Divisibility

For the $GDL(\beta, p_1, p_2)$ random variable Y,

$$\varphi_{y(t)} = \left[\frac{(1-p_1)(1-p_2)}{(1-p_1 e^{it})(1-p_2 e^{-it})}\right]^{\beta}$$

$$= [\varphi_n(t)]^n$$

where $\varphi_n(t) = \left(\frac{(1-p_1)(1-p_2)}{(1-p_1 e^{it})(1-p_2 e^{-it})}\right)^{\frac{\beta}{n}}$ is the characteristic function of the difference of two independent negative binomial random variables with parameters $\left(\frac{\beta}{n}, p_1\right)$ and $\left(\frac{\beta}{n}, p_2\right)$ respectively.

3.5. Convolution Property

Let $X_1, X_2 \ldots X_n$ be i.i.ddistributed $GDL(\beta, p_1, p_2)$variables. Then consider the characteristic function of $Y = X_1 + X_2 + \cdots + X_n$

$$\varphi_Y(t) = \left[\frac{(1-p_1)(1-p_2)}{(1-p_1 e^{it})(1-p_2 e^{-it})}\right]^{n\beta}$$

which follows $GDL(n\beta, p_1, p_2)$. Thus $GDL(\beta, p_1, p_2)$ distribution is closed under convolution.

IV. ESTIMATION

For the generalized discrete Laplace distribution with parameters β, p_1 and p_2 the estimates can be obtained by the method of moments and method of factorial moments.

i) Method of Moments

The estimates of the parameters of $GDL(\beta, p_1, p_2)$ are given by

$$\hat{\beta} = -\frac{3A^2}{3A^2 + An - Bn}, \quad \text{where} \quad A = \sum_{i=1}^{n} x_i^2, \quad B = \sum_{i=1}^{n} x_i^4 \quad (4.1)$$

$$\widehat{p_1} = \frac{3A^2 - 2An - Bn + \sqrt{3(A^2n^2 + 2ABn^2 - 6nA^3)}}{3A^2 + An - Bn} \quad \text{where } 0 < p_1 < 1 \quad (4.2)$$

By substituting the estimates of β and p_1 in equation (3.1) we get , $\widehat{p_2}$, $0 < p_2 < 1$.

ii) Method of Factorial Moments

In this method, the first three factorial moments of $GDL(\beta, p_1, p_2)$ given by (3.5), (3.6) and (3.7) are equated to the corresponding sample factorial moments m_1, m_2 and m_3 and solving the following system of equations simultaneously, we get the estimates for the parameters.

$$m_1 = \beta\vartheta_1 - \beta\vartheta_2$$

$$m_2 = \beta(\beta+1)(\vartheta_1{}^2 + \vartheta_2{}^2) - 2\beta^2\vartheta_1\vartheta_2 + 2\beta\vartheta_2$$

$$m_3 = \beta(\beta+1)(\beta+2)(\vartheta_1{}^3 - \vartheta_2{}^3) - 3\beta^2(\beta+1)\vartheta_1\vartheta_2(\vartheta_1 - \vartheta_2) + 6\beta^2\vartheta_1\vartheta_2 - 6\beta(\beta+1)\vartheta_2{}^2 - 6\beta\vartheta_2$$

where ϑ_1 and ϑ_2 are given in (3.1).

Remark 1

When the observations X_1 and X_2 are observed, the parameters of the $GDL(\beta, p_1, p_2)$ distribution β, p_1 and p_2 can obtained as usual methods of obtaining parameters from two negative binomial (NB) data sets. Here p_1 is obtained from first data set , p_2 is obtained from second data set and β is the common dispersion parameter of the two NB data set. The maximum likelihood estimation of the parameters of the negative binomial distribution have given by Fisher [6]. Method of moment estimator of negative binomial distribution with mean μ is obtained by solving

$$\hat{\mu} = \frac{\beta p_1}{1 - p_1} = \bar{x} \text{ and } \hat{\beta} = \frac{\bar{x}^2}{s^2 - \bar{x}}$$

where $\bar{x}$ and s^2 are the first and second unbiased sample moments.

IV. APPLICATIONS

Generalized discrete Laplace distribution is useful for the customers to find the reliable and customer friendly insurance companies by analyzing the claim count difference.(Consulting actuaries often use NB distribution to model number of claims or aggregate claim count loss) In medical field, difference in the counts of chromosomes indicates the presence of particular deceases. The difference in the counts of retail outlets before and after a particular periods indicates the development of the country. Stock price change data during the financial transactions of any firm can be analyzed using this distribution.

In this session we consider the application of generalized discrete Laplace distribution in modeling currency exchange rates. Many researchers have studied distributions of currency exchange rates. For details, see, Boothe and Glassman [7], Friedman and Vandersteel [8], Inusah [9], McFarland et al.[10], Kozubowski and Podgorski [11], Tucker and Pond [12]. Their results show that currency exchange changes are increasingly leptokurtic and daily changes have fat tails. Kozubowski and Podgorski [11] obtained a good fit to currency exchange data using skew Laplace distribution. Insuah [9] showed that symmetric discrete Laplace distribution is a reasonable model to the data used in [11].

Here, the data consists of daily currency exchange rates quoted in the Indian Rupee for US Dollar for a period from January 2, 2000 to August 1, 2013. The variables of interest are: X_1 , the duration of time in days when the exchange rate is going up, and X_2, the duration of time in days when the exchange rate is going down , and their difference $X = X_1 - X_2$. Figure 3.1 gives the empirical pmf and theoratical pmf of $GDL(\beta, p_1, p_2)$ distribution for the given data.

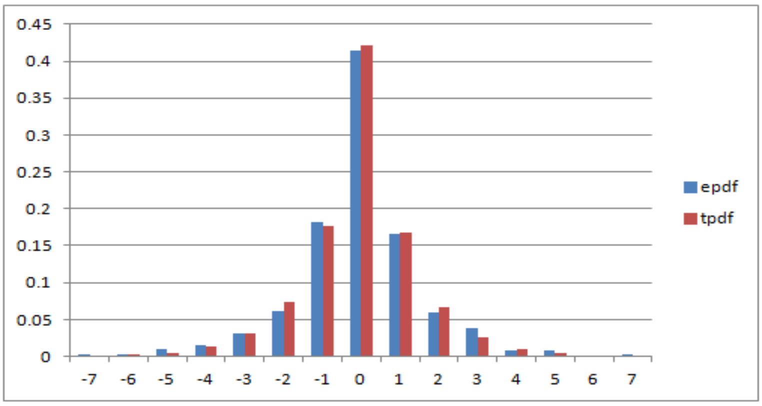

Figure 3.1. The empirical pdf and theoretical pdf of $GDL(\beta, p_1, p_2)$ distribution with $p_1 = 0.3939$, $p_2 = 0.41584$ and $\beta = 1.0054$.

The mean and variance for the data are -0.0624 and 2.6011 respectively. Since the data is over dispersed $GDL(\beta, p_1, p_2)$ distribution is a possible model. The estimates of generalized discrete Laplace parameter is obtained by the method of moments and is given by $p_1 = 0.3939$, $p_2 = 0.41584$ and $\beta = 1.0054$. To test whether there is significant difference between the distribution of observed data and $GDL(\beta, p_1, p_2)$ distribution with parameters $p_1 = 0.3939$, $p_2 = 0.41584$ and $\beta = 1.0054$, we use the Kolmogrov-Smirnov (K.S) test. The calculated value of K.S. test statistic is obtained as 0.015, but the critical value corresponding to the significance level 0.05 is 0.04. Hence we conclude that the generalized discrete Laplace distribution is a suitable model for studying the changes in the exchange rate between U.S. Dollar and Indian Rupee.

V. CONCLUSION

In this paper, we have considered the generalized discrete Laplace distribution and its various distributional properties like mean, variance, moments etc. The suitability of the new model is established by fitting it to a financial data on exchange rate between U.S. Dollar and Indian Rupee from January 2, 2000 to August 1, 2013.

REFERENCES

[1] Kotz, S., Kozubowski, T. J., Podgorski, K.(2001), The Laplace Distribution and Generalizations. A Revisit with Applications to Communications, Economics,Engineering and Finance. Boston: Birkhäuser.
[2] Mathai,A.M.(1993a), On non central generalized Laplacian distribution. J.Statist.Res. 27(1-2):57-80.
[3] Mathai,A.M.(1993b), On noncentral generalized Laplacianness of quadratic forms in normal variables.J.Multivariate Anal. 45(2):239-246.
[4] Kemp, A.W., (1997), Characterizations of a discrete normal distribution. J. Statist. Plann. Inference, 63, 223–229.
[5] Kozubowski, T. J., Inusah, S. (2006), A Skew Laplace Distribution on Integers :Annals of the Institute of Statistical Mathematics, 58, 555-571.
[6] Fisher, R.A. (1941), The Negative Binomial Distribution. Annals of Eugenics 11, pp. 182-187.
[7] Boothe,P.,Glassman,D.(1987),The statistical distribution of exchange rates,J.International Econ.22,297-319.
[8] Friedman,D.,Vandersteel,S.(1982),Short-run fluctuations in foreign exchange rates, J. International Econ.13,171-186.
[9] Inusah (2003), Discrete Laplace Distribution, Unpublished thesis submitted for the degree of science in Mathematics , university of Nevada, Reno.
[10] McFarland,J.W.,Pettit,R.R.,Sung,S.K.,(1982),The distribution of foreign exchange price changes: Trading day effects and risk measurement,J.Finance,37(3),693-715.
[11] Kozubowski,T.J., Podgorski,k.(2000), Asymmetric Laplace laws and modeling financial data,Math.Comput.Modeling 34,1003-1021.
[12] Tucker,A.L.,Pond,L.(1988).Probability distribution of exchange rate changes, Review Econ.Studies,70,638-647.