International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org ǁ Volume 2 ǁ Issue 3 ǁ March 2014 ǁ PP-31-47

The structure of determining matrices for single-delay autonomous linear neutral control systems

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ABSTRACT: *This paper derived and established the structure of determining matrices for single – delay autonomous linear neutral differential systems through a sequence of lemmas, theorems and corollaries and the exploitation of key facts about permutations. The proofs were achieved using ingenious combinations of summation notations, the multinomial distribution,change of variables technique and compositions of signum and max functions. The paper also pointed out some induction pitfalls in the derivation of the main results.*
The paper has extended the results on single–delay models, with more complexity in the structure of the determining matrices.

KEYWORDS: *Delay, Determining, Neutral, Structure, Systems.*

I. INTRODUCTION

The importance of determining matrices stems from the fact that they constitute the optimal instrumentality for the determination of Euclidean controllability and compactness of cores of Euclidean targets. See Gabasov and Kirillova (1976) and Ukwu (1992, 1996 and 2013a). In sharp contrast to determining matrices, the use of indices of control systems on the one hand and the application of controllability Grammians on the other, for the investigation of the Euclidean controllability of systems can at the very best be quite computationally challenging and at the worst, mathematically intractable. Thus, determining matrices are beautiful brides for the interrogation of the controllability disposition of single-delay neutral control systems. See (Ukwu 2013a).

However up-to-date review of literature on this subject reveals that there is currently no correct result on the structure of determining matrices for single-delay neutral systems. This could be attributed to the severe difficulty in identifying recognizable mathematical patterns needed for inductive proof of any claimed result and induction pitfalls in the derivation of determining matrices. This paper establishes valid expressions for the determining matrices in this area of acute research need.

II. ON DETERMINING MATRICES AND CONTROLLABILITY OF SINGLE-DELAY AUTONOMOUS NEUTRAL CONTROL SYSTEMS

We consider the class of neutral systems:

$$\frac{d}{dt}\left[x(t)-A_{-1}x(t-h)\right]=A_0x(t)+A_1x\left(t-h\right)+Bu\left(t\right),\ \ t\geq 0 \qquad (1)$$

where A_{-1}, A_0, A_1 are $n\times n$ constant matrices with real entries and B is an $n\times m$ constant matrix with the real entries. The initial function ϕ is in $C\left(\left[-h,0\right],\mathbf{R}^n\right)$ equipped with sup norm. The control u is in $L_\infty\left(\left[0,t_1\right],\mathbf{R}^n\right)$. Such controls will be called admissible controls. $x\left(t\right), x\left(t-h\right)\in\mathbf{R}^n$ for $t\in\left[0,t_1\right]$. If $x\in C\left(\left[-h,t_1\right],\mathbf{R}^n\right)$, then for $t\in\left[0,t_1\right]$ we define $x_t\in C\left(\left[-h,0\right],\mathbf{R}^n\right)$ by $x_t(s)=x(t+s),\ s\in\left[-h,0\right]$.

2.1 Existence, Uniqueness and Representation of Solutions

If $A_{-1}\neq 0$ and ϕ is continuously differentiable on $\left[-h,0\right]$, then there exists a unique function $x:\left[-h,\infty\right)$ which coincides with ϕ on $\left[-h,0\right]$, is continuously differentiable and satisfies (1) except possibly at the points

$jh;\ j=0,1,2,\ldots$. This solution *x* can have no more derivatives than ϕ and continuously differentiable if and only if the relation:

$$\dot{\phi}(0)=A_{-1}\,\dot{\phi}(-h)+A_0\phi(0)+A_1\,\phi(-h)+Bu(0) \qquad (2)$$

is satisfied. See Bellman and Cooke (1963) and theorem 7.1 in Dauer and Gahl (1977) for complete discussion on existence, uniqueness and representations of solutions of (1).

2.2 Preliminaries on the Partial Derivatives $\frac{\partial^k X(t,\tau)}{\partial \tau^k}, k=0,1,\cdots$

Let $t,\tau\in[0,\ t_1]$ for fixed *t*, let $\tau\to \mathrm{X}(t,\tau)$ be the unique function satisfying the matrix differential equation:

$$\frac{\partial}{\partial\tau}\mathrm{X}(t,\tau)=\frac{\partial}{\partial\tau}\mathrm{X}(t,\tau+h)A_{-1}-\mathrm{X}(t,\tau)A_0-\mathrm{X}(t,\tau+h)A_1 \qquad (3)$$

$0<\tau<t,\ \tau\neq t-jh;\ j=0,1,\ldots$, subject to:

$$\mathrm{X}(t,\tau)=\begin{cases} I_n;\ \tau=t \\ 0,\ \ \tau>t\end{cases} \qquad (4)$$

By Tadmore (1984 , p. 80), $\tau\to\mathrm{X}(t,\tau)$ is analytic on $(t-(j+1)h,\ t-jh),\ j=0,1,\ldots.$ and hence $\tau\to\mathrm{X}(t,\tau)$ is C^∞ on these intervals.

The left and right-hand limits of $\tau\to\mathrm{X}(t,\tau)$ exist at $\tau=t-jh$, so that $\mathrm{X}(t,\tau)$ is of bounded variation on each compact interval. Cf. Banks and Jacobs (1973), Banks and Kent (1972) and Hale (1977). We maintain the notation $\mathrm{X}^{(k)}(t,\tau),\ \Delta\mathrm{X}^{(k)}(t,\tau)$ as in the double-delay systems.

2.3 Investigation of the Expressions and Structure of Determining Matrices for System (1)

In this section we establish the expressions and the structure of the determining matrices for system (1), as well as their relationships with $X^{(k)}(t,\tau)$ through a sequence of lemmas, theorems and corollaries and the exploitation of key facts about permutations.

The process of obtaining necessary and sufficient conditions for the Euclidean controllability of (1) in chapter four will be initiated in the rest of chapter three as follows:

i. Obtaining a workable expression for the determining matrices of (1): $Q_k(jh)$ for $j: t_1-jh>0,\ k=1,2,..$ (5)

ii. Showing that:

$$\Delta\mathrm{X}^{(k)}(t_1-jh,t_1)=(-1)^k\,Q_k(jh) \qquad (6)$$

for $j: t_1-jh>0,\ k=0,1,2,\ldots.,$

iii. Showing that $Q_\infty(t_1)$ is a linear combination of:

$$Q_0(s),Q_1(s),\ldots,Q_{n-1}(s),\ s=0,h,\ldots.,(n-1)h \qquad (7)$$

We now define the determining equation of the $n\times n$ matrices, $Q_k(s)$.

For every integer *k* and real number *s*, define $Q_k(s)$ by:

$$Q_k(s)=A_{-1}Q_k(s-h)+A_0\,Q_{k-1}(s)+A_1\,Q_{k-1}(s-h) \qquad (8)$$

for $k=0,1,\ldots;\ s=0,h,2h,\ldots.$ subject to $Q_0(0)=I_n$, the $n\times n$ identity matrix and $Q_k(s)=0$ for $k<0$ or $s<0$.

2.4 Preliminary Lemma on Determining matrices $Q_k(s), s\in\mathbf{R}$

In (1), assume that $A_{-1}\neq 0$, then for $j,k\geq 1$,

i. $Q_k(0)=A_0^k$

ii. $Q_0\left(jh\right) = A_{-1}^{j}$

iii. $Q_k\left(h\right) = A_0^k\ A_{-1} + \sum_{r=0}^{k-1} A_0^{k-(1+r)}\left(A_1 + A_{-1}\ A_0\right)A_0^r$

iv. $Q_1\left(jh\right) = A_{-1}^j\ A_0 + \sum_{r=0}^{j-1} A_{-1}^r\left(A_0\ A_{-1} + A_1\right)A_{-1}^{j-(1+r)}$

v. $X^{(k)}\left(t_1^-, t_1\right) = \left(-1\right)^k A_0{}^k$

vi. $X^{(k)}\left(t_1^+, t_1\right) = 0$

vii. $\Delta X\left(t_1 - jh, t_1\right) = A_{-1}^j$

Remarks: Notice that A_0^r can switch places with $A_0^{k-(1+r)}$ in (iii); similarly for A_{-1}^r and $A_{-1}^{j-(1+r)}$ in (iv), since *r* and $k-\left(1+r\right)$ have the same range for fixed *k*; similarly in (iv), *r* and $j-\left(1+r\right)$ have the same range for fixed *j*.

Proof

i.From (8) we have

$$Q_k\left(0\right) = A_{-1}Q_k\left(-h\right) + A_0Q_{k-1}\left(0\right) + A_1Q_{k-1}\left(-h\right)$$
$$= 0 + A_0Q_{k-1}\left(0\right) + 0$$

Now, $Q_1\left(0\right) = A_0Q_0\left(0\right) = A_0I_n = A_0$. Assume that $Q_k\left(0\right) = A_0^k$ for some $k > 1$. Then,

$$Q_{k+1}\left(0\right) = A_{-1}\ Q_{k+1}\left(-h\right) + A_0\ Q_k\left(0\right) + A_1\ Q_k\left(-h\right) = 0 + A_1\ Q_k\left(0\right) + 0$$
$$= A_0\ A_0^k\ \left(\text{by the induction hypotheses}\right)$$
$$= A_0^{k+1}$$

Hence, $Q_k\left(0\right) = A_0^k$ for all $k = 1,2,\ldots..$; this proves (i).

ii $\quad Q_0\left(jh\right) = A_{-1}\ Q_0\left(\left(j-1\right)h\right) + A_0\ Q_{-1}\left(jh\right) + A_1\ Q_{-1}\left(\left(j-1\right)h\right)$

$$= A_{-1}\ Q_0\left(\left(j-1\right)h\right) + 0 + 0$$

Hence, $Q_0\left(h\right) = A_{-1}\ Q_0\left(0\right) = A_{-1}$

The rest of the proof is by the principle of mathematical induction: Assume that $Q_0\left(jh\right) = A_{-1}^j$ for some $j > 1$. Then

$$Q_0\left(\left(j+1\right)h\right) = A_{-1}\ Q_0\left(jh\right) + A_0\ Q_{-1}\left(\left(j+1\right)h\right) + A_1\ Q_{-1}\left(\left(j+1\right)h\right),$$
$$= A_{-1}\ Q_0\left(jh\right) + 0 + 0 = A_{-1}\ A_{-1}^j = A_{-1}^{j+1}$$

by induction hypothesis. So $Q_0\left(jh\right) = A_{-1}^j$ for $j = 1, 2, \ldots...$

iii From (8),

$$Q_1(h) = A_{-1}\ Q_1\left(0\right) + A_0\ Q_0\left(h\right) + A_1\ Q_0\left(0\right)$$
$$= A_{-1}A_0 + A_0A_{-1} + A_1.$$

Plugging in the right-hand side of iii yields

$$Q_1(h) = A_0A_{-1} + \sum_{r=0}^{1-1} A_0^{1-(1+r)}\left(A_1 + A_{-1}A_0\right)A_0^r = A_0A_{-1} + A_1 + A_{-1}A_0.$$

Therefore the formula (iii) is valid for $k = 1$.

Assume the validity of (iii) for some $k > 1$. Now

$$Q_{k+1}(h) = A_{-1}Q_{k+1}(0) + A_0 Q_k(k) + A_1 Q_k(0)$$

$$= A_{-1}A_0^{k+1} + A_0 A_0^k A_{-1} + A_0 \sum_{r=0}^{k-1} A_0^{k-(1+r)}\left(A_1 + A_{-1}A_0\right)A_0^r + A_1 A_0^k$$

$$= A_0^{k+1}A_{-1} + A_0 \sum_{r=0}^{k+1} A_0^{k-(1+r)}\left(A_1 + A_{-1}A_0\right)A_0^r + \left(A_1 + A_{-1}A_0\right)A_0^k \quad .$$

$$= A_0^{k+1}A_{-1} + \sum_{r=0}^{k-1} A_0^{k-r}\left(A_1 + A_{-1}A_0\right)A_0^r + A_0^{k-k}\left(A_1 + A_{-1}A_0\right)A_0^k$$

$$= A_0^{k+1}A_{-1} + \sum_{r=0}^{k} A_0^{k-r}\left(A_1 + A_{-1}A_0\right)A_0^r = Q_{k+1}(h)$$

Therefore (iii) is true for $k+1$ and so true for $k = 1, 2, \ldots..$

iv $\quad Q_1(h) = A_0 A_{-1} + A_{-1}A_0 + A_1$.

Plug in the right-hand side of (iv) to obtain

$$A_{-1}A_0 + A_{-1}^0\left(A_0 A_{-1} + A_1\right)A_{-1}^0 = A_{-1}A_0 + A_0 A_{-1} + A_1$$

It follows that (iv) is true for $j = 1.$ Assume that (iv) is valid for some $j > 1$. Now

$$Q_1\left(\left(j+1\right)h\right) = A_{-1}Q_1\left(jh\right) + A_0 Q_0\left(\left(j+1\right)h\right) + A_1 Q_0\left(jh\right)$$

$$= A_{-1}\left[A_{-1}^j A_0 + \sum_{r=0}^{j-1} A_{-1}^r\left(A_0 A_{-1} + A_1\right)A_{-1}^{j-(1+r)}\right]$$

$$+ A_0 A_{-1}^{j+1} + A_1 A_{-1}^j \qquad \text{(by the induction hypothesis)}$$

$$= A_{-1}^{j+1}A_0 + \sum_{r=0}^{j-1} A_{-1}^{r+1}\left(A_0 A_{-1} + A_1\right)A_{-1}^{j-(1+r)} + \left(A_0 A_{-1} + A_1\right)A_{-1}^j$$

$$= A_{-1}^{j+1}A_0 + \sum_{r=1}^{j} A_{-1}^r\left(A_0 A_{-1} + A_1\right)A_{-1}^{j-r} + A_{-1}^r\left(A_0 A_{-1} + A_1\right)A_{-1}^{j-r}\Big|_{r=0}$$

$$= A_{-1}^{j+1}A_0 + \sum_{r=0}^{j} A_{-1}^r\left(A_0 A_{-1} + A_1\right)A_{-1}^{j-r} = Q_1\left(\left(j+1\right)h\right)$$

So, the formula (iv) is valid for $j+1$. This completes the proof that (iv) holds for all $j = 1, 2, \ldots$

v. For $k = 1$,

$$\mathrm{X}^{(k)}\left(t_1^-, t_1\right) = \mathrm{X}^{(1)}\left(t_1^-, t_1\right) = \frac{\partial}{\partial\tau}\mathrm{X}\left(\tau, t_1\right)\Big|_{\tau=t_1^-}$$

$$= -\mathrm{X}\left(t_1^-, t_1\right)A_0 - \mathrm{X}\left(\left(t_1+h\right)^-, t_1\right)A_1 + \frac{\partial}{\partial\tau}\mathrm{X}\left(\tau+h, t_1\right)A_{-1}\Big|_{\tau=t_1^-} = -A_0 - 0.A_1 + \frac{\partial}{\partial\tau}[0]\Bigg|_{\tau=t_1^-} = -A_0$$

by (3), noting that for τ sufficiently close to $t_1, \tau + h > t_1$; so $\left(t_1 + h\right)^- > t_1$.

Assume that $\mathrm{X}^{(k)}\left(t_1^-, t_1\right) = \left(-1\right)^k A_0^k$ for some $k > 1$. Then

$$X^{(k+1)}\left(t_1^-, t_1\right) = \left[\frac{\partial}{\partial\tau}\left(X^{(k)}\left(\tau, t_1\right)\right)\right]_{\tau=t^-}$$

$$= -X^{(k)}(t_1^-, t_1)A_0 - X^{(k)}\left(\left(t_1+h\right)^-, t_1\right)A_1 + X^{(k+1)}\left(\left(t_1+h\right)^-, t_1\right)A_{-1}$$

$$= -\left(-1\right)^k A_0^k A_0 - 0 - 0 = \left(-1\right)^{k+1} A_0^{k+1}$$

Therefore, $X^{(k)}\left(t_1^-, t_1\right) = \left(-A_0\right)^k$ for all $k = 1, 2, \ldots$.

vi. $X^{(k)}\left(t_1^+,t_1\right)=\lim_{\substack{\tau\to t_1\\ t_1<\tau<t_1+h}} X^{(k)}\left(\tau,t_1\right)=0$, since $\tau > t_1$

vii Let j be a nonnegative integer such that $t_1 - jh > 0$. We integrate (3) and apply (4). Thus

$$\int_0^{(t_1-jh)^-}\frac{\partial}{\partial\tau}\left[X\left(\tau,t_1\right)-X\left(\tau+h,t_1\right)A_{-1}\right]d\tau=-\int_0^{(t_1-jh)^-}\left[X\left(\tau,t_1\right)A_0+X\left(\tau+h,t_1\right)A_{-1}\right]d\tau$$

Hence,

$$\mathrm{X}\left(\left(t_1-jh\right)^-,t_1\right)-\mathrm{X}\left(\left(t_1-(j-1)h\right)^-,t_1\right)A_{-1}-\mathrm{X}\left(0,t_1\right)+\mathrm{X}\left(h,t_1\right)A_{-1}$$

$$=-\int_0^{(t_1-jh)^-}\left[\mathrm{X}\left(\tau,t_1\right)A_0+\mathrm{X}\left(\tau+h,t_1\right)A_{-1}\right]d\tau$$

Similarly

$$X\left(\left(t_1-jh\right)^+,t_1\right)-X\left(t_1,\left(t_1-(j-1)h\right)^+,\ t_1\right)A_{-1}-X\left(0,t_1\right)+\mathrm{X}\left(h,t_1\right)A_{-1}$$

$$=-\int_0^{(t_1-jh)^+}\left[\mathrm{X}\left(\tau,t_1\right)A_0+\mathrm{X}\left(\tau+h,t_1\right)A_{-1}\right]d\tau$$

Therefore

$$\left[X\left(\left(t_1-jh\right)^-,t_1\right)-X\left(\left(t_1-(j-1)h\right)^-,t_1\right)A_{-1}\right]-\left[X\left(\left(t_1-jh\right)^+,t_1\right)-X\left(\left(t_1-(j-1)h\right)^+,t_1\right)A_{-1}\right]$$

$$=\int_{(t_1-jh)^-}^{(t_1-jh)^+}\left[X\left(\tau,t_1\right)A_0+X\left(\tau+h,t_1\right)A_{-1}\right]d\tau=0,$$

since $\lim_{\varepsilon\to 0}\int_{a-\varepsilon}^{a+\varepsilon} f\left(t\right)dt=0$, for any bounded integrable function *f*, and $\tau\to\mathrm{X}\left(\tau,t_1\right)A_0+\mathrm{X}\left(\tau+h,t_1\right)A_1$ is bounded and integrable. Therefore we have deduced that

$$X\left(\left(t_1-jh\right)^-,t_1\right)-X\left(\left(t_1-jh\right)^+,t_1\right)=\left[X\left(\left(t_1-(j-1)h\right)^-,t_1\right)-X\left(\left(t_1-(j-1)h\right)^+,t_1\right)\right]A_{-1}$$

for any $j\geq 1$, j integer.

For $j=1$, we have

$$X\left(\left(t_1-h\right)^-,t_1\right)-X\left(\left(t_1-h\right)^+,t_1\right)=\left[X\left(t_1^-,t_1\right)-X\left(t_1^+,t_1\right)\right]A_{-1}=I_nA_{-1}=A_{-1}$$

by (vi).

The rest of the proof is by induction on j.

Assume the validity of (vii) for $j=1,\dots,p$ for some $p>1$.

Then

$$\Delta X\left(t_1-(p+1)h,t_1\right)=X\left(\left(t_1-(p+1)h\right)^-,t_1\right)-X\left(\left(t_1(p+1)h\right)^+,t_1\right)$$

$$=\left[X\left(\left(t_1-ph\right)^-,t_1\right)-X\left(\left(t_1-ph\right)^+,t_1\right)\right]A_{-1}=A_{-1}^pA_{-1}$$

by the induction hypothesis. So $\Delta\mathrm{X}\left(t_1-(p+1)h,t_1\right)=A_{-1}^{p+1}$ and hence (vii) is valid.

2.5 Lemma on $Q_k(h)$ and $Q_1(jh)$ as sums of products of permutations

For any nonnegative integers j and k,

(i) $Q_k(h)=\sum_{(v_1,\cdots,v_{k+1})\in P_{-1(1),0(k)}} A_{v_1}\cdots A_{v_{k+1}}+\sum_{(v_1,\cdots,v_k)\in P_{0(k-1),1(1)}} A_{v_1}\cdots A_{v_k}$

(ii) $Q_1(jh)=\sum_{(v_1,\cdots,v_{j+1})\in P_{-1(j),0(1)}} A_{v_1}\cdots A_{v_{j+1}}+\sum_{(v_1,\cdots,v_j)\in P_{-1(j-1),\,1(1)}} A_{v_1}\cdots A_{v_j}$

Proof of (i)

By lemma 2.4, $Q_0(jh) = A_{-1}^j$ and $Q_k(h) = A_0^k A_{-1} + \sum_{r=0}^{k-1} A_0^{k-(1+r)} (A_1 + A_{-1} A_0) A_0^r$

$k = 0, j = 1, \Rightarrow Q_0(h) = A_{-1};\ rhs = \sum_{v_1 \in P_{-1(1),0(0)}} A_{v_1} + 0 = A_{-1}$

$k = 1 \Rightarrow Q_1(h) = A_0 A_{-1} + A_{-1} A_0 + A_1;\ rhs = \sum_{(v_1, v_2) \in P_{-1(1),0(1)}} A_{v_1} A_{v_2} + \sum_{v_1 \in P_{0(1-1),1(1)}} A_{v_1} A_0 A_{-1} + A_{-1} A_0 + A_1$

So, the lemma is valid for $k \in \{0,1\}$.

Assume the validity of the lemma for $2 \le k \le p$, for some integer p. Then,

$$Q_p(h) = A_0^p A_{-1} + \sum_{r=0}^{p-1} A_0^{p-(1+r)} (A_1 + A_{-1} A_0) A_0^r$$

$$= \sum_{(v_1,\cdots,v_{p+1}) \in P_{-1(1),0(p)}} A_{v_1} \cdots A_{v_{p+1}} + \sum_{(v_1,\cdots,v_p) \in P_{0(p-1),1(1)}} A_{v_1} \cdots A_{v_p}$$

Now $Q_{p+1}(h) = A_0^{p+1} A_{-1} + \sum_{r=0}^{p+1-1} A_0^{p+1-(1+r)} (A_1 + A_{-1} A_0) A_0^r$

$$= A_0^{p+1} A_{-1} + \sum_{r=0}^{p} A_0^{p-r} (A_1 + A_{-1} A_0) A_0^r$$

$$= A_0^{p+1} A_{-1} + (A_1 + A_{-1} A_0) A_0^p + \sum_{r=0}^{p-1} A_0^{p-r} (A_1 + A_{-1} A_0) A_0^r$$

$$= A_0^{p+1} A_{-1} + (A_1 + A_{-1} A_0) A_0^p + A_0 \sum_{r=0}^{p-1} A_0^{p-(1+r)} (A_1 + A_{-1} A_0) A_0^r$$

$$= A_0 \left(A_0^p A_{-1} + \sum_{r=0}^{p-1} A_0^{p-(1+r)} (A_1 + A_{-1} A_0) A_0^r \right) + (A_1 + A_{-1} A_0) A_0^p$$

$$= A_0 \left(\sum_{(v_1,\cdots,v_{p+1}) \in P_{-1(1),0(p)}} A_{v_1} \cdots A_{v_{p+1}} + \sum_{(v_1,\cdots,v_p) \in P_{0(p-1),1(1)}} A_{v_1} \cdots A_{v_p} \right) + A_{-1} A_0^{p+1} + A_1 A_0^p$$

$$= A_0 \sum_{(v_1,\cdots,v_{p+1}) \in P_{-1(1),0(p)}} A_{v_1} \cdots A_{v_{p+1}} + A_{-1} A_0^{p+1} + A_0 \sum_{(v_1,\cdots,v_p) \in P_{0(p-1),1(1)}} A_{v_1} \cdots A_{v_p} + A_1 A_0^p$$

$$= \sum_{(v_1,\cdots,v_{p+2}) \in P_{-1(1),0(p+1)}} A_{v_1} \cdots A_{v_{p+2}} \text{ (with a leading } A_0) + \sum_{(v_1,\cdots,v_{p+2}) \in P_{-1(1),0(p+1)}} A_{v_1} \cdots A_{v_{p+2}} \text{ (with a leading } A_{-1})$$

$$+ \sum_{(v_1,\cdots,v_{p+1}) \in P_{0(p+1-1),1(1)}} A_{v_1} \cdots A_{v_{p+1}} \text{ (with a leading } A_0) + \sum_{(v_1,\cdots,v_{p+1}) \in P_{0(p+1-1),1(1)}} A_{v_1} \cdots A_{v_{p+1}} \text{ (with a leading } A_1)$$

$$= \sum_{(v_1,\cdots,v_{p+2}) \in P_{-1(1),0(p+1)}} A_{v_1} \cdots A_{v_{p+2}} + \sum_{(v_1,\cdots,v_{p+1}) \in P_{0(p),1(1)}} A_{v_1} \cdots A_{v_{p+1}}$$

Therefore, part (i) of the lemma is valid for $k = p+1$, and hence valid for every nonnegative integer k.

Proof of (ii)

Using similar reasoning (ii) can be established by induction; but let us do it differently by using the expression for $Q_1(jh)$ from lemma 2.4.

$$Q_1(jh) = A_{-1}^j A_0 + \sum_{r=0}^{j-1} A_{-1}^r \left(A_0 A_{-1} + A_1\right) A_{-1}^{j-(1+r)}$$

$$= \sum_{r=0}^{j} A_{-1}^r A_0 A_{-1}^{j-r} + \sum_{r=0}^{j-1} A_{-1}^r A_1 A_{-1}^{j-1-r}$$

$$= \sum_{(v_1,\cdots,v_{j+1})\in P_{-1(j),0(1)}} A_{v_1}\cdots A_{v_{j+1}} + \sum_{(v_1,\cdots,v_j)\in P_{-1(j-1),1(1)}} A_{v_1}\cdots A_{v_j},$$

noting that in the first summation over r A_0 occupies positions $1,2,\cdots,j+1$ for $r=0,1,\cdots,j$ respectively. A_0 leads only once and trails only once.; in the second summation A_1 occupies positions $1,2,\cdots,j$ for $r=0,1,\cdots,j-1$ respectively. A_1 leads only once and trails only once.

Remarks: Observe that $Q_1(jh)$ may be deduced from $Q_k(h)$ by (a) replacing k by j in the expression for $Q_k(h)$ (b) interchanging -1 and 0 in the expression for $Q_k(h)$ (c) replacing 0 by -1 in the summations or permutations involving only the indices 0 and 1.

2.6 Lemma on $\dfrac{\partial X(\tau,t_1)}{\partial \tau}$ of uncontrolled part of (1)

(i) $X^{(1)}((t_1-jh)^-,t_1) = -\sum_{r=0}^{j}\left[\sum_{i=0}^{1} X((t_1-(j-(r+i))h)^-,t_1)A_i\right]A_{-1}^r$

(ii) $X^{(1)}((t_1-jh)^+,t_1) = -\sum_{r=0}^{j}\left[\sum_{i=0}^{1} X((t_1-(j-(r+i))h)^+,t_1)A_i\right]A_{-1}^r$

(iii) $\Delta X^{(1)}(t_1-jh,t_1) = -\sum_{r=0}^{j}\left[\sum_{i=0}^{1} \Delta X(t_1-(j-(r+i))h,t_1)A_i\right]A_{-1}^r$

Proof of (i)

$$X^{(1)}(t_1,(t_1-jh)^-) = -X((t_1-jh)^-,t_1)A_0 - X((t_1-(j-1)h)^-,t_1)A_1 + X^{(1)}((t_1-(j-1)h)^-,t_1)A_{-1}$$

$$X^{(1)}((t_1-(j-1)h)^-,t_1)A_{-1} = -[X((t_1-(j-1)h)^-,t_1)A_0 + X((t_1-(j-2)h)^-,t_1)A_1]A_{-1}$$
$$+ X^{(1)}((t_1-(j-2)h)^-,t_1)A_{-1}^2$$

$$j=0 \Rightarrow \text{lhs} = X^{(1)}(t_1^-,t_1) = -A_0 = \text{rhs};\ j=1 \Rightarrow \text{lhs} = X^{(1)}((t-h)^-,t_1)$$
$$= -X((t-h)^-,t_1)A_0 - A_1 - A_0A_{-1}$$

(By (v) of lemma 2.4)

Let us examine $-\sum_{r=0}^{1}\left[\sum_{i=0}^{1} X((t_1-(1-(r+i))h)^-,t_1)A_i\right]A_{-1}^r$; $r=0,\ i=0$ yield $-(X(t_1,(t_1-h)^-)A_0$

$r=0, i=1$ yield $-(X(t_1^-,t_1)A_1 = -A_1; r=1,\ i=0$ yield $-A_0A_{-1};\ r=1,\ i=1$ yield 0.

Sum the results for these contingencies to get

$$-\sum_{r=0}^{1}\left[\sum_{i=0}^{1} X((t_1-(1-(r+i))h)^-,t_1)A_i\right]A_{-1}^r = -(X((t_1-h)^-,t_1)A_0 - A_1 - A_0A_{-1}$$

So, the lemma is valid for $j=1$. Let us explore $j=2$.

$$j=2 \Rightarrow \text{lhs} = X^{(1)}((t-2h)^-,t_1) = -X((t-2h)^-,t_1)A_0 - X((t-h)^-,t_1)A_1 + X^{(1)}((t-h)^-,t_1)A_{-1}$$
$$= -X((t-2h)^-,t_1)A_0 - X((t-h)^-,t_1)A_1 - X((t-h)^-,t_1)A_0A_{-1} - A_1A_{-1} - A_0A_{-1}^2$$

Let us examine $-\sum_{r=0}^{2}\left[\sum_{i=0}^{1} X((t_1-(2-(r+i))h)^-,\, t_1)A_i\right]A_{-1}^r$;

$r=0,\ i=0$ yield $-(X((t_1-2h)^-,t_1)A_0$

$r=0, i=1$ yield $-(X((t_1-h)^-,t_1)A_1; r=1,\ i=0$ yield $-(X((t_1-h)^-,t_1)A_0A_{-1};$

$r=1,\ i=1$ yield $-A_1A_{-1};$

$r=2, i=0$ yield $-A_0A_{-1}^2;\ r=2, i=0$ yield $0.$

Sum the results for these contingencies to get

$$-\sum_{r=0}^{2}\left[\sum_{i=0}^{1} X((t_1-(2-(r+i))h)^-,t_1)A_i\right]A_{-1}^r$$

$$=-(X((t_1-2h)^-,t_1)A_0-(X((t_1-h)^-,t_1)A_1-(X((t_1-h)^-,t_1)A_0A_{-1}-A_1A_{-1}-A_0A_{-1}^2.$$

So, (i) of the lemma is valid for $j\in\{0,1,2\}$. Assume that (i) of the lemma is valid for $3\le j\le n$, for some integer *n.* Then

$$X^{(1)}((t_1-nh)^-,t_1)=-\sum_{r=0}^{n}\left[\sum_{i=0}^{1} X((t_1-(n-(r+i))h)^-,t_1)A_i\right]A_{-1}^r,$$ by the induction hypothesis.

Now, $$X^{(1)}((t_1-(n+1)h)^-,t_1)=-\left[X((t_1-(n+1)h)^-,t_1)A_0+X((t_1-nh)^-,t_1)A_1\right]$$
$$+X^{(1)}((t_1-nh)^-,t_1)A_{-1}$$

$$=X^{(1)}((t_1-nh)^-,t_1)A_{-1}-\sum_{i=0}^{1} X((t_1-(n+1-i)h)^-,t_1)A_i$$

$$=-\sum_{r=0}^{n}\left[\sum_{i=0}^{1} X((t_1-(n-(r+i))h)^-,t_1)A_i\right]A_{-1}^{r+1}-\sum_{i=0}^{1} X((t_1-(n+1-i)h)^-,t_1)A_i$$

$$=-\sum_{r=0}^{n+1}\left[\sum_{i=0}^{1} X((t_1-(n+1-(r+i))h)^-,t_1)A_i\right]A_{-1}^r,$$ as desired. So (i) is true for $j = n$, and hence valid for all j.

Proof of (ii)

$$X^{(1)}((t_1-jh)^+,t_1)=-X((t_1-jh)^+,t_1)A_0-X((t_1-(j-1)h)^+,t_1)A_1+X^{(1)}((t_1-(j-1)h)^+,t_1)A_{-1}$$

$$X^{(1)}((t_1-(j-1)h)^+,t_1)A_{-1}=-[X((t_1-(j-1)h)^+,t_1)A_0+X((t_1-(j-2)h)^+,t_1)A_1]A_{-1}$$
$$+X^{(1)}((t_1-(j-2)h)^+,t_1)A_{-1}^2$$

$j=0\Rightarrow$ lhs $=X^{(1)}(t_1^+,t_1)=0=$ rhs;

$j=1\Rightarrow$ lhs $=X^{(1)}((t_1-h)^+,t_1)=-X((t_1-h)^+,t_1)A_0-X(t_1^+,t_1)A_1+X^{(1)}(t_1^+,t_1)A_{-1}$

$$=-X((t_1-h)^+,t_1)A_0-0+0$$

Let us examine $-\sum_{r=0}^{1}\left[\sum_{i=0}^{1} X((t_1-(1-(r+i))h)^+,t_1)A_i\right]A_{-1}^r$; $r=0,\ i=0$ yield $-(X(t_1,(t_1-h)^+,t_1)A_0$

$r=0, i=1$ yield $-(X(t_1^+,t_1)A_1=0; r=1,\ i=0$ yield $-X(t_1^+,t_1)A_0A_{-1}=0;\ r=1,\ i=1$ yield $0.$

Sum the results for these contingencies to get

$$-\sum_{r=0}^{1}\left[\sum_{i=0}^{1} X((t_1-(1-(r+i))h)^+,t_1)A_i\right]A_{-1}^r=-(X((t_1-h)^+,t_1)A_0$$

So, (ii) of the lemma is valid for $j=1$. Let us explore $j=2$.

$$j=2 \Rightarrow \text{lhs} = X^{(1)}((t-2h)^+,t_1)=-X((t-2h)^+,t_1)A_0-X((t-h)^+,t_1)A_1+X^{(1)}((t-h)^+,t_1)A_{-1}$$

$$=-X((t-2h)^+,t_1)A_0-X((t-h)^-,t_1)A_1-X((t-h)^+,t_1)A_0A_{-1}-X(t_1^+,t_1)A_1A_{-1}$$

$$+X^{(1)}(t^+,t_1)A_{-1}^2$$

$$=-X((t-2h)^+,t_1)A_0-X((t-h)^+,t_1)A_1-X((t-h)^+,t_1)A_0A_{-1}$$

Let us examine $-\sum_{r=0}^{2}\left[\sum_{i=0}^{1} X((t_1-(2-(r+i))h)^+,t_1)A_i\right]A_{-1}^r$;

$r=0,\ i=0$ yield $-(X((t_1-2h)^+,t_1)A_0;\ r=0, i=1$ yield $-(X((t_1-h)^+,t_1)A_1;$

$r=1,\ i=0$ yield $-(X((t_1-h)^+,t_1)A_0A_{-1};\ r=1,\ i=1$ yield $0; r=2, i=0$ yield $0;$

$r=2, i=1$ yield $0.$

Sum the results for these contingencies to get

$$-\sum_{r=0}^{2}\left[\sum_{i=0}^{1} X((t_1-(2-(r+i))h)^+,t_1)A_i\right]A_{-1}^r$$

$$=-(X((t_1-2h)^+,t_1)A_0-(X((t_1-h)^+,t_1)A_1-(X((t_1-h)^+,t_1)A_0A_{-1}.$$

So, (ii) of the lemma is valid for $j\in\{0,1,2\}$. Assume that (ii) of the lemma is valid for $3\le j\le n$, for some integer n. Then

$$X^{(1)}((t_1-nh)^-,t_1)=-\sum_{r=0}^{n}\left[\sum_{i=0}^{1} X((t_1-(n-(r+i))h)^+,t_1)A_i\right]A_{-1}^r,$$ by the induction hypothesis.

Now, $$X^{(1)}((t_1-(n+1)h)^+,t_1)=-\left[X((t_1-(n+1)h)^+,t_1)A_0+X((t_1-nh)^+,t_1)A_1\right]$$
$$+X^{(1)}((t_1-nh)^+,t_1)A_{-1}$$

$$=X^{(1)}((t_1-nh)^+,t_1)A_{-1}-\sum_{i=0}^{1}X((t_1-(n+1-i)h)^+,t_1)A_i$$

$$=-\sum_{r=0}^{n}\left[\sum_{i=0}^{1} X((t_1-(n-(r+i))h)^+,t_1)A_i\right]A_{-1}^{r+1}-\sum_{i=0}^{1}X((t_1-(n+1-i)h)^+,t_1)A_i$$

$$=-\sum_{r=0}^{n+1}\left[\sum_{i=0}^{1} X((t_1-(n+1-(r+i))h)^+,t_1)A_i\right]A_{-1}^r,$$ as desired. So (ii) is true for $j=n+1$, and

hence true for all j.

Proof of (iii)

The proof follows by noting that $\Delta X^{(1)}(t_1-jh,t_1)=X^{(1)}((t_1-jh)^-,t_1)-X^{(1)}((t_1-jh)^+,t_1)$ and then using (i) and (ii) to deduce that

$$\Delta X^{(1)}(t_1,(t_1-jh))=-\sum_{r=0}^{j}\left[\sum_{i=0}^{1}\left[X((t_1-(j-(r+i))h)^-,t_1)-X((t_1-(j-(r+i))h)^+,t_1)\right]A_i\right]A_{-1}^r$$

$$=-\sum_{r=0}^{j}\left[\sum_{i=0}^{1}\Delta X((t_1-(j-(r+i))h),t_1)A_i\right]A_{-1}^r=-\sum_{r=0}^{j}\left[\sum_{i=0}^{1}A_{-1}^{j-(r+i)}A_iA_{-1}^r\right]$$

2.7 Lemma on $\Delta X^{(k)}(t_1,((t_1-jh))$ recursions

$$\Delta X^{(k)}(t_1-jh,t_1)=-\sum_{r=0}^{j}\left[\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(j-(r+i))h),t_1)A_i\right]A_{-1}^r$$

Proof

If $j=0,$ then $r=0;$ $\text{lhs}=\Delta X^{(k)}(t_1,t_1)=(-1)^kA_0^k,\ \text{rhs}=-\left[\sum_{i=0}^{1}\Delta X^{(k-1)}(t_1+ih,t_1)A_i\right]$

$=\Delta X^{(k-1)}(t_1,t_1)A_0-\Delta X^{(k-1)}(t_1+h,t_1)A_1=(-1)(-1)^{k-1}A_0^{k-1}A_0-0=(-1)^kA_0^k$ (by lemma 2.4)

If $j=1,$ then using lemma 2.4,

$\text{lhs}=\Delta X^{(k)}(t_1-(1-1)h,t_1)A_{-1}-\left[\Delta X^{(k-1)}(t_1-h,t_1)A_0+\Delta X^{(k-1)}(t_1,t_1)A_1\right]$

$=\Delta X^{(k)}(t_1-(1-1)h,t_1)A_{-1}-\left[\Delta X^{(k-1)}(t_1-h,t_1)A_0+(-1)^{k-1}A_0^{k-1}A_1\right]$

$=(-1)^kA_0^kA_{-1}-[\Delta X^{(k-1)}(t_1-h,t_1)A_0+(-1)^{k-1}A_0^{k-1}A_1$ (by 2.4).

$\text{Rhs}=-\sum_{r=0}^{j}\left[\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(j-(r+i))h),t_1)A_i\right]A_{-1}^r$

$=-\left[\Delta X^{(k-1)}((t_1-h),t_1)A_0A_{-1}^0+\Delta X^{(k-1)}(t_1,t_1)A_1\right]-\left[\Delta X^{(k-1)}(t_1,t_1)A_0A_{-1}^1+\Delta X^{(k-1)}(t_1+h,t_1)A_1A_{-1}\right]$

$=-\left[\Delta X^{(k-1)}((t_1-h),t_1)A_0+(-1)^{k-1}A_0^{k-1}A_1\right]-\left[(-1)^{k-1}A_0^{k-1}A_0A_{-1}^1+0\right]$

$=(-1)^kA_0^kA_{-1}-\left[\Delta X^{(k-1)}((t_1-h),t_1)A_0+(-1)^{k-1}A_0^{k-1}A_1\right]$

So, lhs = rhs. Therefore the lemma is valid for j = 1. Assume the validity of the lemma for $2\le j\le m$ for some integer m. Then $\Delta X^{(k)}((t_1-mh),t_1)=-\sum_{r=0}^{m}\left[\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(m-(r+i))h),t_1)A_i\right]A_{-1}^r$

Now $\Delta X^{(k)}((t_1-(m+1)h),t_1)$

$$=-\left[\Delta X^{(k-1)}((t_1-(m+1)h),t_1)A_0+\Delta X^{(k-1)}(t_1-mh,t_1)A_1\right]$$
$$+\Delta X^{(k)}(t_1-mh,t_1)A_{-1}$$

$$=\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(m+1-i)h),t_1)A_i-\sum_{r=0}^{m}\left[\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(m-(r+i))h,t_1)A_iA_{-1}^{r+1}\right]$$

$$=\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(m+1-i)h),t_1)A_i-\sum_{r=1}^{m+1}\left[\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(m-(r-1+i))h,t_1)A_iA_{-1}^{r+1-1}\right]$$

$$=\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(m+1-i)h),t_1)A_i-\sum_{r=0}^{m+1}\left[\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(m+1-(r+i))h,t_1)A_iA_{-1}^{r}\right]$$

$$-\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(m+1-i)h),t_1)A_i=\sum_{r=0}^{m+1}\left[\sum_{i=0}^{1}\Delta X^{(k-1)}((t_1-(m+1-(r+i))h,t_1)A_iA_{-1}^{r}\right]$$

(We used the change of variables $r \to r-1$ in the $r-$summand, $r \to r+1$ in the $r-$limits).
So the lemma is valid for $j = m+1$. This completes the desired proof.
The following major theorem gives explicit computable expressions for $Q_k(jh)$, based on painstakingly and rigorously observed $Q_k(jh)$ patterns for various j, k contingencies.

III. THEOREM ON EXPLICIT COMPUTABLE EXPRESSION FOR DETERMINING MATRICES OF (1)

Let j and k be nonnegative integers. If $j \ge k \ge 1$, then

$$Q_k(jh) = \sum_{(v_1,\cdots,v_{j+k}) \in P_{-1(j),0(k)}} A_{v_1} \cdots A_{v_{j+k}} + \sum_{(v_1,\cdots,v_j) \in P_{-1(j-k),1(k)}} A_{v_1} \cdots A_{v_j} + \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j-k),0(r),1(k-r)}} A_{v_1} \cdots A_{v_{j+r}}$$

If $k \ge j \ge 1$, then

$$Q_k(jh) = \sum_{(v_1,\cdots,v_{j+k}) \in P_{-1(j),0(k)}} A_{v_1} \cdots A_{v_{j+k}} + \sum_{(v_1,\cdots,v_k) \in P_{0(k-j),1(j)}} A_{v_1} \cdots A_{v_k} + \sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{k+r}) \in P_{-1(r),0(r+k-j),1(j-r)}} A_{v_1} \cdots A_{v_{k+r}}$$

Proof

Case $j \ge k$.

If $j = k = 0$, then $Q_k(jh) = Q_0(0) = I_n$, by (4); $k = 0 \Rightarrow Q_k(jh) = Q_0(jh) = A_{-1}^j$ (by (ii) of lemma 2.4).

$j \ge 1,\ k = 1 \Rightarrow Q_k(jh) = Q_1(jh) = A_{-1}^j A_0 + \sum_{r=0}^{j-1} A_{-1}^r (A_0 A_{-1} + A_1) A_{-1}^{j-(r+1)}$, (by (iv) of lemma 2.4)

$$= A_{-1}^j A_0 + \sum_{r=0}^{j-1} A_{-1}^r A_0 A_{-1}^{j-r} + \sum_{r=0}^{j-1} A_{-1}^r A_1 A_{-1}^{j-(r+1)} = \sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j),0(1)}} A_{v_1} \cdots A_{v_{j+1}} + \sum_{(v_1,\cdots,v_j) \in P_{-1(j-1),1(1)}} A_{v_1} \cdots A_{v_j}$$

If $j = k = 2$, then $Q_k(jh) = Q_2(2h) = A_{-1}Q_2(h) + A_0 Q_1(2h) + A_1 Q_1(h)$

$$= A_{-1} \sum_{(v_1,\cdots,v_3) \in P_{-1(1),0(2)}} A_{v_1} \cdots A_{v_3} + A_{-1} \sum_{(v_1,v_2) \in P_{0(1),1(1)}} A_{v_1} A_{v_2}$$

$$+ A_0 \sum_{(v_1,\cdots,v_3) \in P_{-1(2),0(1)}} A_{v_1} \cdots A_{v_3} + A_0 \sum_{(v_1,v_2) \in P_{-1(1),1(1)}} A_{v_1} A_{v_2} + A_1 \sum_{(v_1,v_2) \in P_{-1(1),0(1)}} A_{v_1} A_{v_2} + A_1 \sum_{v_1 \in P_{1(1)}} A_{v_1}$$

$$\sum_{(v_1,\cdots,v_4) \in P_{-1(2),0(2)}} A_{v_1} \cdots A_{v_4} + A_1^2 + \sum_{(v_1,\cdots,v_3) \in P_{-1(1),0(1),1(1)}} A_{v_1} \cdots A_{v_3}$$

$$= \sum_{(v_1,\cdots,v_4) \in P_{-1(2),0(2)}} A_{v_1} \cdots A_{v_4} + \sum_{(v_1,v_2) \in P_{-1(2-2),1(2)}} A_{v_1} A_{v_2} + \sum_{r=1}^{2-1} \sum_{(v_1,\cdots,v_{2r+s}) \in P_{-1(r),0(r),1(2-r)}} A_{v_1} \cdots A_{v_{2+r}}$$

Therefore the theorem is valid for $j, k \in \{0, 1, 2\}$. In particular it is valid for $0 \le k \le j \le 2$.
The cases : $Q_1(jh)$ and $Q_k(h)$ have been established already. We need only prove the remaining cases.

Case $j \ge k$: Assume that the theorem is valid for all pairs of triples $j,k,Q_k(jh);\ \tilde{j},\tilde{k},Q_{\tilde{k}}(\tilde{j}h)$ for which $\tilde{j}+\tilde{k} \le j+k$, for some $j \ge 3$, and $k \ge 2$. Then apply the induction principle to the right-hand side of the determining equation:

$Q_k([j+1]h) = A_0Q_{k-1}([j+1]h) + A_1Q_{k-1}(jh) + A_{-1}Q_k(jh)$, noting that $j \ge k \Rightarrow j+1 > k$, ,

$j+1 > k-1,\ j > k-1.$

Therefore:

$$Q_k([j+1]h)$$

$$= \begin{cases} A_0 \sum_{(v_1,\cdots,v_{j+k}) \in P_{-1(j+1),0(k-1)}} A_{v_1}\cdots A_{v_{j+k}} + A_0 \sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j+1-(k-1)),1(k-1)}} A_{v_1}\cdots A_{v_{j+1}} \\ + A_0 \sum_{r=1}^{k-1-1} \sum_{(v_1,\cdots,v_{j+1+r}) \in P_{-1(r+j+1-(k-1)),0(r),1(k-1-r)}} A_{v_1}\cdots A_{v_{j+1+r}} \\ A_1 \sum_{(v_1,\cdots,v_{j+k-1}) \in P_{-1(j),0(k-1)}} A_{v_1}\cdots A_{v_{j+k-1}} + A_1 \sum_{(v_1,\cdots,v_j) \in P_{-1(j-(k-1)),1(k-1)}} A_{v_1}\cdots A_{v_j} \\ + A_1 \sum_{s=1}^{k-1-r} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j-(k-1)),0(r),1(k-1-r)}} A_{v_1}\cdots A_{v_{j+r}} \\ A_{-1} \sum_{(v_1,\cdots,v_{j+k}) \in P_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}} + A_{-1} \sum_{(v_1,\cdots,v_j) \in P_{-1(j-k),1(k)}} A_{v_1}\cdots A_{v_j} \\ + A_{-1} \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j-k),0(r),1(k-r)}} A_{v_1}\cdots A_{v_{j+r}} \end{cases}$$

$$= \sum_{i=-1}^{0} \sum_{(v_1,\cdots,v_{j+1}) \in P^{iL}_{-1(j+1),0(k)}} A_{v_1}\cdots A_{v_{j+1+k}} + \sum_{i\in\{-1,1\}} \sum_{(v_1,\cdots,v_{j+1}) \in P^{iL}_{-1(j+1-k),1(k)}} A_{v_1}\cdots A_{v_{j+1}} \tag{9}$$

$$+ A_0 \sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j+2-k),1(k-1)}} A_{v_1}\cdots A_{v_{j+1}} + A_0 \sum_{r=1}^{k-2} \sum_{(v_1,\cdots,v_{j+1+r}) \in P_{-1(r+j+2-k),0(r),1(k-1-r)}} A_{v_1}\cdots A_{v_{j+1+r}} \tag{10}$$

$$A_1 \sum_{(v_1,\cdots,v_{j+k-1}) \in P_{-1(j),0(k-1)}} A_{v_1}\cdots A_{v_{j+k-1}} + A_1 \sum_{r=1}^{k-2} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j+1-k),0(r),1(k-1-r)}} A_{v_1}\cdots A_{v_{j+r}} \tag{11}$$

$$+ A_{-1} \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j-k),0(r),1(k-r)}} A_{v_1}\cdots A_{v_{j+r}} \tag{12}$$

The expression (9) yields:

$$\sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j+1),0(k)}} A_{v_1}\cdots A_{v_{j+1+k}} + \sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j+1-k),1(k)}} A_{v_1}\cdots A_{v_{j+1}} \tag{13}$$

The expression: $\sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+r}) \in P^{-1L}_{-1(r+j+1-k),0(r),1(k-r)}} A_{v_1}\cdots A_{v_{j+1+r}}$ (14)

is immediate from (12)

We use the change of variables technique in the expression (10) to get:

$$A_0 \sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j+2-k),1(k-1)}} A_{v_1} \cdots A_{v_{j+1}} + A_0 \sum_{r=1}^{k-2} \sum_{(v_1,\cdots,v_{j+1+r}) \in P_{-1(r+j+2-k),0(r),1(k-1-r)}} A_{v_1} \cdots A_{v_{j+1+r}} \quad (15)$$

$$= A_0 \sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j+2-k),1(k-1)}} A_{v_1} \cdots A_{v_{j+1}} + A_0 \sum_{r=2}^{k-1} \sum_{(v_1,\cdots,v_{j+1+r-1}) \in P_{-1(r-1+j+2-k),0(r-1),1(k-1-(r-1))}} A_{v_1} \cdots A_{v_{j+1+r-1}}$$

$$= A_0 \sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j+2-k),1(k-1)}} A_{v_1} \cdots A_{v_{j+1}} + A_0 \sum_{r=2}^{k-1} \sum_{(v_1,\cdots,v_{j+1+r-1}) \in P_{-1(r+j+1-k),0(r-1),1(k-r)}} A_{v_1} \cdots A_{v_{j+1+r-1}}$$

$$= A_0 \sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j+2-k),1(k-1)}} A_{v_1} \cdots A_{v_{j+1}} + A_0 \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+1+r-1}) \in P_{-1(r+j+1-k),0(r-1),1(k-r)}} A_{v_1} \cdots A_{v_{j+1+r-1}}$$

$$-A_0 \sum_{(v_1,\cdots,v_{j+1+1-1}) \in P_{-1(1+j+1-k),0(1-1),1(k-1)}} A_{v_1} \cdots A_{v_{j+1+1-1}} = A_0 \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+1+r-1}) \in P_{-1(r+j+1-k),0(r-1),1(k-r)}} A_{v_1} \cdots A_{v_{j+1+r-1}}$$

$$= \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+1+r}) \in P^{0L}_{-1(r+j+1-k),0(r),1(k-r)}} A_{v_1} \cdots A_{v_{j+1+r}} \quad (16)$$

We are left with expression (11), which yields:

$$A_1 \sum_{(v_1,\cdots,v_{j+k-1}) \in P_{-1(j),0(k-1)}} A_{v_1} \cdots A_{v_{j+k-1}} + A_1 \sum_{r=1}^{k-2} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j+1-k),0(r),1(k-1-r)}} A_{v_1} \cdots A_{v_{j+r}} \quad (17)$$

$$= A_1 \sum_{(v_1,\cdots,v_{j+k-1}) \in P_{-1(j),0(k-1)}} A_{v_1} \cdots A_{v_{j+k-1}} + A_1 \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j+1-k),0(r),1(k-1-r)}} A_{v_1} \cdots A_{v_{j+r}}$$

$$- A_1 \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(k-1+j+1-k),0(k-1),1(k-1-(k-1))}} A_{v_1} \cdots A_{v_{j+r}} = A_1 \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j+1-k),0(r),1(k-1-r)}} A_{v_1} \cdots A_{v_{j+r}}$$

$$= \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+1+r}) \in P^{1L}_{-1(r+j+1-k),0(r),1(k-r)}} A_{v_1} \cdots A_{v_{j+1+r}} \quad (18)$$

The resulting expressions (13), (14), (16) and (18) add up to yield:

$$Q_k([j+1]h)$$

$$= \sum_{(v_1,\cdots,v_{j+1+k}) \in P_{-1(j+1),0(k)}} A_{v_1} \cdots A_{v_{j+k}} + \sum_{(v_1,\cdots,v_j) \in P_{-1(j+1-k),1(k)}} A_{v_1} \cdots A_{v_j}$$

$$+ \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+1+r}) \in P_{-1(r+j+1-k),0(r),1(k-r)}} A_{v_1} \cdots A_{v_{j+1+r}} \quad (19)$$

This concludes the inductive proof for the case $j \geq k$.

We proceed to furnish the inductive proof for the case $k \geq j$.

Theorem 3 is already proved for $j, k \in \{1, 2, 3, 4.\}, k \geq j$.

Therefore the theorem is valid for $j,\ k \in \{0,\ 1,\ 2,\ 3, 4\}$. Assume that the theorem is valid for all pairs of triples $j, k, Q_k(jh)$; $\tilde{j}, \tilde{k}, Q_{\tilde{k}}(\tilde{j}h)$ for which $\tilde{j} + \tilde{k} \leq j + k$, for some $j \geq 3$, and $k \geq 4$. Then apply the induction principle to the right-hand side of the determining equation:

$Q_{k+1}(jh) = A_0 Q_k(jh) + A_1 Q_k([j-1]h) + A_{-1} Q_{k+1}([j-1]h)$, noting that $k \ge j \Rightarrow k+1 > j$, $k+1 > j-1,\ k > j-1$. Therefore:

$$
\mathrm{Q}_{k+1}(jh) = \begin{cases}
A_0 \sum\limits_{(v_1,\cdots,v_{j+k}) \in P_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}} + A_0 \sum\limits_{(v_1,\cdots,v_k) \in P_{0(k-j)),1(j)}} A_{v_1}\cdots A_{v_k} \\
+ A_0 \sum\limits_{r=1}^{j-1} \sum\limits_{(v_1,\cdots,v_{j+r}) \in P_{-1(r),0(r+k-j),1(j-r)}} A_{v_1}\cdots A_{v_{k+r}} \\
A_1 \sum\limits_{(v_1,\cdots,v_{j-1+k}) \in P_{-1(j-1),0(k)}} A_{v_1}\cdots A_{v_{j+k-1}} + A_1 \sum\limits_{(v_1,\cdots,v_{j-1}) \in P_{0(k-(j-1)),1(j-1)}} A_{v_1}\cdots A_{v_k} \\
+ A_1 \sum\limits_{r=1}^{j-1-r} \sum\limits_{(v_1,\cdots,v_{k+r}) \in P_{-1(r),0(r+k-(j-1)),1(j-1-r)}} A_{v_1}\cdots A_{v_{k+r}} \\
A_{-1} \sum\limits_{(v_1,\cdots,v_{j+k}) \in P_{-1(j-1),0(k+1)}} A_{v_1}\cdots A_{v_{j+k}} + A_{-1} \sum\limits_{(v_1,\cdots,v_j) \in P_{0(k+1-(j-1)),1(j-1)}} A_{v_1}\cdots A_{v_{k+1}} \\
+ A_{-1} \sum\limits_{r=1}^{j-2} \sum\limits_{(v_1,\cdots,v_{k+1+r}) \in P_{-1(r),0(r+k+1-(j-1)),1(j-1-r)}} A_{v_1}\cdots A_{v_{k+1+r}}
\end{cases}
$$

$$
= \sum_{i=-1}^{0} \sum_{(v_1,\cdots,v_{j+k+1}) \in P^{iL}_{-1(j),0(k+1)}} A_{v_1}\cdots A_{v_{j+k+1}} + \sum_{i=0}^{1} \sum_{(v_1,\cdots,v_{j+1}) \in P^{iL}_{0(k+1-j),1(j)}} A_{v_1}\cdots A_{v_{k+1}} \tag{20}
$$

$$
+ A_0 \sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{k+r}) \in P_{-1(r),0(r+k-j),1(j-r)}} A_{v_1}\cdots A_{v_{k+r}} \tag{21}
$$

$$
A_1 \sum_{(v_1,\cdots,v_{j-1+k}) \in P_{-1(j-1),0(k)}} A_{v_1}\cdots A_{v_{j-1+k}} + A_1 \sum_{r=1}^{j-2} \sum_{(v_1,\cdots,v_{k+r}) \in P_{-1(r),0(r+k+1-j),1(j-1-r)}} A_{v_1}\cdots A_{v_{k+r}} \tag{22}
$$

$$
+ A_{-1} \sum_{(v_1,\cdots,v_{k+1}) \in P_{0(k+2-j),0(j-1)}} A_{v_1}\cdots A_{v_{k+2-j}} + A_{-1} \sum_{r=1}^{j-2} \sum_{(v_1,\cdots,v_{k+1+r}) \in P_{-1(r),0(r+k+2-j),1(j-1-r)}} A_{v_1}\cdots A_{v_{k+1+r}} \tag{23}
$$

The expression (20) yields:

$$
\sum_{(v_1,\cdots,v_{j+1}) \in P_{-1(j),0(k+1)}} A_{v_1}\cdots A_{v_{j+1+k}} + \sum_{(v_1,\cdots,v_{j+1}) \in P_{0(k+1-j),1(j)}} A_{v_1}\cdots A_{v_{j+1}} \tag{24}
$$

The expression: $$\sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{k+1+r}) \in P^{0L}_{-1(r),0(r+k+1-j),1(j-r)}} A_{v_1}\cdots A_{v_{k+1+r}} \tag{25}$$

is immediate from (21)

We use the addition and subtraction technique in (22) to get:

$$A_1 \sum_{(v_1,\cdots,v_{j-1+k}) \in P_{-1(j-1),0(k)}} A_{v_1} \cdots A_{v_{j-1+k}} + A_1 \sum_{r=1}^{j-2} \sum_{(v_1,\cdots,v_{k+r}) \in P_{-1(r),0(r+k+1-j),1(j-1-r)}} A_{v_1} \cdots A_{v_{k+r}}$$

$$= A_1 \sum_{(v_1,\cdots,v_{j-1+k}) \in P_{-1(j-1),0(k)}} A_{v_1} \cdots A_{v_{j-1+k}} + A_1 \sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{k+r}) \in P_{-1(r),0(r+k+1-j),1(j-1-r)}} A_{v_1} \cdots A_{v_{k+r}}$$

$$- A_1 \sum_{(v_1,\cdots,v_{k+j-1}) \in P_{-1(j-1),0(j-1+k+1-j),1(j-1-(j-1))}} A_{v_1} \cdots A_{v_{k+j-1}}$$

$$= \sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{k+r}) \in P^{1L}_{-1(r),0(r+k+1-j),1(j-r)}} A_{v_1} \cdots A_{v_{k+1+r}} \tag{26}$$

We are left with expression (23), which, using the change of variables technique yields:

$$A_{-1} \sum_{(v_1,\cdots,v_{k+1}) \in P_{0(k+2-j),0(j-1)}} A_{v_1} \cdots A_{v_{k+2-j}} + A_{-1} \sum_{r=1}^{j-2} \sum_{(v_1,\cdots,v_{k+1+r}) \in P_{-1(r),0(r+k+2-j),1(j-1-r)}} A_{v_1} \cdots A_{v_{k+1+r}}$$

$$= A_{-1} \sum_{(v_1,\cdots,v_{k+1}) \in P_{0(k+2-j),0(j-1)}} A_{v_1} \cdots A_{v_{k+2-j}} + A_{-1} \sum_{r=2}^{j-1} \sum_{(v_1,\cdots,v_{k+1+r-1}) \in P_{-1(r-1),0(r-1+k+2-j),1(j-1-(r-1))}} A_{v_1} \cdots A_{v_{k+1+r-1}}$$

$$= A_{-1} \sum_{(v_1,\cdots,v_{k+1}) \in P_{0(k+2-j),0(j-1)}} A_{v_1} \cdots A_{v_{k+2-j}} + A_{-1} \sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{k+1+r-1}) \in P_{-1(r-1),0(r-1+k+2-j),1(j-1-(r-1))}} A_{v_1} \cdots A_{v_{k+1+r-1}}$$

$$- A_{-1} \sum_{(v_1,\cdots,v_{k+1+1-1}) \in P_{-1(1-1),0(1-1+k+2-j),1(j-1-(1-1))}} A_{v_1} \cdots A_{v_{k+1+1-1}}$$

$$= \sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{k+1+r}) \in P^{-1L}_{-1(r),0(r+k+1-j),1(j-r)}} A_{v_1} \cdots A_{v_{k+1+r}} \tag{27}$$

The resulting expressions (24), (25), (26) and (27) add up to yield:

$$Q_{k+1}(jh)$$

$$= \sum_{(v_1,\cdots,v_{j+k+1}) \in P_{-1(j),0(k+1)}} A_{v_1} \cdots A_{v_{j+k}} + \sum_{(v_1,\cdots,v_{k+1}) \in P_{0(k+1-j),1(j)}} A_{v_1} \cdots A_{v_{k+1}}$$

$$+ \sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{j+1+r}) \in P_{-1(r),0(r+k+1-j),1(j-r)}} A_{v_1} \cdots A_{v_{k+1+r}} \tag{28}$$

This concludes the inductive proof for the case $k \geq j$, completing the proof of the theorem.

The cases $j \geq k$ and $k \geq j$, in the preceding theorem can be unified by using a composition of the max and the signum functions as follows:

IV. INDUCTION PITFALLS IN THE DERIVATION OF THE DETERMINING MATRICES

In the development of $Q_k(jh)$ for the single – delay neutral system (1) we had proposed, based on the observed pattern for $Q_k(jh)$ for $0 \leq \min\{j,k\} \leq 2,$ that for $j \geq k \geq 1,$

$$\mathrm{Q}_k(jh) \quad = \sum_{(v_1,\cdots,v_{j+k})\in P_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}} + \sum_{(v_1,\cdots,v_j)\in P_{-1(j-k),1(k)}} A_{v_1}\cdots A_{v_j}; k\geq 1$$

$$+\sum_{r=1}^{k-1}\sum_{s=1}^{k-r}\sum_{(v_1,\cdots,v_{2r+j-k+s})\in P_{-1(r+j-k),0(r),1(s)}} A_{v_1}\cdots A_{v_{j+r}}\ ; k\geq 2$$

and for $k\geq j\geq 1,$

$$\mathrm{Q}_k(jh) \quad = \sum_{(v_1,\cdots,v_{j+k})\in P_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}} + \sum_{(v_1,\cdots,v_k)\in P_{0(k-j),1(j)}} A_{v_1}\cdots A_{v_k}; j\geq 1$$

$$+\sum_{r=1}^{j-1}\sum_{s=1}^{j-r}\sum_{(v_1,\cdots,v_{2r+k-j+s})\in P_{-1(r),0(r+k-j),1(s)}} A_{v_1}\cdots A_{v_{2r+k-j+s}}\ ; j\geq 2$$

This formula is valid for all $j,k:1\leq \min\{j,k\}\leq 2.$ However it turns out to be false for all $j,k:\min\{j,k\}\geq 3.$ This revelelation came to the fore following the discovery of some fallacy in the proof of the expressions for $Q_k(jh),$ which could never be overcome; it became imperative to examine $Q_3(3h)$ from the determining equations for $Q_k(jh).$ The formula was found to be inconsistent with the true result from the determining equations. Therefore the formula had to be abandoned after so much effort.
Further long and sustained research effort led to the conjecture

$$\mathrm{Q}_k(jh) \quad = \sum_{(v_1,\cdots,v_{j+k})\in P_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}} + \sum_{(v_1,\cdots,v_j)\in P_{-1(j-k),1(k)}} A_{v_1}\cdots A_{v_j}; k\geq 1$$

$$+\begin{cases}\displaystyle\sum_{r=1}^{k-1}\sum_{s=1}^{k-r}\sum_{(v_1,\cdots,v_{2r+j-k+s})\in P_{-1(r+j-k),0(r),1(s)}} A_{v_1}\cdots A_{v_{j+r}}\ ; k\geq 2, j\neq k,\\ \displaystyle\sum_{r=1}^{k-1}\sum_{s=\max\{1,k+1-2r\}}^{k-r}\sum_{(v_1,\cdots,v_{2r+j-k+s})\in P_{-1(r+j-k),0(r),1(s)}} A_{v_1}\cdots A_{v_{j+r}}\ ; j=k\geq 2\end{cases}$$

as well as the analogous expression for $Q_k(jh)$ for $k\geq j\geq 1.$

The formula was found to be valid for $j,k:1\leq \min\{j,k\}\leq 2,$ as in the aborted formula. Furthermore $Q_3(3h)$ obtained from the postulated formula agreed with the corresponding result (sum of 63 permutation products) obtained directly from the determining equations. Unfortunately, the proof for arbitrary j and k could not push through; there was no way to apply induction on the lower limit for *s*, not to talk of dealing with the piece–wise nature of the third component summations.

Above pitfalls provided object lessons that integer–based mathematical claims should never be accepted without error-free proofs by mathematical induction. There is no other way to ascertain that a pattern will persist –no matter the number of terms for which it has remained valid.

V. CONCLUSION

The results in this article bear eloquent testimony to the fact that we have comprehensively extended the previous single-delay result by Ukwu (1992) together with appropriate embellishments through the unfolding of intricate inter–play multiple summations and permutation objects in the course of deriving the expressions for the determining matrices.

By using the change of variables technique, deft application of mathematical induction principles and careful avoidance of some induction pitfalls, we were able to obtain the structure of the determining matrices for the single–delay neutral control model, without which the computational investigation of Euclidean controllability would be impossible.

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