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Fixed Point Results for Weakly Compatible Mappings in Convex G-Metric Space

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ABSTRACT*: In the present paper ,we establish different convex structures on G-metric space and deduce fixed point results for weakly compatible mappings.*
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KEYWORDS*: Takahashi convex structure, Convex Metric Space, convex G-metric space, Weakly compatible mappings .*

I. INTRODUCTION

Various generalizations of the usual notion of a metric space are proposed by several mathematicians. In 1963, Gahler [3] introduced the notion of 2-metric spaces but different authors proved that there is no relation between these two functions and there is no easy relationship between results obtained in the two settings. Ha et.al. [5] have pointed out that the results given by Gahler are independent, rather than generalizations of the corresponding results in metric spaces. In 1992, Dhage [2] introduced a new concept of D-metric space for the measure of nearness between three or more objects. But topological structure of so called D-metric spaces was proved to be incorrect. In 2006, Mustafa along with Sims introduced a new notion of generalized metric space called G-metric space [8]. This is a generalization of metric spaces in which a non-negative real number is assigned to every triplet of elements. Fixed point theory in these spaces was initiated in [9], Banach contraction mapping principle being the main tool. After that several fixed point results were proved in these spaces. Mustafa et al. studied many fixed point results for a self-mapping in G-metric space.[6]-[12]can be cited for reference. Takahashi [13] introduced the concept of convex structure in metric spaces and established some fixed point theorems. Inspired by this Thangavelu et.al. [14] introduced the concept of convexity structure in D-metric space. They further extended this concept to get strong convex D-metric space, J-convex D-metric spaces, weak convex D-metric spaces and quasi convex D-Metric Spaces. We extend this to G-metric space by providing different convex structures to G-metric space analogous to Thangavelu et.al. [14] and use it to prove some fixed point results .

II. PRELIMINARIES

DEFINITION 2.1. (G-Metric Space [8]). Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

$(G1)$ $G(x, y, z) = 0$, if $= y = z$;
$(G2)$ $0 < G(x, x, y)$, for all $x, y \in X$ with $\neq y$;
$(G3)$ $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
$(G4)$ $G(x, y, z) = G(y, x, z) = G(z, y, x) = \cdots$ (all permutations of x, y, z), (symmetry in all three variables);
$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

DEFINITION 2.2. Let (X, d) be a metric space and $I = [0,1]$. A mapping $W: X \times X \times I$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d\big(u, W(x, y, \lambda)\big) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

If (X, d) is equipped with a Takahashi convex structure, then it is called a convex metric space denoted by (X, d, W). A Banach space , or any convex subset of it is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$.

DEFINITION 2.3. Let X be a convex metric space. A nonempty subset M of X is said to be convex if $W(x, y, \lambda) \in M$ whenever$(x, y, \lambda) \in M \times M \times [0,1]$.

DEFINITION 2.4. Let (X, G) be a G-metric space. A mapping $W: X \times X \times X \times (0, 1] \to X$ is said to be a convex structure on (X, G) if for each $(x, y, z, \lambda) \in X \times X \times X \times (0, 1]$and for all $u, v \in X$ the condition

$$G(u,v,W(x,y,z,\lambda)) \leq \frac{\lambda}{3}\, G(u,v,x) + \frac{\lambda}{3}\, G(u,v,y) + \frac{\lambda}{3} G(u,v,z)$$

holds. If W is convex structure on a G-metric space (X,G), then the triplet (X,G,W) is called a convex G-metric space.

DEFINITION 2.5. A subset M of a convex G-metric space (X,G,W) is said to be a convex set if $W(x,y,z,\lambda) \in M$ for all $x,y,z \in M$ and for all λ with $0 < \lambda \leq 1$.

DEFINITION 2.6. Let (X,G) be a G-metric space. A mapping $W: X \times X \times X \times (0,1] \to X$ is said to be a strong convex structure on (X,G) if for each $(x,y,z,\lambda) \in X \times X \times X \times (0,1]$ and for all $u,v \in X$ the condition

$$G(u,v,W(x,y,z,\lambda)) \leq max\{\frac{\lambda}{3} G(u,v,x), \frac{\lambda}{3} G(u,v,y), \frac{\lambda}{3} G(u,v,z)\}$$

holds. If W is strong convex structure on a G-metric space (X,G), then the triplet (X,G,W) is called a strong convex G-metric space.

DEFINITION 2.7. A subset M of a strong convex G-metric space (X,G,W) is said to be a convex set if $W(x,y,z,\lambda) \in M$ for all $x,y,z \in M$ and for all λ with $0 < \lambda \leq 1$.

DEFINITION 2.8. Let (X,G) be a G-metric space. A mapping $W: X \times X \times X \times (0,1] \to X$ is said to be a β-convex structure on a G-metric space (X,G) if for each
$(x,y,z,\lambda) \in X \times X \times X \times I$ and for all $u,v \in X$ the condition

$$D(u,v,W(x,y,z,\lambda)) \leq min\{\frac{\lambda}{3} G(u,v,x), \frac{\lambda}{3} G(u,v,y), \frac{\lambda}{3} G(u,v,z)\}$$

holds. If W is a β-convex structure on a G-metric space (X,G), then the triplet (X,G,W) is called a β-convex G-metric space.

DEFINITION 2.9. A subset M of a β-convex G-metric space (X,D,W) is said to be a convex set if $W(x,y,z,\lambda) \in M$ for all $x,y,z \in M$ and for all λ with $0 < \lambda \leq 1$.

DEFINITION 2.10. ([8]). Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$ and the sequence $\{x_n\}$ is said to be G-convergent to x.
Thus, if $x_n \to x$ in a G-metric space (X,G), then for any $\varepsilon > 0$, there exists a positive integer N such that $G(x,x_n,x_m) < \varepsilon$ for all $n,m \geq N$.

DEFINITION 2.11. ([8]). Let (X,G) be a G-metric space. A sequence $\{x_n\}$ in X is called G-Cauchy if for every $\varepsilon > 0$, there is a positive integer N such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geq N$, that is, if $G(x_n,x_m,x_l) \to 0$, as $n,m,l \to \infty$.

EXAMPLE 2.1 (see [8]) Let R be the set of all real numbers. Define $G: R \times R \times R \to R^+$ by
$G(x,y,z) = |x-y| + |y-z| + |z-x|$, for all $x,y,z \in X$.
Then (R,G) is a G-metric space.
EXAMPLE 2.2 ([8]). Let (X,d) be a usual metric space. Then (X,G_1) and (X,G_2) are G-metric spaces where

$$G_1(x,y,z) = d(x,y) + d(y,z) + d(x,z)$$

for all $x,y,z \in X$ and

$$G_2(x,y,z) = max\{d(x,y), d(y,z), d(x,z)\}$$

for all $x,y,z \in X$.

LEMMA 2.1. (see [8]) Let (X,G) be a G-metric space. Then for any x,y,z, and $a \in X$, it follows that
(1) if $G(x,y,z) = 0$ then $x = y = z$,
(2) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,
(3) $G(x,y,y) \leq 2G(y,x,x)$,

(4) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,
(5) $G(x,y,z) \leq \frac{2}{3}(G(x,y,a) + G(x,a,z) + G(a,y,z)$

LEMMA 2.2. ([8]). If (X,G) is a G-metric space, then the following are equivalent:
(1) $\{x_n\}$ is G-convergent to x;
(2) $G(x_n,x_n,x) \to 0$, as $n \to \infty$
(3) $G(x_n,x,x) \to 0$, as $n \to \infty$;
(4) $G(x,x_n,x_m) \to 0$, as $m,n \to \infty$.

LEMMA 2.3. ([7]). If (X,G) is a G-metric space, then the following are equivalent:
(1) The sequence $\{x_n\}$ is G-Cauchy.
(2) For every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n,x_m,x_m) < \varepsilon$ for all $n,m \geq N$.

DEFINITION 2.12. ([8]). Let (X,G) and (X',G') be two G-metric spaces. Then a function $f: X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f\{x_n\}\}$ is G'-convergent to $f(x)$.

DEFINITION 2.13. ([8]). A G-metric space (X,G) is called symmetric if
$G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

DEFINITION 2.14. ([8]). A G-metric space (X,G) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence in (X,G) is convergent in X.
DEFINITION 2.15. Let A and B be two self-mappings on X. A point x of X is called
(i) a fixed point of A if $Ax = x$,
(ii) a common fixed point of the pair (A,B) if $Ax = Bx = x$ and
(iii) a coincidence point of the pair (A,B) if $Ax = Bx$.

DEFINITION 2.16. Let S and T be two self-mappings on X. The pair (S,T) is weakly compatible if S and T commute on the set of their coincidence points i.e. $SBx = BAx$ for all $x \in C(A,B)$.

III. MAIN RESULTS

THEOREM 3.1.1. Let (X,G,W) be a convex G-metric space. Let S and T be two self-mappings of X such that $S(X)$ is complete and $T(X) \subset S(X)$. If (S,T) is such that

$$G(Sx,Sy,Tz) \leq aG(Sx,Sy,Sz), \qquad (3.1.1.1)$$

then S and T have a unique coincidence point. Moreover if S and T are weakly compatible, then S and T have a unique common fixed point.
PROOF . Let x_0 be an arbitrary member of X. Since $T(X) \subset S(X)$ we can construct a sequence $\{Sx_n\}$ in $S(X)$ by defining

$$Sx_n = W(Sx_{n-1},Sx_{n-1},Tx_{n-1},\lambda), \quad n \in N \qquad (3.1.1.2)$$

Using $(G4)$ and $(G3)$ we have

$$\begin{aligned} G(Sx_n,Sx_{n+1},Sx_{n+1}) &= G(Sx_{n+1},Sx_{n+1},Sx_n) \\ &\leq G(Sx_{n+1},Sx_n,Sx_{n-1}) \\ &= G(Sx_{n-1},Sx_n,Sx_{n+1}) \end{aligned}$$

Now applying (3.1.1.2),we get

$$G(Sx_n,Sx_{n+1},Sx_{n+1}) \leq (Sx_{n-1},Sx_n,W(Sx_n,Sx_n,Tx_n,\lambda))$$

From the definition of convex G- metric space ,we infer

$$\begin{aligned} G(Sx_n,Sx_{n+1},Sx_{n+1}) &\leq \frac{\lambda}{3}G(Sx_{n-1},Sx_n,Sx_n) + \frac{\lambda}{3}G(Sx_{n-1},Sx_n,Sx_n) \\ &\quad + \frac{\lambda}{3}G(Sx_{n-1},Sx_n,Tx_n) \\ &= \frac{2\lambda}{3}G(Sx_{n-1},Sx_n,Sx_n) + \frac{\lambda}{3}G(Sx_{n-1},Sx_n,Tx_n) \\ &= \frac{2\lambda}{3}G(Sx_{n-1},Sx_n,Sx_n) + a\frac{\lambda}{3}G(Sx_{n-1},Sx_n,Sx_n) \end{aligned}$$

This implies that

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq (2+a)\frac{\lambda}{3} G(Sx_{n-1}, Sx_n, Sx_n) \qquad (3.1.1.3)$$

Let $r = (2+a)\frac{\lambda}{3}$. Since $0 \leq a < 1$, hence $0 \leq r < 1$.
By repeated application of (3.1.1.3), we have
$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq r^n G(Sx_0, Sx_1,$ [illegible] (3.1.1.4)
Then, for all $n, m \in N, n < m$, we have [illegible] peated us[illegible] he rectangle inequality and (3.1.1.4) that
$G(Sx_n, Sx_m, Sx_m) \leq G(Sx_n, Sx_{n+1}, S$ [illegible] $) + G(Sx_{n+}$ [illegible] $_{n+2}, Sx_{n+2})$
$+G($ [illegible] $_{n+2}, Sx_{n+3}, S$ [illegible] $) + \cdots G(Sx_{m-1}, Sx_m, Sx_m)$
$\leq (r^n + r^{n+1} + r$ [illegible] $+ \cdots + r^m$ [illegible] (Sx_0, Sx_1, Sx_1)
$\leq \frac{r^n}{1-r} G(Sx_0, Sx_1, Sx$ [illegible]
Letting $n, m \to \infty$,we obtain, $\lim_{n,m}$ [illegible] $Sx_m, Sx_m) = 0$, as

$$\lim_{n,m \to \infty} \text{[illegible]} G(Sx_0, Sx\text{[illegible]}x_1) = 0$$

For $n, m, l \in N$, $(G5)$ implies [illegible]
$G(Sx$ [illegible] $Sx_l) \leq G(S$ [illegible] $x_m) + G(Sx_l, Sx_m, Sx_m)$
Taking limit as $n, m, l \to \infty$ we [illegible] (Sx_n, Sx_m, Sx_l) [illegible] $\{Sx_n\}$ is a G-Cauchy sequence. By completeness of S[illegible] exists [illegible] is
G-convergent to Su i.e. $G(Sx_n, Sx_n, S$ [illegible] ∞.
Now with $x = z = x_n, y = u$ in (3.1.1.1), we get

$$G(Sx_n, Su, Tx_n) \leq aG(Sx_n, Su, Sx_n)$$

Letting $n \to \infty$ and and using the fact that the function G is continuous on its variables, we obtain that $G(Su, Su, Tx_n) \to 0$ which implies that $\{Tx_n\}$ is
G-convergent to Su.
Also $G(Sx_n, Su$

$$\frac{\lambda}{3} G(Sx_{n-1}, Sx_n, Tx_n)\}$$

Making use of (3.1.1.1) we obtain,

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq \frac{\lambda}{3} G(Sx_{n-1}, Sx_n, Sx_n)$$

Since $\lambda \in (0,1]$, repeating the steps as in theorem 3.1.1 we can construct Cauchy sequence $\{Sx_n\}$ converging to Su. Rest of the proof can be framed analogically.

COROLLARY 3.1.2. Let (X, G, W) be a strong convex G-metric space. Let S and T be two self-mappings of X such that $S(X)$ is complete and $T(X) \subset S(X)$. If (S, T) satisfies (3.1.1.1) then S and T have a unique coincidence point. Moreover if S and T are weakly compatible, then S and T have a unique common fixed point.

PROOF. After constructing a sequence $\{Sx_n\}$ in $S(X)$ defined by (3.1.1.2), i.e.

$$Sx_n = W(Sx_{n-1}, Sx_{n-1}, Tx_{n-1}, \lambda), \quad n \in N$$

and using $(G4)$, $(G3)$ and (3.1.1.2), we have

$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq (Sx_{n-1}, Sx_n, W(Sx_n, Sx_n, Tx_n, \lambda))$

From the definition of β-convex G- metric space ,we infer

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq min\{\frac{\lambda}{3} G(Sx_{n-1}, Sx_n, Sx_n), \frac{\lambda}{3} G(Sx_{n-1}, Sx_n, Sx_n),$$

$$\frac{\lambda}{3} G(Sx_{n-1}, Sx_n, Tx_n)\}$$

Making use of (3.1.1.1) we obtain,

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq \frac{\lambda}{3} G(Sx_{n-1}, Sx_n, Tx_n) \leq a\frac{\lambda}{3} G(Sx_{n-1}, Sx_n, Sx_n)$$

Since $0 \leq a < 1$ and $\lambda \in (0,1]$ we can construct Cauchy sequence $\{Sx_n\}$ converging to Su repeating the steps as in theorem 3.1.1 . Rest of the proof can be framed similarly.

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