Inference on The Doubly Truncated Gamma Distribution For Lifetime Data

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ABSTRACT: *Truncation in probability distributions may occur in many studies such as life testing and reliability. Truncation arises because, in many situations, failure of a unit is observed only if it fails before and/or after a certain period. In this paper, we discuss the distributional properties of, and make statistical inferences on, the two-parameter doubly truncated gamma distribution. Formulae for the moments and the moment-generating function of the distribution are given, and the estimates of the parameters and their properties are discussed. A simulation study has been conducted on the obtained results, and applications on lifetime datasets are illustrated.*

KEY WORDS: *Kernel density; maximum likelihood estimators; moments; truncation.*

I.   INTRODUCTION

Lifetime data pertain to the lifetimes of units, either industrial or biological. An industrial or a biological unit cannot be in operation forever. Such a unit cannot continue to operate in the same condition forever. Any random variable is said to be truncated if it can be observed over part of its range. Truncation occurs in various situations. For example, right truncation occurs in the study of life testing and reliability of items such as an electronic component, light bulbs, etc. Left truncation arises because, in many situations, failure of a unit is observed only if it fails after a certain period. Often, study units may not be followed at the beginning of an experiment until all of them fail, and the experimenter may have to start at a certain time and stop at a certain time when some of the units may still be working.

Broeder (l955), Chapman (l956), and Gross (l971) are some early papers that studied estimating the parameters of a truncated Gamma distribution. Later, Barndorff-Nielsen (1978) gave a set of general conditions for the existence and the uniqueness of the maximum likelihood estimators in a minimal exponential family. Hegde and Dahiya (1989) presented the estimation of the parameters of a truncated gamma distribution. Zaninetti (2014) presents a right and left truncated gamma distribution with application to the stars that introduces an upper and a lower boundary. The parameters which characterize the truncated gamma distribution are evaluated. The literature review indicated the estimation problem in the case of two-parameter truncated gamma distribution needs to be studied in greater depth. In this paper, we discuss the two-parameter doubly truncated Gamma distribution in detail. We start with the probability distribution function (pdf) and the cumulative probability distribution function (cdf) of the doubly truncated Gamma distribution and show their shape. We then obtain the moments and the moment-generating function of the distribution. Finally, we discuss the estimation problems of the parameters and discuss their properties.

II.   THE GAMMA DISTRIBUTION

Let X be a random variable taking values in the interval $[0, \infty]$ following the Gamma distribution. The pdf of the Gamma distribution with $\beta > 0$ as the scale and $\alpha > 0$ as the shape can be expressed as:

$$f(x; \alpha, \beta) = \frac{\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\beta \Gamma(\alpha)}, \tag{1}$$

(See Johnson, *et. al.*, 1994), where

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \tag{2}$$

The expected value of this distribution is:

$$E(X; \alpha, \beta) = \alpha\beta, \qquad (3)$$

and its variance is

$$Var(X; \alpha, \beta) = \alpha\beta^2. \qquad (4)$$

The cdf is denoted by:

$$F(x; \alpha, \beta) = \frac{\gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma(\alpha)}, \qquad (5)$$

where $\gamma(s, x)$ is the lower incomplete Gamma function and is given by:

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t}\, dt. \qquad (6)$$

Note that the term $\Gamma(s, x)$ is known as the upper incomplete Gamma function. See Abramowitz & Stegun, (1965) and Olver, et al., (2010) and is given by:

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t}\, dt,$$

$$= \Gamma(s) - \gamma(s, x),$$

by integration by parts, $\Gamma(s, x)$ may be expressed as

$$\Gamma(s, x) = (s-1)\Gamma(s-1, x) + x^{s-1} \qquad (7)$$

The mode of the Gamma distribution is at

$$m(x; \alpha, \beta) = \alpha\beta - \beta \qquad (8)$$

when $\alpha > 1$ (See Zaninetti, 2014).

The two parameters of the distribution can be estimated through the method of moments (MoM) by matching the moments to obtain the following estimates:

$$\hat{\alpha} = \left(\frac{\bar{x}}{s}\right)^2 ,$$

$$\hat{\beta} = \frac{s^2}{\bar{x}}, \qquad (9)$$

where $\bar{x}$ and s^2 are the sample mean and the sample variance, respectively (See Evans, et al. 2000).

The maximum likelihood (ML) estimators of both parameters α and β of the Gamma distribution cannot be obtained in a closed form. However, a fast algorithm for ML estimation of both parameters of a Gamma distribution may be obtained numerically using Newton's method (Minka, 2002) and generalized Newton's method (Minka, 2013). To start the iterative numerical procedure, we may use the initial values obtained from the method of moments given in Eq. (9). Alternatively, we may use the initial values proposed by Minka (2002) and given in Eq. (10) below.

$$\hat{\alpha} = \frac{1}{2\left(\ln(\bar{x}) - \frac{\sum_{i=1}^n \ln(x)}{n}\right)},$$

$$\hat{\beta} = 2\bar{x}\left(\ln(\bar{x}) - \frac{\sum_{i=1}^{n} \ln(x)}{n}\right). \tag{10}$$

III.   THE DISTRIBUTION OF THE DOUBLY TRUNCATED GAMMA

The probability distribution function (pdf):

   Let X be a random variable having the doubly truncated Gamma distribution in the interval $[a, b]$. The truncated pdf of any variable takes the form:

$$f(x; \alpha, \beta | a < X < b) = \frac{f(x; \alpha, \beta)}{\int_a^b f(x; \alpha, \beta) dx}$$

(See Block, et. al., 2010).

$$= \frac{f(x; \alpha, \beta)}{F(x; \alpha, \beta)|_a^b}, \tag{11}$$

$$f(x; \alpha, \beta, a, b) = \frac{f(x; \alpha, \beta)}{F(b; \alpha, \beta) - F(a; \alpha, \beta)}. \tag{12}$$

The pdf of the doubly truncated Gamma distribution then takes the form:

$$f(x; \alpha, \beta, a, b) = \frac{\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\beta \Gamma(\alpha)\left[\frac{\gamma\left(\alpha, \frac{b}{\beta}\right)}{\Gamma(\alpha)} - \frac{\gamma\left(\alpha, \frac{a}{\beta}\right)}{\Gamma(\alpha)}\right]}$$

$$= \frac{\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\beta\left[\gamma\left(\alpha, \frac{b}{\beta}\right) - \gamma\left(\alpha, \frac{a}{\beta}\right)\right]} \tag{13}$$

Using the property $\gamma(s, x) = \Gamma(s) - \Gamma(s, x)$, we obtain

$$f(x; \alpha, \beta, a, b) = \frac{\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\beta\left[\Gamma(\alpha) - \Gamma\left(\alpha, \frac{b}{\beta}\right) - \Gamma(\alpha) + \Gamma\left(\alpha, \frac{a}{\beta}\right)\right]}$$

$$= \frac{\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\beta\left[\Gamma\left(\alpha, \frac{a}{\beta}\right) - \Gamma\left(\alpha, \frac{b}{\beta}\right)\right]} \tag{14}$$

and can be expressed as:

$$f(x; \alpha, \beta, a, b) = K\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}, \tag{15}$$

where the constant K is

$$K = \frac{1}{\beta\left[\Gamma\left(\alpha, \frac{a}{\beta}\right) - \Gamma\left(\alpha, \frac{b}{\beta}\right)\right]}. \tag{16}$$

In Figure (1), we present the shape of the truncated Gamma distribution function using Eq. (15) together with the distribution of a simulated sample of 1000 observations generated from the truncated Gamma distribution superimposed on the empirical distribution generated from the Gamma distribution using Eq. (1). However, Figure (2) presents the shape of the same distributions of truncated gamma distribution function using Eq. (15) together with the distribution of a simulated sample of 1000 observations, but this time generated from the Gamma distribution and then truncated by omitting values outside the truncation range superimposed on the empirical distribution generated from the Gamma distribution using Eq. (1). It is clear the empirical distribution and theoretical one generated from Eq. (15) are identical.
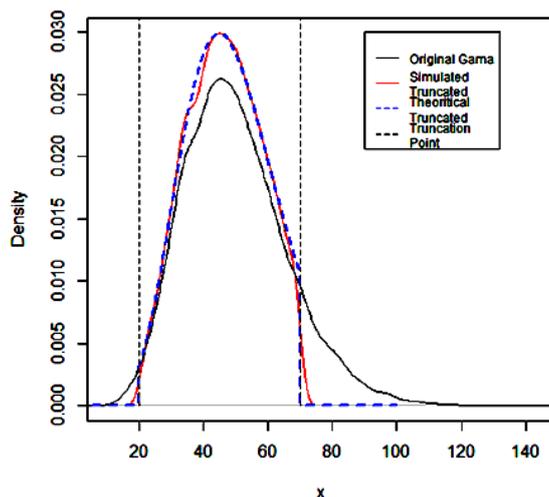
Figure (1): Distribution of Truncated Data Generated from the Gamma Distribution with (alpha=5, beta=10)
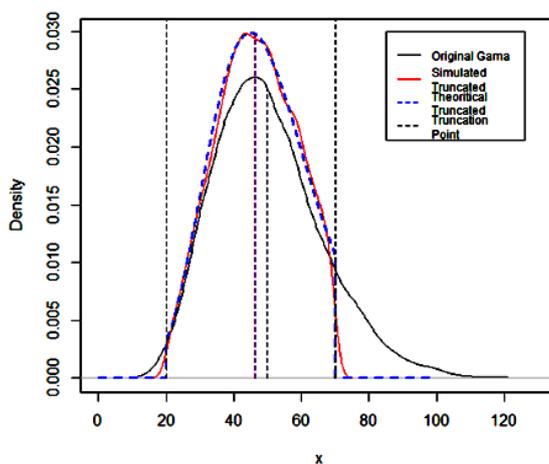
Figure (2): Distribution of Data Generated from Doubly Truncated Gamma Distribution with (alpha=5, beta=10)

Figure (3) presents the shape of the truncated Gamma distribution function using Eq. (15) with different values of left truncation points (a=20,35,50) and a fixed right truncation point at (b=70) together with the original Gamma distribution. However, Figure (4) presents the shape of the truncated Gamma distribution function with different values of right truncation points (b=50,60,70,80) and a fixed left truncation point (a=30) together with the original Gamma distribution.
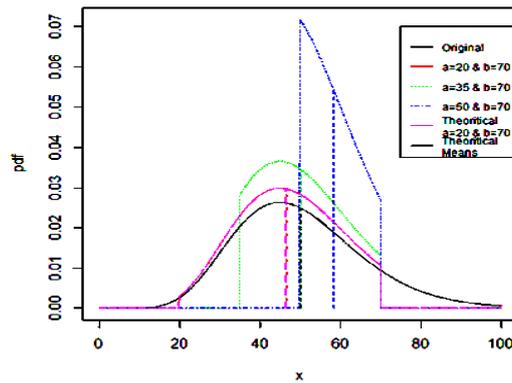
Figure (3): Doubly Truncated Gamma Distribution (alpha=5, beta=10) with different left truncation points and fixed right truncation point
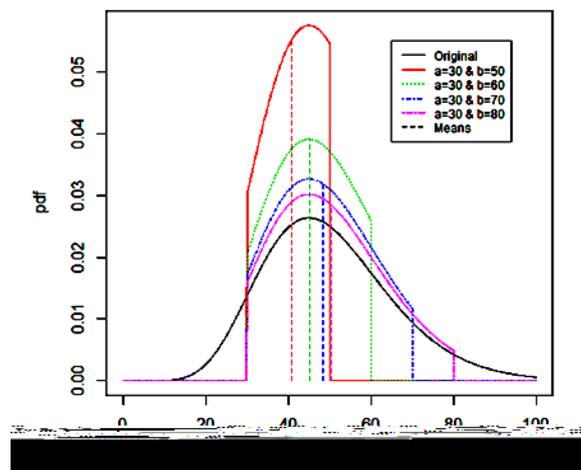
Figure (4): Doubly Truncated Gamma Distributions (alpha=5, beta=10) with different right truncation points (b=50,60,70,80) and fixed left truncation point

Special Cases : Let X be a random variable having the left truncated Gamma distribution and taking values in the interval   . Using Eq. (15) of the doubly truncated Gamma distribution above and the fact that $\Gamma(\alpha, \infty) = \int_{\infty}^{\infty} t^{\alpha-1} e^{-t} \, dt = 0$, we obtain:

$$f(x; \alpha, \beta, a) = \frac{\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\beta \left[\Gamma\left(\alpha, \frac{a}{\beta}\right)\right]}$$

$$= \frac{x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\left[\int_{a}^{\infty} u^{\alpha-1} e^{-\frac{u}{\beta}} \, du\right]} \tag{17}$$

Equation (17) is identical to a result of (Koning & Franses, 2003).

Now, Let X be a random variable having the right truncated Gamma distribution and taking values in the interval $[0, b]$. Using Eq. (15) we get:

$$f(x; \alpha, \beta, b) = \frac{\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\beta \left[\Gamma(\alpha) - \Gamma\left(\alpha, \frac{b}{\beta}\right)\right]}$$

$$= \frac{x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\beta^\alpha \left[\gamma\left(\alpha, \frac{b}{\beta}\right)\right]}$$

$$= \frac{x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}}{\left[\int_0^b u^{\alpha-1} e^{-\frac{u}{\beta}} du\right]}. \tag{18}$$

This is identical to Eq. (1) in Hegde and Dahiya (1989).

The Cumulative Distribution Function

The cdf of the truncated Gamma distribution takes the form:

$$F(x; \alpha, \beta, a, b) = \int_a^x K \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)} dx$$

$$= K \left[\int_a^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)} dx - \int_x^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)} dx\right]$$

$$= K\beta \left[\Gamma\left(\alpha, \frac{a}{\beta}\right) - \Gamma\left(\alpha, \frac{x}{\beta}\right)\right]. \tag{19}$$

Figure (5) presents the shape of the cdf of the doubly truncated Gamma distribution using Eq. (19) together with the cdf of a simulated sample of 1000 observations generated from the truncated Gamma distribution and superimposed on the Gamma distribution of Eq. (1). The Figure shows the three cdf curves of the doubly truncated Gamma distributions are identical. Figure (6) also presents the shape of the cdf of the truncated Gamma distribution using Eq. (19) with different values of truncation points together with the cdf of the original Gamma distribution.
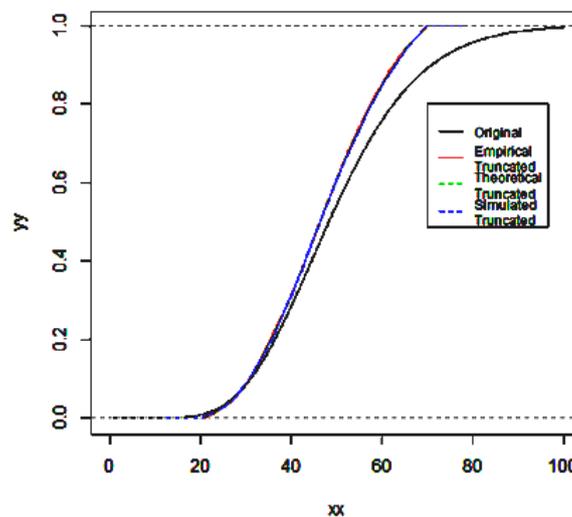
Figure (5): Empirical cdf for doubly truncated Gamma Distribution (alpha=5, beta=10) with truncation points at a=20, b=70
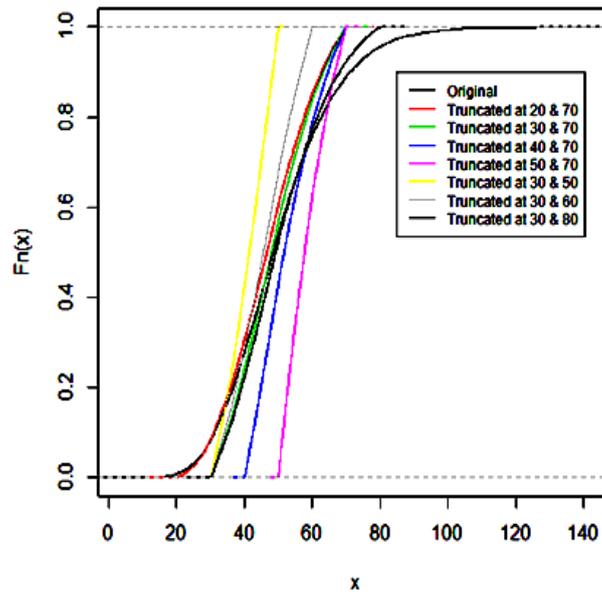
Figure (6): Empirical CDF for Gamma Distribution (alpha=5, beta=10) with truncation points a=20, b=70

IV.   THE MOMENTS OF THE DOUBLY TRUNCATED GAMMA DISTRIBUTION

The mth moment:

The mth moment of the truncated Gamma distribution is

$$E(X^m; \alpha, \beta, a, b) = \int_a^b x^m f(x; \alpha, \beta, a, b) \, dx$$

$$= \int_a^b x^m K \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)} dx$$

$$= K\beta^m \int_a^b \left(\frac{x}{\beta}\right)^{(\alpha+m)-1} e^{-\left(\frac{x}{\beta}\right)} dx$$

$$= K\beta^m \left[\int_a^\infty \left(\frac{x}{\beta}\right)^{(\alpha+m)-1} e^{-\left(\frac{x}{\beta}\right)} dx - \int_b^\infty \left(\frac{x}{\beta}\right)^{(\alpha+m)-1} e^{-\left(\frac{x}{\beta}\right)} dx\right]$$

$$= K\beta^m \left[\int_{\frac{a}{\beta}}^\infty t^{(\alpha+m)-1} e^{-t} \beta \, dt - \int_{\frac{b}{\beta}}^\infty t^{(\alpha+m)-1} e^{-t} \beta \, dt\right]$$

$$= K\beta^{m+1} \left[\Gamma\left(\alpha + m, \frac{a}{\beta}\right) - \Gamma\left(\alpha + m, \frac{b}{\beta}\right)\right]. \qquad (20)$$

The Mode of the Truncated Gamma Distribution :

The mode of the truncated Gamma distribution is the value x at which its pdf has the maximum value.

$$\frac{df(x; \alpha, \beta, a, b)}{dx} = \frac{d}{dx} \left[K \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}\right] = 0$$

Therefore,

$$\frac{K}{\beta^{\alpha-1}} \frac{d}{dx}\left[x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)}\right] = 0$$

$$\frac{K}{\beta^{\alpha-1}}\left[(\alpha-1)x^{\alpha-2}e^{-\left(\frac{x}{\beta}\right)} + x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)}\left(\frac{-1}{\beta}\right)\right] = 0$$

$$(\alpha-1)x^{\alpha-2}e^{-\left(\frac{x}{\beta}\right)} - x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)}\left(\frac{1}{\beta}\right) = 0$$

$$(\alpha-1) = x\left(\frac{1}{\beta}\right)$$

$$x = \beta(\alpha-1) \tag{21}$$

where $\alpha > 1$. Therefore, we get the mode of the doubly truncated Gamma distribution at $x = \beta(\alpha-1)$.

The Moment Generating Function of the Truncated Gamma Distribution

For random variable X which follows a gamma distribution, $\Gamma(\alpha, \beta)$, the moment-generating function (mgf) is given by

$$M_X(t) = (1-\beta t)^{-\alpha}, \qquad t \in \left[0, \frac{1}{\beta}\right) \tag{22}$$

Now we consider Y a random variable which follows a doubly truncated version of X with lower truncation point, a, and upper truncation point, b. The mgf of Y is

$$M_Y(t) = E[e^{tY}] = \int_a^b e^{ty} f(y; \alpha, \beta, a, b)\, dy$$

$$= \int_a^b e^{ty} \frac{f(y; \alpha, \beta)}{F(b; \alpha, \beta) - F(a; \alpha, \beta)}\, dy$$

$$= \frac{1}{F(b; \alpha, \beta) - F(a; \alpha, \beta)} \int_a^b e^{ty} \frac{y^{\alpha-1}e^{-\left(\frac{y}{\beta}\right)}}{\beta^{\alpha}\Gamma(\alpha)}\, dy$$

$$= \frac{1}{F(b; \alpha, \beta) - F(a; \alpha, \beta)} \int_a^b \frac{y^{\alpha-1}e^{-\left(\frac{(1-\beta t)y}{\beta}\right)}}{\beta^{\alpha}\Gamma(\alpha)}\, dy \tag{23}$$

$$= \frac{(1-\beta t)^{1-\alpha}}{F(b; \alpha, \beta) - F(a; \alpha, \beta)} \int_a^b \frac{[(1-\beta t)y]^{\alpha-1}e^{-\left(\frac{(1-\beta t)y}{\beta}\right)}}{\beta^{\alpha}\Gamma(\alpha)}\, dy.$$

$$= \frac{(1-\beta t)^{-\alpha}}{F(b; \alpha, \beta) - F(a; \alpha, \beta)} \int_{(1-\beta t)a}^{(1-\beta t)b} \frac{r^{\alpha-1}e^{-\left(\frac{r}{\beta}\right)}}{\beta^{\alpha}\Gamma(\alpha)}\, dr\,.$$

$$= \frac{(1-\beta t)^{-\alpha}}{F(b; \alpha, \beta) - F(a; \alpha, \beta)} [F(b(1-\beta t); \alpha, \beta)$$
$$- F(a(1-\beta t); \alpha, \beta)] \tag{24}$$

We can rewrite that definition as

$$F(b(1-\beta t);\alpha,\beta) - F(a(1-\beta t);\alpha,\beta) = 1 - \frac{\Gamma\left(\alpha,\frac{b(1-\beta t)}{\beta}\right)}{\Gamma(\alpha)} - 1 + \frac{\Gamma\left(\alpha,\frac{a(1-\beta t)}{\beta}\right)}{\Gamma(\alpha)},$$

$$= \frac{\Gamma\left(\alpha,\frac{a(1-\beta t)}{\beta}\right) - \Gamma\left(\alpha,\frac{b(1-\beta t)}{\beta}\right)}{\Gamma(\alpha)}, \tag{25}$$

Now, substituting Eq. (25) in Eq. (24), we obtain

$$M_Y(t) = \left[\frac{F(b(1-\beta t);\alpha,\beta) - F(a(1-\beta t);\alpha,\beta)}{F(b;\alpha,\beta) - F(a;\alpha,\beta)}\right]M_X(t),$$

$$= \left[\frac{\Gamma\left(\alpha,\frac{a(1-\beta t)}{\beta}\right) - \Gamma\left(\alpha,\frac{b(1-\beta t)}{\beta}\right)}{\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)}\right]M_X(t). \tag{26}$$

Thus, the moment-generating function of Y which follows a doubly truncated Gamma distribution equals the product of the moment-generating function of a random variable X which follows the Gamma distribution and a factor which accounts for the truncation.

Computing the Moments from the Moment Generating Function

The expected value of the random variable Y which follows the doubly truncated gamma distribution is

$$E(Y;\alpha,\beta,a,b) = M_Y'(t)|_{t=0}$$

$$M_Y'(t) = \frac{1}{\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)}\left[\left(\Gamma\left(\alpha,\frac{a(1-\beta t)}{\beta}\right) - \Gamma\left(\alpha,\frac{b(1-\beta t)}{\beta}\right)\right)M_X'(t)\right.$$

$$+ \left(\left(\frac{a^\alpha(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}}e^{-\frac{a(1-\beta t)}{\beta}}\right)\right.$$

$$\left.\left.- \left(\frac{b^\alpha(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}}e^{-\frac{b(1-\beta t)}{\beta}}\right)\right)M_X(t)\right], \tag{27}$$

Thus, we have

$$E(Y;\alpha,\beta,a,b) = M_Y'(t)|_{t=0}$$

$$= \frac{1}{\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)}\left[\left(\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)\right)E(X;\alpha,\beta)\right.$$

$$\left.+ \left(\left(\frac{a^\alpha}{\beta^{\alpha-1}}e^{-\frac{a}{\beta}}\right) - \left(\frac{b^\alpha}{\beta^{\alpha-1}}e^{-\frac{b}{\beta}}\right)\right)(1-\beta t)^{-\alpha}|_{t=0}\right]$$

$$= \alpha\beta + \left(\frac{\left(\frac{a^\alpha}{\beta^{\alpha-1}}e^{-\frac{a}{\beta}}\right) - \left(\frac{b^\alpha}{\beta^{\alpha-1}}e^{-\frac{b}{\beta}}\right)}{\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)}\right)$$

$$= E(X) + \left(\frac{a^\alpha e^{-\frac{a}{\beta}} - b^\alpha e^{-\frac{b}{\beta}}}{\beta^{\alpha-1}\left[\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)\right]}\right). \tag{28}$$

$$= \frac{\alpha\beta^\alpha\Gamma\left(\alpha,\frac{a}{\beta}\right) - \alpha\beta^\alpha\Gamma\left(\alpha,\frac{b}{\beta}\right) + a^\alpha e^{-\frac{a}{\beta}} - b^\alpha e^{-\frac{b}{\beta}}}{\beta^{\alpha-1}\left[\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)\right]}$$

$$= K\beta^{2-\alpha}\left[\alpha\beta^\alpha\Gamma\left(\alpha,\frac{a}{\beta}\right) - \alpha\beta^\alpha\Gamma\left(\alpha,\frac{b}{\beta}\right) + a^\alpha e^{-\frac{a}{\beta}} - b^\alpha e^{-\frac{b}{\beta}}\right]$$

$$= K\beta^2\left[\Gamma\left(\alpha+1,\frac{a}{\beta}\right) - \Gamma\left(\alpha+1,\frac{b}{\beta}\right)\right]. \tag{29}$$

Now, to obtain the variance of the doubly truncated gamma distribution, we have

$$E(Y^2; \alpha, \beta, a, b) = M_Y''(t)|_{t=0}$$

$$M_Y''(t) = \frac{1}{\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)}\left[\left(\Gamma\left(\alpha,\frac{a(1-\beta t)}{\beta}\right) - \Gamma\left(\alpha,\frac{b(1-\beta t)}{\beta}\right)\right)M_X''(t)\right.$$

$$+ \left(\left(-\left(\frac{a(1-\beta t)}{\beta}\right)^{\alpha-1} e^{-\frac{a(1-\beta t)}{\beta}} * -a\right)\right.$$

$$\left.- \left(-\left(\frac{b(1-\beta t)}{\beta}\right)^{\alpha-1} e^{-\frac{b(1-\beta t)}{\beta}} * -b\right)\right)M_X'(t)$$

$$\left.+ \left(\left(\frac{a^\alpha(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}} e^{-\frac{a(1-\beta t)}{\beta}}\right) - \left(\frac{b^\alpha(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}} e^{-\frac{b(1-\beta t)}{\beta}}\right)\right)M_X'(t)\right]$$

$$+ \left(\left(\frac{a^\alpha(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}} e^{-\frac{a(1-\beta t)}{\beta}} * a\right) + \left(\frac{-a^\alpha(\alpha-1)(1-\beta t)^{\alpha-2}}{\beta^{\alpha-2}} e^{-\frac{a(1-\beta t)}{\beta}}\right)\right.$$

$$\left.- \left(\frac{b^\alpha(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}} e^{-\frac{b(1-\beta t)}{\beta}} * b\right) + \left(\frac{b^\alpha(\alpha-1)(1-\beta t)^{\alpha-2}}{\beta^{\alpha-2}} e^{-\frac{b(1-\beta t)}{\beta}}\right)\right)M_X(t)$$

$$M_Y''(t) = \frac{1}{\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)}\left[\left(\Gamma\left(\alpha,\frac{a(1-\beta t)}{\beta}\right) - \Gamma\left(\alpha,\frac{b(1-\beta t)}{\beta}\right)\right)M_X''(t)\right.$$

$$+ 2\left(\left(\frac{a^\alpha(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}} e^{-\frac{a(1-\beta t)}{\beta}}\right) - \left(\frac{b^\alpha(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}} e^{-\frac{b(1-\beta t)}{\beta}}\right)\right)M_X'(t)$$

$$+ \left(\left(\frac{a^{\alpha+1}(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}} e^{-\frac{a(1-\beta t)}{\beta}}\right) - \left(\frac{a^\alpha(\alpha-1)(1-\beta t)^{\alpha-2}}{\beta^{\alpha-2}} e^{-\frac{a(1-\beta t)}{\beta}}\right)\right.$$

$$\left.- \left(\frac{b^{\alpha+1}(1-\beta t)^{\alpha-1}}{\beta^{\alpha-1}} e^{-\frac{b(1-\beta t)}{\beta}}\right) + \left(\frac{b^\alpha(\alpha-1)(1-\beta t)^{\alpha-2}}{\beta^{\alpha-2}} e^{-\frac{b(1-\beta t)}{\beta}}\right)\right)M_X(t)\right]$$

Therefore, we have

$$E(Y^2; \alpha, \beta, a, b) = M_Y''(t)|_{t=0}$$

$$= \frac{1}{\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)} \left[\left(\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right) \right) E(x^2;\alpha,\beta) + 2\left(\left(\frac{a^\alpha}{\beta^{\alpha-1}} e^{-\frac{a}{\beta}}\right) - \left(\frac{b^\alpha}{\beta^{\alpha-1}} e^{-\frac{b}{\beta}}\right) \right) E(x;\alpha,\beta) \right.$$

$$\left. + \left(\left(\frac{a^{\alpha+1}}{\beta^{\alpha-1}} e^{-\frac{a}{\beta}}\right) - \left(\frac{a^\alpha(\alpha-1)}{\beta^{\alpha-2}} e^{-\frac{a}{\beta}}\right) - \left(\frac{b^{\alpha+1}}{\beta^{\alpha-1}} e^{-\frac{b}{\beta}}\right) + \left(\frac{b^\alpha(\alpha-1)}{\beta^{\alpha-2}} e^{-\frac{b}{\beta}}\right) \right)(1-\beta t)^{-\alpha}\big|_{t=0} \right]$$

$$= \frac{1}{\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)} \left[\left(\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right) \right)(\alpha\beta^2 + \alpha^2\beta^2) + \left(\frac{2\alpha\beta a^\alpha}{\beta^{\alpha-1}} e^{-\frac{a}{\beta}}\right) \right.$$

$$- \left(\frac{2\alpha\beta b^\alpha}{\beta^{\alpha-1}} e^{-\frac{b}{\beta}}\right) + \left(\frac{a^{\alpha+1}}{\beta^{\alpha-1}} e^{-\frac{a}{\beta}}\right) - \left(\frac{a^\alpha(\alpha-1)}{\beta^{\alpha-2}} e^{-\frac{a}{\beta}}\right) - \left(\frac{b^{\alpha+1}}{\beta^{\alpha-1}} e^{-\frac{b}{\beta}}\right)$$

$$\left. + \left(\frac{b^\alpha(\alpha-1)}{\beta^{\alpha-2}} e^{-\frac{b}{\beta}}\right) \right]$$

$$= E(x^2) + \left[\frac{e^{-\frac{a}{\beta}}(2\alpha\beta a^\alpha + a^{\alpha+1} - \beta a^\alpha(\alpha-1)) + e^{-\frac{b}{\beta}}(-2\alpha\beta b^\alpha - b^{\alpha+1} + \beta b^\alpha(\alpha-1))}{\beta^{\alpha-1}\left[\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)\right]} \right]$$

$$= K\beta^3 \left[\alpha\Gamma\left(\alpha,\frac{a}{\beta}\right) - \alpha\Gamma\left(\alpha,\frac{b}{\beta}\right) + \alpha^2\Gamma\left(\alpha,\frac{a}{\beta}\right) - \alpha^2\Gamma\left(\alpha,\frac{b}{\beta}\right) + \alpha\beta^{-\alpha}a^\alpha e^{-\frac{a}{\beta}} + \beta^{-(\alpha+1)}a^{\alpha+1}e^{-\frac{a}{\beta}} \right.$$

$$\left. + \beta^{-\alpha}a^\alpha e^{-\frac{a}{\beta}} - \alpha\beta^{-\alpha}b^\alpha e^{-\frac{b}{\beta}} - \beta^{-(\alpha+1)}b^{\alpha+1}e^{-\frac{b}{\beta}} - \beta^{-\alpha}b^\alpha e^{-\frac{b}{\beta}} \right]$$

$$= K\beta^3 \left[\Gamma\left(\alpha+1,\frac{a}{\beta}\right) - \Gamma\left(\alpha+1,\frac{b}{\beta}\right) + \alpha^2\Gamma\left(\alpha,\frac{a}{\beta}\right) - \alpha^2\Gamma\left(\alpha,\frac{b}{\beta}\right) + \alpha\left(\frac{a}{\beta}\right)^\alpha e^{-\frac{a}{\beta}} + \left(\frac{a}{\beta}\right)^{\alpha+1} e^{-\frac{a}{\beta}} - \alpha\left(\frac{b}{\beta}\right)^\alpha e^{-\frac{b}{\beta}} \right.$$

$$\left. - \left(\frac{b}{\beta}\right)^{\alpha+1} e^{-\frac{b}{\beta}} \right]$$

$$= K\beta^3 \left[\Gamma\left(\alpha+2,\frac{a}{\beta}\right) - \Gamma\left(\alpha+2,\frac{b}{\beta}\right) + \alpha^2\left[\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)\right] - \alpha^2\left[\Gamma\left(\alpha,\frac{a}{\beta}\right) - \Gamma\left(\alpha,\frac{b}{\beta}\right)\right] \right]$$

$$= K\beta^3 \left[\Gamma\left(\alpha+2,\frac{a}{\beta}\right) - \Gamma\left(\alpha+2,\frac{b}{\beta}\right) \right]. \tag{30}$$

Therefore,

$$Var(X;\alpha,\beta,a,b) = E(X^2;\alpha,\beta,a,b) - \left(E(X;\alpha,\beta,a,b)\right)^2.$$

V.   ESTIMATION OF THE PARAMETERS OF THE DOUBLY TRUNCATED GAMMA DISTRIBUTION

Maximum Likelihood Estimators:

The likelihood function is denoted by

$$L(\alpha,\beta) = K^n \left(\prod_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{\alpha-1} \right) e^{\frac{-\sum_{i=1}^{n} x_i}{\beta}}$$

$$= K^n e^{(\alpha-1)\sum_{i=1}^{n} \ln\left(\frac{x}{\beta}\right) - \frac{\sum_{i=1}^{n} x_i}{\beta}}. \tag{31}$$

The log-likelihood function is

$$\ell(\alpha, \beta) = n \ln K + (\alpha - 1) \sum_{i=1}^{n} \ln \left(\frac{x}{\beta}\right) - \frac{\sum_{i=1}^{n} x_i}{\beta}. \tag{32}$$

Now, to obtain the maximum likelihood estimators, we get the derivative of equation (32) with respect to α and β.

$$\frac{\partial \ell(\alpha, \beta)}{\partial \alpha} = \frac{n}{K} \left[\beta K^2 \left[\ln \left(\frac{b}{\beta} \Gamma \left(\alpha, \frac{b}{\beta}\right)\right) + \frac{b}{\beta} T \left(3, \alpha, \frac{b}{\beta}\right) - \ln \left(\frac{a}{\beta} \Gamma \left(\alpha, \frac{a}{\beta}\right)\right) - \frac{a}{\beta} T \left(3, \alpha, \frac{a}{\beta}\right) \right] \right] + \sum_{i=1}^{n} \ln \left(\frac{x}{\beta}\right)$$

$$= 0$$

$$n \beta K \left[\ln \left(\frac{b}{\beta} \Gamma \left(\alpha, \frac{b}{\beta}\right)\right) + \frac{b}{\beta} T \left(3, \alpha, \frac{b}{\beta}\right) - \ln \left(\frac{a}{\beta} \Gamma \left(\alpha, \frac{a}{\beta}\right)\right) - \frac{a}{\beta} T \left(3, \alpha, \frac{a}{\beta}\right) \right]$$

$$+ \sum_{i=1}^{n} \ln \left(\frac{x}{\beta}\right) = 0, \tag{33}$$

where the function

$$T(m, s, x) = G_{m-1, m}^{m, 0} \left(\begin{matrix} 0, 0, \dots, 0 \\ s - 1, -1, \dots, -1 \end{matrix} \middle| x \right).$$

is a special case of the Meijer G-function (See Prudnikov, et al., 1992).

$$\frac{\partial \ell(\alpha, \beta)}{\partial \beta} = \frac{n}{K} \left[K^2 \left[\Gamma \left(\alpha, \frac{b}{\beta}\right) - \Gamma \left(\alpha, \frac{a}{\beta}\right) + \left(\frac{b}{\beta}\right)^{\alpha} e^{-\frac{b}{\beta}} - \left(\frac{a}{\beta}\right)^{\alpha} e^{-\frac{a}{\beta}} \right] \right] - \frac{n(\alpha - 1)}{\beta} + \frac{n\bar{x}}{\beta^2} = 0$$

hence we have

$$K \left[\left(\Gamma \left(\alpha, \frac{b}{\beta}\right) - \Gamma \left(\alpha, \frac{a}{\beta}\right) \right) + \left(\Gamma \left(\alpha + 1, \frac{b}{\beta}\right) - \Gamma \left(\alpha + 1, \frac{a}{\beta}\right) \right) - \alpha \left(\Gamma \left(\alpha, \frac{b}{\beta}\right) - \Gamma \left(\alpha, \frac{a}{\beta}\right) \right) \right] - \frac{(\alpha - 1)}{\beta} + \frac{\bar{x}}{\beta^2}$$

$$= 0$$

$$\frac{\beta \left[\Gamma \left(\alpha + 1, \frac{b}{\beta}\right) - \Gamma \left(\alpha + 1, \frac{a}{\beta}\right) \right] + \bar{x} \left[\Gamma \left(\alpha, \frac{a}{\beta}\right) - \Gamma \left(\alpha, \frac{b}{\beta}\right) \right]}{\beta^2 \left[\Gamma \left(\alpha, \frac{a}{\beta}\right) - \Gamma \left(\alpha, \frac{b}{\beta}\right) \right]} = 0$$

$$\beta \left[\Gamma \left(\alpha + 1, \frac{b}{\beta}\right) - \Gamma \left(\alpha + 1, \frac{a}{\beta}\right) \right] + \bar{x} \left[\Gamma \left(\alpha, \frac{a}{\beta}\right) - \Gamma \left(\alpha, \frac{b}{\beta}\right) \right] = 0 \tag{34}$$

ML estimators of α and β can be found by solving equations (33) and (34) numerically using the Newton-Raphson iteration method. ML estimation with grouped data has been discussed by Rosaiah et al. (1991) in the context of choosing optimal groups. Numerically, the ML estimates are found by maximizing a function of incomplete Gamma integrals (See Brawn and Upton, 2007). Moreover, the ML estimates of α and β can be obtained by maximizing the log-likelihood in Eq. (32) and can be accomplished by using the R software. The function maxLik of the maxLik library in R software can be used to find the ML estimates of α and β for data from a truncated Gamma distribution.

Alternative estimation procedures: Chapman (1956) studied the estimation problem of three-parameter truncated Gamma distribution and suggested a procedure based on deliberate grouping of the data. His procedure depends on reducing the number of parameters from three to two. Chapman chose to work with the logarithms of the ratios of the counts in successive bins (so that N cancels). The inversion of the $(r - 1) \times (r - 1)$ variance–covariance matrix is not straightforward. Chapman gave the form of the inverse in the case of equiprobable bins and suggested omitting every second ratio if the inversion is infeasible, though that results in much less efficient estimates. Dahiya and Gurland (1978) wished to circumvent the non-linear maximum likelihood equations. They developed generalized minimum chi-squared estimators that were the solutions resulting from a lengthy sequence of simple matrix operations, with two of the three critical matrices involved containing differences of estimated moments and the third containing estimated cumulates. Their study concluded that, theoretically, these were efficient estimators.

VI. SIMULATION STUDY

A simulation has been conducted to study the properties of the ML estimators of α and β of the truncated Gamma distribution at different sample sizes (n=20,50,100,200,500) when the true parameters equal ($\alpha = 5$ and $\beta = 10$). Figure (7) presents bootstrap distributions for 100 samples of truncated Gamma distribution with a (red) solid line together with 100 samples of Gamma distribution with a (black) solid line. We observe that the sampling distribution and bootstrap distribution are the same. Figure (8) presents the bootstrap distribution of the means of truncated gamma distribution together with the original distribution, and the dotted lines correspond to the means. Based on the bootstrap distribution, the 95% confidence interval for the mean by percentile bootstrap method is (46.31, 46.76).
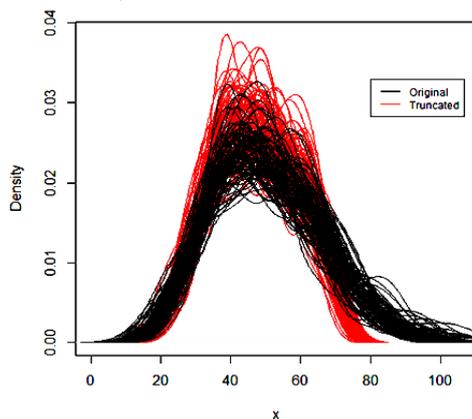
Figure (7) : The Densities of 100 Bootstrapped Samples for Gamma Distribution (alpha=5, beta=10).
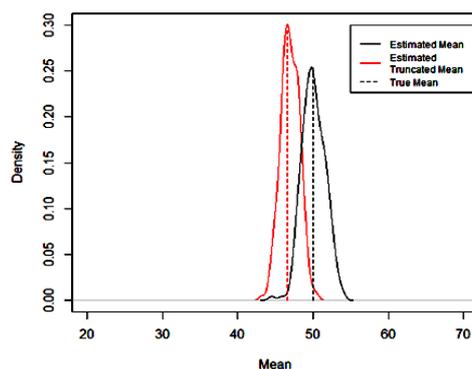
Figure (8) : Comparison of Bootstrapped Truncated Means for Gamma Distribution (alpha=5, beta=10).

Table (1): Expected Values and Standard Errors of the Estimate of α and β of the Truncated Gamma Distribution at Different Sample Sizes When the True Parameter (alpha=5, beta=10).

Sample size (n)	For α		For β	
	Mean	Standard Error	Mean	Standard Error
20	1570.772	36692.64	9.560196	7.759768
20 (trimmed)	6.070658	6.756753	12.11214	6.663903
50	18.23002	1110.056	10.74494	3.753756
50 (trimmed)	5.658334	3.732658	10.74494	3.753756
100	5.291514	2.145619	10.35315	2.563726
200	5.116577	1.122945	10.17575	1.773892
500	5.050782	0.6633553	10.0587	1.109506

Figure (9) represents the sampling distributions of the estimate of scale parameter ($\hat{\beta}$) of the truncated gamma distribution for different sample sizes (n=20,50,100,200,500) with a (red) solid line together with the dotted line, which represents the mean of distribution. Figure (10) presents the sampling distributions of the estimate of shape parameter ($\hat{\alpha}$) of the truncated gamma distribution for different sample sizes (n=20,50,100,200,500) with a (blue) solid line together with the dotted line, which represents the mean of the distribution.

From figure (9), figure (10), and table (1) above we can observe that the estimates of the parameters α and β of the truncated Gamma distribution are asymptotically unbiased for large samples. As can be seen in Table (1), the standard error decreases as the sample size increases, which indicates the estimators of the parameters, $\hat{\alpha}$ and $\hat{\beta}$, are consistent estimators.
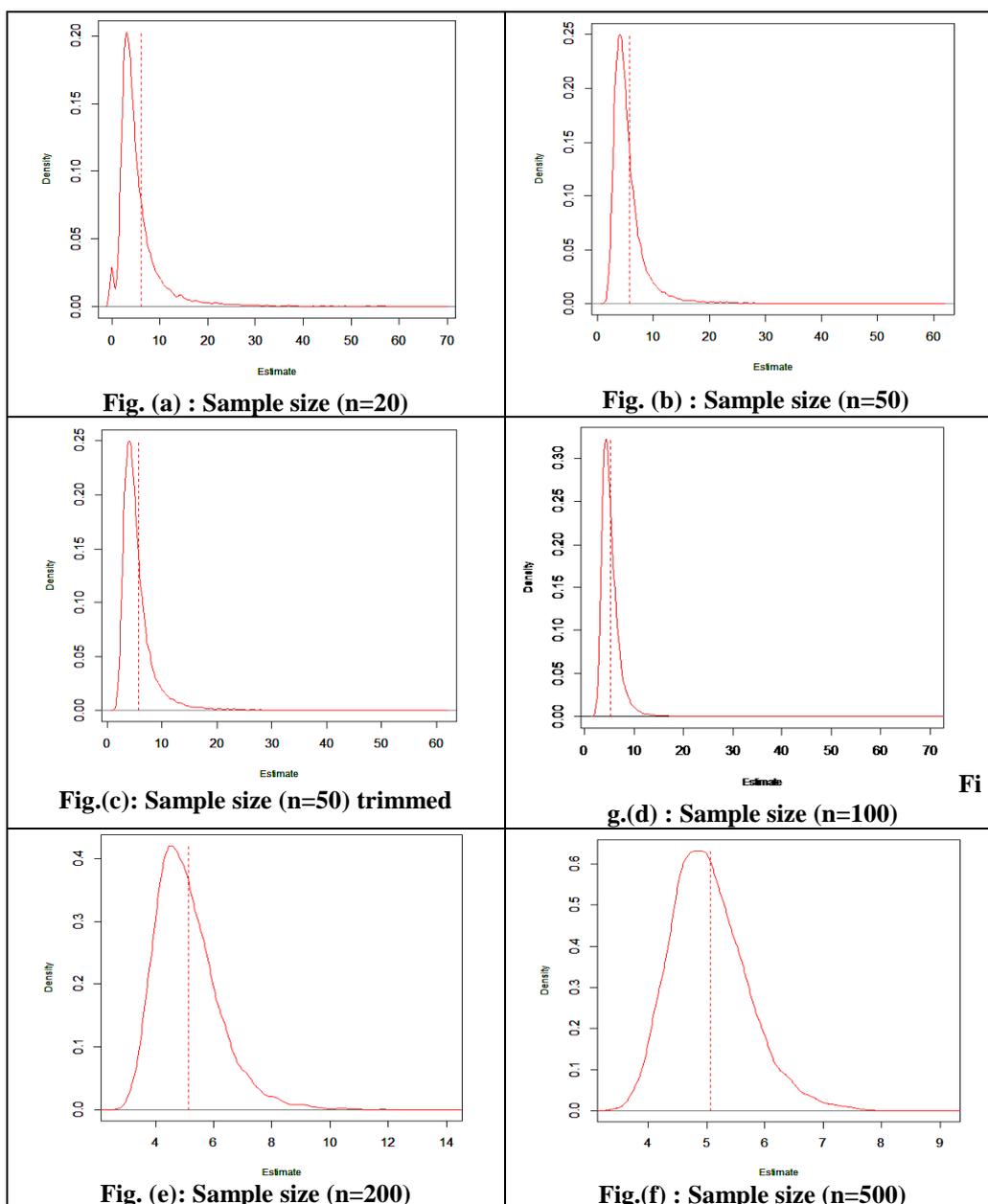
Figure (9): Sampling Distribution of the Estimate of Scale Parameter for Gamma Distribution (alpha=5, beta=10).
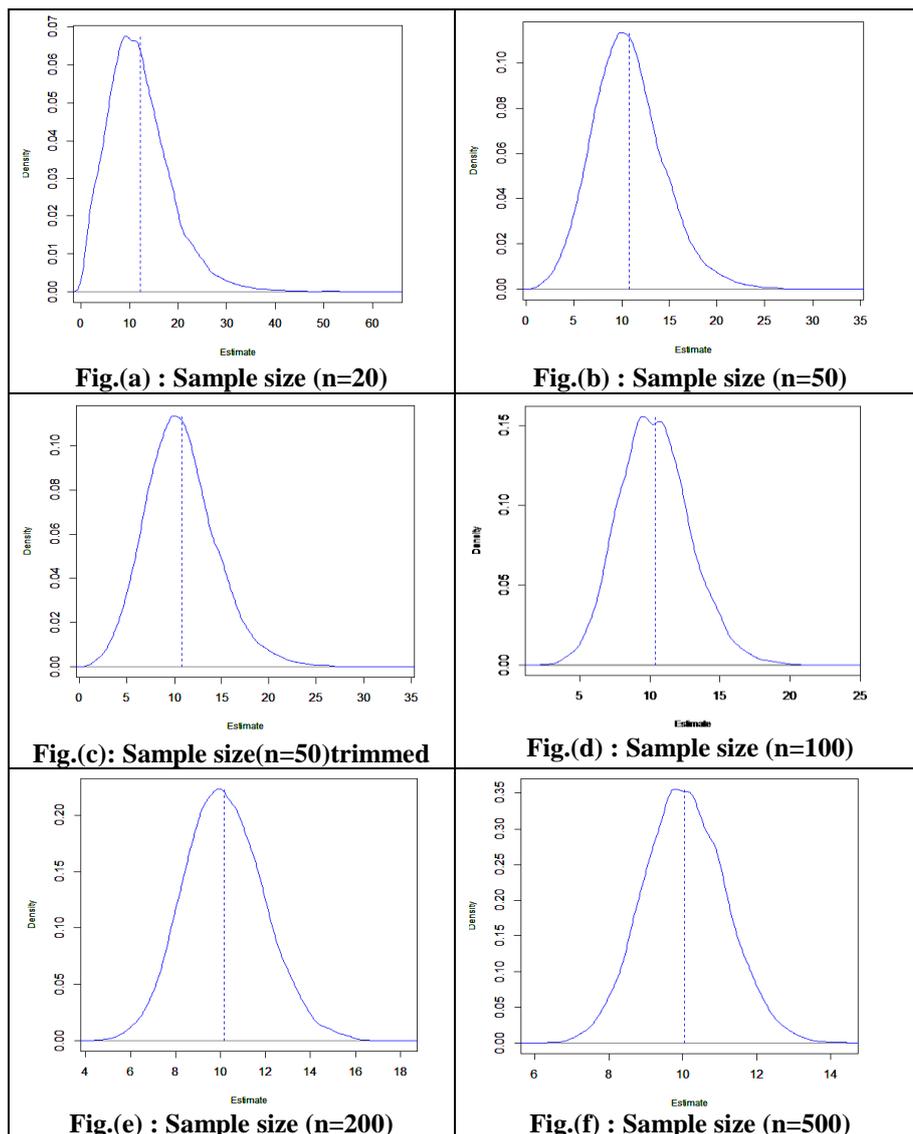
Figure (10) : Sampling Distribution of the Estimate of the Shape Parameter for Gamma Distribution (alpha=5, beta=10).
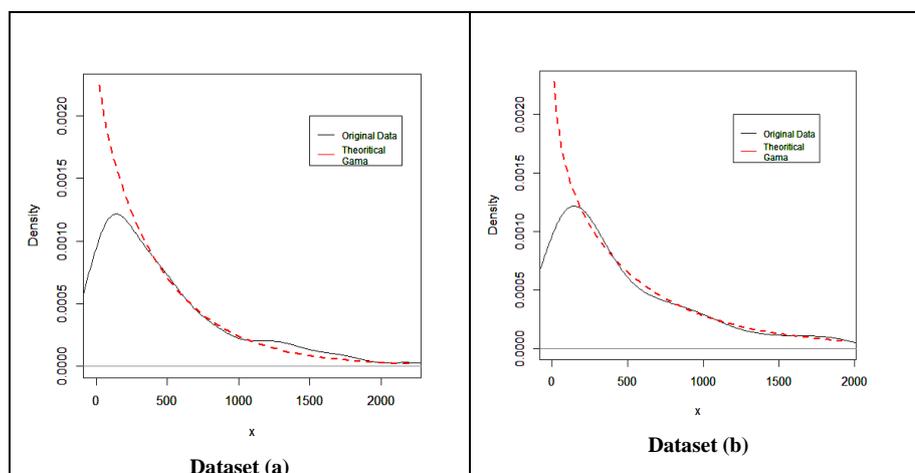
Figure 11: The Kernel density function of the datasets and the theoretical gamma density function with values of the estimated parameters values superimposed

VII.   APPLICATIONS

Life data are sometimes modeled with the gamma distribution. Data that represent failure times of machine parts, some of which are manufactured by manufacturer A and some by manufacturer B, are given by Lawless (2003) for applications of the gamma distribution to life data. Results of the above sections of this paper were applied on both datasets. The datasets involve 90 and 111 observations, respectively; and each dataset clearly follows a two-parameter gamma distribution with unknown parameters. Figure (11) presents the Kernel density function of the two datasets and the theoretical gamma density function with the values of the parameters equal to their estimated values from the data superimposed. The figure indicates that both datasets follow the gamma distributions.Using results of section (2), the estimated values of the shape and scale parameters of the first dataset were 550.60 and 0.85 with standard errors equal 110.42 and 0.12 and the t-values equal 4.986 and 7.053, respectively. The second dataset has estimated values of the shape and scale parameters 568.01 and 0.809 with standard errors equal 114.72 and 0.118 and 4.95 and the t-values equal 4.95 and 6.84, respectively. This indicates the two estimates of both datasets were significantly different from zero.

Now, to illustrate the results of the doubly truncated gamma distribution in sections 3, 4, and 5 above both datasets were truncated at two truncation points, a=100 and b=1000. The truncated datasets involve 55 and 66 observations, respectively; and each dataset clearly follows a two-parameter doubly truncated gamma distribution. Figure (12) presents the Kernel density function of the truncated datasets and the theoretical doubly truncated gamma density function presented in Eq. (15) with values of the parameters equal to their estimated values from the data superimposed. The figure indicates that both datasets follow the doubly truncated gamma distributions.Again, using results of sections 3, 4, and 5, the estimated values of the shape and scale parameters of the doubly truncated gamma distribution from the first dataset were 484.26 and 0.944, respectively. The estimated values of the shape and scale parameters from the second dataset were 686.73 and 0.840, respectively. Comparing the estimation results of the parameters of the gamma distribution and the doubly truncated gamma distribution in both datasets, we observe the ML estimation tends to overestimate the parameters of the doubly truncated gamma distribution.
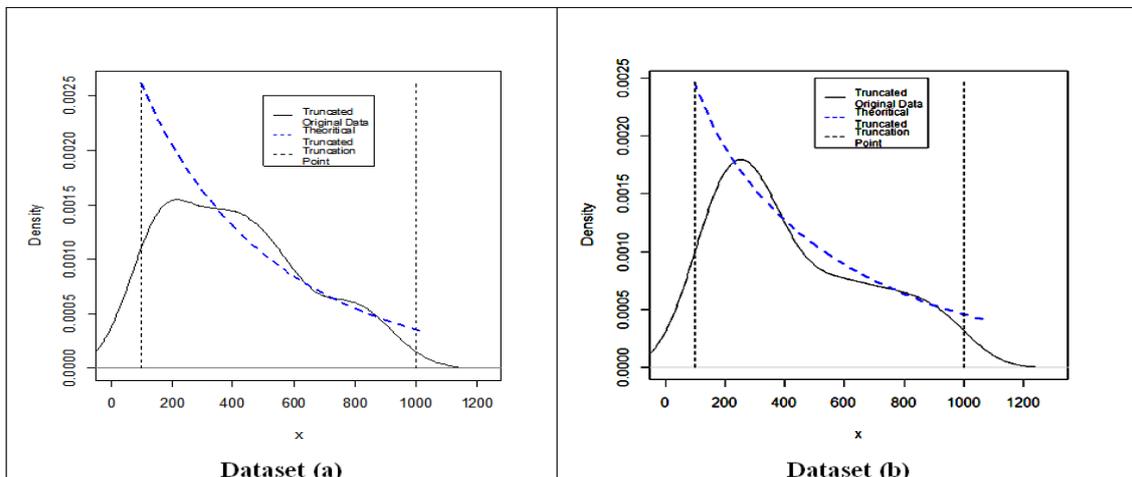
Figure 12: The Kernel density function of the two datasets and the theoretical doubly truncated gamma density functions with estimated parameter values superimposed

Moreover, the estimated lifetime mean and variance of the gamma distribution were computed from the original first dataset and found to be 468.74 and 399.06, respectively; and the corresponding estimates computed from the second dataset were 459.53 and 371.81, respectively. However, the estimated lifetime mean and variance of the doubly truncated gamma distribution after truncating the data at (a=100 and b=1000) were computed from the original first dataset and found to be 410.07 and 56537.30, respectively; and the corresponding estimates computed from the second dataset were 430.58 and 60584.11, respectively. The high values of the estimates of the variance of the doubly truncated gamma distribution from the two datasets are unsurprising. This is because the sample size of both datasets was very small for the ML estimation of the parameters of the doubly truncated gamma distribution as appears from the simulation study in the previous section.

VIII.   CONCLUSIONS AND RECOMMENDATIONS

In this study, we constructed inferences on the doubly truncated gamma distribution to establish some results that can be useful for the analysis of lifetime data. Therefore, we studied the probability density functions of the doubly truncated gamma distribution, its cdf, mean, variance, and mgf. We also attempted to provide a good estimate for the distribution's parameters. From all the discussion above, we may conclude it is possible to obtain good estimates of the parameters of the original distribution based on data from the truncated distributions based on a large sample from the truncated data. From the discussion above, we can recommend further research should be conducted on inference on different truncated distributions, particularly the truncated gamma distribution such as bias reduction of the ML estimates and hypothesis testing of the parameters, and on the application of the results of doubly truncated distributions on various fields, especially economics, survival analyses, quality assurance, and environmental applications.

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