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Spectral Properties of Unitary and Normal Bimatrices

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ABSTRACT*: A spectral theory for unitary and normal bimatrices is developed. Some basic results are derived.*

KEYWORDS*: eigen bivalues, eigen bivectors, unitary bimatrix, normal bimatrix, unitary similarity*

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I. INTRODUCTION

Let $C_{n\times n}$ be the space of $n\times n$ complex matrices of order n . Let C_n be the space of all complex n-tuples. A matrix $A_B = A_1 \cup A_2$ is called a bimatrix if A_1 and A_2 are matrices of same or different orders [4]. we consider here only matrices of same order. For $A_B \in C_{n\times n}$, let $A^T_{B,}\ A^{-1}_B, A^*_B$ and $\sigma(A_B)$ denote the transpose, inverse, conjugate transpose and spectrum of A_B . A bimatrix A_B is called normal if $A_B A^*_B = A^*_B A_B$ and unitary if $A_B A^*_B = A^*_B A_B = I_B$ [5]. Let A_B be an $n\times n$ bimatrix. An eigen bivector of A_B is a non-zero bivector $x_B \in C_n$ such that $A_B x_B = \lambda_B x_B$. The scalar λ_B is called an eigen bivalue of A_B [5]. The characteristic polynomial of A_B is the polynomial f_{A_B} defined by $f_{A_B}(x) = \det(x_B I_B - A_B)$. The set $\sigma(A_B)$ of all eigenvalues of A_B is called spectrum of A_B . In this section we have given a characterization of eigen bivalues of normal bimatrices analogous to that of the results found in [3].

Theorem: 1.1

A normal bimatrix is unitary if and only if its eigen bivalues all have absolute value of 1.

Proof

Let $A_B x_{B_i} = \lambda_{B_i} x_{B_i}$, where $\left|\lambda_{B_i}\right| = 1$ and $(x_{B_i}, x_{B_j}) = \delta_{B_{ij}}\ (1 \le i, j \le n)$. For any bivector $x_B \in C_n$, write $x_B = \sum_{i=1}^{n} \alpha_{B_i} x_{B_i}$ then it is found that

$$A^*_B A_B x_B = A^*_B A_B \left(\sum_{i=1}^{n} \alpha_{B_i} x_{B_i} \right)$$

$$= A^*_B \left(\sum_{i=1}^{n} \alpha_{B_i} A_B x_{B_i} \right)$$

$$= A^*_B \left(\sum_{i=1}^{n} \alpha_{B_i} (A_1 \cup A_2)(x_{1_i} \cup x_{2_i}) \right)$$

$$= A^*_B \left(\sum_{i=1}^{n} \alpha_{B_i} (A_1 x_{1_i} \cup A_2 x_{2_i}) \right)$$

$$= A^*_B \left(\sum_{i=1}^{n} \alpha_{B_i} (\lambda_{1_i} x_{1_i} \cup \lambda_{2_i} x_{2_i}) \right)$$

$$= A_B^* \left(\sum_{i=1}^{n} \alpha_{B_i} (\lambda_{1_i} \cup \lambda_{2_i})(x_{1_i} \cup x_{2_i}) \right)$$

$$A_B^* A_B x_B = A_B^* \left(\sum_{i=1}^{n} \alpha_{B_i} \lambda_{B_i} x_{B_i} \right) \quad (1)$$

We know that $A_B^* x_{B_i} = \overline{\lambda}_{B_i} x_{B_i} \ (i = 1, 2, 3, .., n)$ so (1) becomes

$$A_B^* A_B x_B = \sum_{i=1}^{n} \alpha_{B_i} \lambda_{B_i} (A_B^* x_{B_i})$$

$$= \sum_{i=1}^{n} \alpha_{B_i} (\lambda_{B_i} \overline{\lambda}_{B_i}) x_{B_i}$$

$$= \sum_{i=1}^{n} \alpha_{B_i} \left| \lambda_{B_i} \right|^2 x_{B_i}$$

$$= \sum_{i=1}^{n} \alpha_{B_i} (1) x_{B_i}$$

$$= \sum_{i=1}^{n} \alpha_{B_i} x_{B_i} = x_B$$

Thus $A_B^* A_B x_B = x_B$ for every $x_B \in C_n$ and therefore, $A_B^* A_B = I_B$. The relation $A_B A_B^* = I_B$ now follows since a left inverse is also a right inverse. Hence the bimatrix A_B is unitary.

Conversely, if $A_B x_B = \lambda_B x_B$ and $\langle x_B, x_B \rangle = 1$.

$$\langle A_B x_B, A_B x_B \rangle = \langle (A_1 \cup A_2)(x_1 \cup x_2), (A_1 \cup A_2)(x_1 \cup x_2) \rangle$$

$$= \langle A_1 x_1 \cup A_2 x_2, A_1 x_1 \cup A_2 x_2 \rangle$$

$$= \langle A_1 x_1, A_1 x_1 \rangle_1 \cup \langle A_2 x_2, A_2 x_2 \rangle_2$$

$$= \langle x_1, A_1^* A_1 x_1 \rangle_1 \cup \langle x_2, A_2^* A_2 x_2 \rangle_2$$

$$= \langle x_1, I_1 x_1 \rangle_1 \cup \langle x_2, I_2 x_2 \rangle_2$$

$$= \langle x_1, x_1 \rangle_1 \cup \langle x_2, x_2 \rangle_2$$

$$= \langle x_1 \cup x_2, x_1 \cup x_2 \rangle$$

$$= \langle x_B, x_B \rangle$$

$$\langle A_B x_B, A_B x_B \rangle = 1 \quad (2)$$

On the other hand, $\langle A_B x_B, A_B x_B \rangle = \langle \lambda_B x_B, \lambda_B x_B \rangle$

$$= \langle (\lambda_1 \cup \lambda_2)(x_1 \cup x_2), (\lambda_1 \cup \lambda_2)(x_1 \cup x_2) \rangle$$

$$= \langle (\lambda_1 x_1 \cup \lambda_2 x_2, \lambda_1 x_1 \cup \lambda_2 x_2 \rangle$$

$$= \langle \lambda_1 x_1, \lambda_1 x_1 \rangle_1 \cup \langle \lambda_2 x_2, \lambda_2 x_2 \rangle_2$$

$$= \overline{\lambda}_1 \lambda_1 \langle x_1, x_1 \rangle_1 \cup \overline{\lambda}_2 \lambda_2 \langle x_2, x_2 \rangle_2$$

$$= \left| \lambda_1 \right|^2 I_1 \cup \left| \lambda_2 \right|^2 I_2$$

$$= \left(\left| \lambda_1 \right|^2 \cup \left| \lambda_2 \right|^2 \right) (I_1 \cup I_2)$$

$$= \left(\left| \lambda_1 \right| \cup \left| \lambda_2 \right| \right)^2 (I_1 \cup I_2)$$

$$= \left| \lambda_B \right|^2 I_B$$

$$\langle A_B x_B , A_B x_B \rangle = \left|\lambda_B\right|^2 \qquad (3)$$

Form (2) and (3) ,we get $\left|\lambda_B\right|^2 = 1$. Thus $\left|\lambda_B\right| = 1$.

II. UNITARY SIMILARITY

In this section we have generalized some important results of unitary and normal matrices found in [2] to unitary and normal bimatrices. Also we have given a generalization of a result found in [1]. Here we define unitary similarity of bimatrices and have proved some theorems on unitary similarity.

Definition: 2.1

Two bimatrices A_B and B_B in $C_{n\times n}$ are said to be unitarily similar if there exists a unitary bimatrix $C_B \in C_{n\times n}$ such that $B_B = C_B^{-1} A_B C_B = C_B^{*} A_B C_B$.

That is, $B_1 \cup B_2 = C_1^{-1} A_1 C_1 \cup C_2^{-1} A_2 C_2 = C_1^{*} A_1 C_1 \cup C_2^{*} A_2 C_2$.

Lemma: 2.2

Unitary similarity on bimatrices is an equivalence relation.

Proof

Let A_B and B_B be any two unitary bimatrices. Bimatrix B_B is unitarily similar to A_B , if there exists a unitary bimatrix C_B such that $B_B = C_B^{-1} A_B C_B$. To show that this is an equivalence relation , we have to prove that it is reflexive, symmetric and transitive.

(i) Reflexive

Let I_B be the unit bimatrix, we have $A_B = I_B^{-1} A_B I_B$. Thus as I_B is invertible, A_B is unitarily similar to itself, showing that the relation of unitary similarity is reflexive.

(ii) Symmetric

If A_B is unitarily similar to B_B then to show that B_B is unitarily similar to A_B . We have

$$A_B = C_B^{-1} B_B C_B$$

$$A_1 \cup A_2 = C_1^{-1} B_1 C_1 \cup C_2^{-1} B_2 C_2$$

$$C_B \left(A_1 \cup A_2 \right) C_B^{-1} = C_B \left(C_1^{-1} B_1 C_1 \cup C_2^{-1} B_2 C_2 \right) C_B^{-1}$$

$$\left(C_1 \cup C_2 \right)\left(A_1 \cup A_2 \right)\left(C_1 \cup C_2 \right)^{-1} = \left(C_1 \cup C_2 \right)\left(C_1^{-1} B_1 C_1 \cup C_2^{-1} B_2 C_2 \right)\left(C_1 \cup C_2 \right)^{-1}$$

$$\left(C_1 \cup C_2 \right)\left(A_1 \cup A_2 \right)\left(C_1^{-1} \cup C_2^{-1} \right) = \left(C_1 \cup C_2 \right)\left(C_1^{-1} B_1 C_1 \cup C_2^{-1} B_2 C_2 \right)\left(C_1^{-1} \cup C_2^{-1} \right)$$

$$C_1 A_1 C_1^{-1} \cup C_2 A_2 C_2^{-1} = C_1 C_1^{-1} B_1 C_1 C_1^{-1} \cup C_2 C_2^{-1} B_2 C_2 C_2^{-1}$$

$$= I_1 B_1 I_1 \cup I_2 B_2 I_2$$

$$= B_1 \cup B_2$$

$$\left(C_1 \cup C_2 \right)\left(A_1 \cup A_2 \right)\left(C_1 \cup C_2 \right)^{-1} = B_1 \cup B_2$$

$$C_B A_B C_B^{-1} = B_B$$

Therefore, B_B is unitarily similar to A_B that is, the relation unitary similarity is symmetric.

(iii) Transitive

Let A_B be unitarily similar to B_B and B_B be unitarily similar to C_B that is, for unitary bimatrices P_B and Q_B , we have

$$A_B = P_B^{-1} B_B P_B \qquad (4)$$

and $$B_B = Q_B^{-1} C_B Q_B \qquad (5)$$

From (4), we have $A_B = P_B^{-1} B_B P_B$

$= P_B^{-1} (Q_B^{-1} C_B Q_B) P_B$ (by 5)

$$= (P_1 \cup P_2)^{-1}(Q_1^{-1}C_1Q_1 \cup Q_2^{-1}C_2Q_2)(P_1 \cup P_2)$$
$$= (P_1^{-1} \cup P_2^{-1})(Q_1^{-1}C_1Q_1 \cup Q_2^{-1}C_2Q_2)(P_1 \cup P_2)$$
$$= P_1^{-1}Q_1^{-1}C_1Q_1P_1 \cup P_2^{-1}Q_2^{-1}C_2Q_2P_2$$
$$= (P_1^{-1}Q_1^{-1} \cup P_2^{-1}Q_2^{-1})(C_1 \cup C_2)(Q_1P_1 \cup Q_2P_2)$$
$$= ((Q_1P_1)^{-1} \cup (Q_2P_2)^{-1})(C_1 \cup C_2)(Q_1 \cup Q_2)(P_1 \cup P_2)$$
$$= (Q_1P_1 \cup Q_2P_2)^{-1} C_B Q_B P_B$$
$$= ((Q_1 \cup Q_2)(P_1 \cup P_2))^{-1} C_B (Q_B P_B)$$
$$A_B = (Q_B P_B)^{-1} C_B (Q_B P_B) \begin{bmatrix} \because P_B \ and \ Q_B \ are \ unitary \\ \Rightarrow P_B Q_B \ is \ unitary \\ \Rightarrow (Q_B P_B)^{-1} = P_B^{-1} Q_B^{-1} \end{bmatrix}$$

Therefore, A_B is unitarily similar to C_B. That is, the relation of unitary similarity is transitive.

Hence, unitary similarity on bimatrices is an equivalence relation.

Lemma: 2.3

Two unitarily similar bimatrices have the same determinant.

Proof

Let A_B be unitarily similar to B_B .Therefore, there exists a unitary bimatrix P_B such that

$$A_B = P_B^{-1} B_B P_B$$
$$\det A_B = \det(P_B^{-1} B_B P_B)$$
$$= \det(P_B^{-1}) \det(B_B) \det(P_B)$$
$$= \det(B_B) \det(P_B^{-1}) \det(P_B)$$
$$= \det(B_B P_B^{-1} P_B)$$
$$= \det(B_1 P_1^{-1} P_1 \cup B_2 P_2^{-1} P_2)$$
$$= \det(B_1 I_1 \cup B_2 I_2)$$
$$= \det((B_1 \cup B_2)(I_1 \cup I_2))$$
$$= \det(B_B I_B)$$
$$= \det(B_B) \det(I_B)$$
$$= \det(B_B) I_B$$
$$= \det(B_B)$$

Hence, unitary similarity bimatrices have the same determinant.

Lemma: 2.4

Let A_B and B_B be two bimatrices. If B_B is unitarily similar to A_B then B_B^T is unitarily similar to A_B^T .

Proof

Since B_B is unitarily similar to A_B we can find a unitary bimatrix P_B such that $B_B = P_B^{-1} A_B B_B$. (6)

N_B is the bimatrix whose column vectors are normalized eigen bivectors of P_B then (6) can be written as $B_B = N_B^T A_B N_B$

$$B_1 \cup B_2 = (N_1 \cup N_2)^T (A_1 \cup A_2)(N_1 \cup N_2)$$
$$= (N_1^T \cup N_2^T)(A_1 \cup A_2)(N_1 \cup N_2)$$
$$= N_1^T A_1 N_1 \cup N_2^T A_2 N_2$$

$$(B_1 \cup B_2)^T = (N_1^T A_1 N_1 \cup N_2^T A_2 N_2)^T$$
$$B_B^T = (N_1^T A_1 N_1)^T \cup (N_2^T A_2 N_2)^T$$
$$B_B^T = (N_1^T A_1^T N_1) \cup (N_2^T A_2^T N_2)$$
$$B_B^T = (N_1^T \cup N_2^T)(A_1^T \cup A_2^T)(N_1 \cup N_2)$$
$$B_B^T = (N_1 \cup N_2)^T (A_1 \cup A_2)^T (N_1 \cup N_2)$$
$$B_B^T = N_B^T A_B^T N_B$$

Therefore, B_B^T is unitarily similar to A_B^T.

Lemma: 2.5

Let A_B and B_B be unitarily similar. Then A_B is normal iff B_B is normal.

Proof

Let P_B be unitary bimatrix with $B_B = P_B^{-1} A_B P_B$ .Then $B_B^* = (P_B^{-1} A_B P_B)^* = P_B^* A_B^* P_B$ and $P_B^* = P_B^{-1}$

So $B_B^* B_B - B_B B_B^* = (P_B^* A_B^* P_B)(P_B^* A_B P_B) - (P_B^* A_B P_B)(P_B^* A_B^* P_B)$

$= (P_1 \cup P_2)^* (A_1 \cup A_2)^* (P_1 \cup P_2)(P_1 \cup P_2)^* (A_1 \cup A_2)(P_1 \cup P_2) - (P_1 \cup P_2)^* (A_1 \cup A_2)(P_1 \cup P_2)(P_1 \cup P_2)^* (A_1 \cup A_2)^* (P_1 \cup P_2)$

$= (P_1^* \cup P_2^*)(A_1^* \cup A_2^*)(P_1 \cup P_2)(P_1^* \cup P_2^*)(A_1 \cup A_2)(P_1 \cup P_2) - (P_1^* \cup P_2^*)(A_1 \cup A_2)(P_1 \cup P_2)(P_1^* \cup P_2^*)(A_1^* \cup A_2^*)(P_1 \cup P_2)$

$= ((P_1^* A_1^* P_1 P_1^* A_1 P_1) \cup (P_2^* A_2^* P_2 P_2^* A_2 P_2)) - ((P_1^* A_1 P_1 P_1^* A_1^* P_1) \cup (P_2^* A_2 P_2 P_2^* A_2^* P_2))$

$= (P_1^* A_1^* I_1 A_1 P_1 \cup P_2^* A_2^* I_2 A_2 P_2) - (P_1^* A_1 I_1 A_1^* P_1 \cup P_2^* A_2 I_2 A_2^* P_2)$

$= (P_1^* A_1^* A_1 P_1 \cup P_2^* A_2^* A_2 P_2) - (P_1^* A_1 A_1^* P_1 \cup P_2^* A_2 A_2^* P_2)$

$= (P_1^* \cup P_2^*)(A_1^* \cup A_2^*)(A_1 \cup A_2)(P_1 \cup P_2) - (P_1^* \cup P_2^*)(A_1 \cup A_2)(A_1^* \cup A_2^*)(P_1 \cup P_2)$

$= (P_1 \cup P_2)^* (A_1 \cup A_2)^* (A_1 \cup A_2)(P_1 \cup P_2) - (P_1 \cup P_2)^* (A_1 \cup A_2)(A_1 \cup A_2)^* (P_1 \cup P_2)$

$= P_B^* A_B^* A_B P_B - P_B^* A_B A_B^* P_B$

$= P_B^* (A_B^* A_B - A_B A_B^*) P_B$

Thus $B_B^* B_B - B_B B_B^* = 0$ iff $A_B^* A_B - A_B A_B^* = 0$

$B_B^* B_B = B_B B_B^*$ iff $A_B^* A_B = A_B A_B^*$

Hence, A_B is normal iff B_B is normal.

III. SPECTRAL THEORY FOR NORMAL BIMATRICES

In this section the spectral theorem for normal bimatrices basically states that a bimatrix A_B is normal if and only if it is unitarily diagonolizable that is, there exists a unitary bimatrix C_B and a diagonal bimatrix D_B such that $C_B^* A_B C_B = D_B$ analogous to that of the results found in [6]. It is important to note that the latter is equivalent to saying that there exists an orthonormal basis (the columns of C_B) of eigen bivectors of A_B (the corresponding eigen bivalues being the diagonal elements of D_B).

Theorem: 3.1

Let $A_B \in C_{n\times n}$. Then A_B is unitarily similar to an upper triangular bimatrix.

Proof

The proof is by induction on n. Since every 1×1 bimatrix is upper triangular, then n=1 case is trivial.

Assume $n > 1$. Let λ_B be an eigen bivalue of A_B afforded by the eigen bivector $x_B \in C_{n\times 1}$.

Since $x_B \neq 0$. Therefore, we may assume $\|x_B\| = 1$.

By theorem 2.9 of [6], $\{x_B\}$ can be extended to an orthonormal basis $\{x_B, y_{B_2}, \ldots, y_{B_n}\}$ of $C_{n\times 1}$. Let C_B be the bimatrix whose first column is x_B and whose j^{th} column is y_{B_j}, $1 < j \le n$. Then $C_B^* C_B = I_B$, the $n \times n$ identity bimatrix. Moreover, the first column of $C_B^* A_B C_B$ is $C_B^* A_B x_B = \lambda_B C_B^* x_B = \lambda_B C_{B_1}$, where C_{B_1} is the first column of I_B, that is

$$C_B^* A_B C_B = \begin{pmatrix} \lambda_B & \# & \# & \cdots & \# \\ 0 & & & & \\ 0 & & A_{B_1} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

Where A_{B_1} is an $((n-1)\times(n-1))$ bimatrix and #'s stands for unspecified entries. It follows from the induction hypothesis that there is an (n-1) square unitary bimatrix C_{B_1} such that $C_{B_1}^* A_{B_1} C_{B_1}$ is upper triangular. Let $D_B = (1) \oplus C_{B_1}$ that is,

$$D_B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & C_{B_1} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}.$$

Then $(C_B D_B)^* A_B (C_B D_B) = \begin{pmatrix} \lambda & \# & \# & \cdots & \# \\ 0 & & & & \\ 0 & & C_{B_1}^* A_{B_1} C_{B_1} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$ is upper triangular. Since it is a product of unitary bimatrices. Hence $C_B D_B$ is unitary.

Theorem: 3.2

Let $A_B \in C_{n\times n}$. Then A_B is unitarily similar to a diagonal bimatrix if and only if $A_B^* A_B = A_B A_B^*$.

Proof

Let C_B be a unitary bimatrix such that $C_B^* A_B C_B = D_B$ is diagonal. Then

$$\begin{aligned} A_B^* A_B &= (C_B D_B C_B^*)^* (C_B D_B C_B^*) \\ &= (C_B D_B^* C_B^*)(C_B D_B C_B^*) \\ &= \left[(C_1 \cup C_2)(D_1 \cup D_2)^* (C_1 \cup C_2)^*\right]\left[(C_1 \cup C_2)(D_1 \cup D_2)(C_1 \cup C_2)^*\right] \\ &= \left[(C_1 \cup C_2)(D_1^* \cup D_2^*)(C_1^* \cup C_2^*)\right]\left[(C_1 \cup C_2)(D_1 \cup D_2)(C_1^* \cup C_2^*)\right] \\ &= \left(C_1 D_1^* C_1^* \cup C_2 D_2^* C_2^*\right)\left(C_1 D_1 C_1^* \cup C_2 D_2 C_2^*\right) \\ &= \left(C_1 D_1^* C_1^* C_1 D_1 C_1^*\right) \cup \left(C_2 D_2^* C_2^* C_2 D_2 C_2^*\right) \\ &= \left(C_1 D_1^* I_1 D_1 C_1^*\right) \cup \left(C_2 D_2^* I_2 D_2 C_2^*\right) \\ &= \left(C_1 D_1 D_1^* C_1^*\right) \cup \left(C_2 D_2 D_2^* C_2^*\right) \end{aligned}$$

$$= \left(C_1 D_1 C_1^* C_1 D_1^* C_1^*\right) \cup \left(C_2 D_2 C_2^* C_2 D_2^* C_2^*\right)$$
$$= (C_1 \cup C_2)(D_1 \cup D_2)\left(C_1^* \cup C_2^*\right)(C_1 \cup C_2)\left(D_1^* \cup D_2^*\right)\left(C_1^* \cup C_2^*\right)$$
$$= (C_1 \cup C_2)(D_1 \cup D_2)(C_1 \cup C_2)^* (C_1 \cup C_2)(D_1 \cup D_2)^* (C_1 \cup C_2)^*$$
$$= \left(C_B D_B C_B^*\right)\left(C_B D_B^* C_B^*\right)$$
$$= \left(C_B D_B C_B^*\right)\left(C_B D_B C_B^*\right)^*$$
$$= A_B A_B^*$$

Therefore, $A_B^* A_B = A_B^* A_B$.

Hence, A_B is a normal bimatrix.

Conversely, by theorem (3.1) there is a unitary bimatrix C_B such that $C_B^* A_B C_B$ and $C_B^* A_B^* C_B$ are both upper triangular. Since $C_B^* A_B^* C_B$ is upper triangular, its conjugate transpose, $C_B^* A_B C_B$ must be lower triangular.

Therefore, $C_B^* A_B C_B$ is both upper and lower triangular. That is, $C_B^* A_B C_B$ is a diagonal bimatrix. The following result provides an easy -to-check necessary and sufficient condition for normality.

Theorem: 3.3

Given $A_B \in C_n$, the following statements are equivalent

(i) A_B is normal bimatrix

(ii) A_B is unitarily diagonalizable

(iii) $\sum_{1 \le i, j \le n} \left|a_{i,j}^1 \cup a_{i,j}^2\right|^2 = \sum_{1 \le i \le n} \left|\lambda_i^1 \cup \lambda_i^2\right|^2$, $\lambda_1^1 \cup \lambda_1^2, \ldots, \lambda_n^1 \cup \lambda_n^2$ are the eigen bivalues of A_B, counting multiplicities.

Proof

$(i) \Leftrightarrow (ii)$: By theorem (3.1), A_B is unitarily similar to a triangular bimatrix T_B . Then

A_B is normal bimatrix iff T_B is normal bimatrix

iff T_B is diagonal bimatrix

iff A_B is unitarily diagonalizable.

$(ii) \Rightarrow (iii)$

suppose that A_B is unitarily similar to a diagonal bimatrix D_B . Note that the diagonal entries of D_B are the eigen bivalues $\lambda_1^1 \cup \lambda_1^2, \ldots, \lambda_n^1 \cup \lambda_n^2$ of A_B .Then

$$\sum_{1 \le i, j \le n} \left|a_{i,j}^1 \cup a_{i,j}^2\right|^2 = tr(A_B^* A_B)$$

$$= tr(D_B^* D_B)$$

$$= \sum_{1 \le i \le n} \left| \lambda_i^1 \cup \lambda_i^2 \right|^2 \qquad (7)$$

$(iii) \Rightarrow (ii)$

By theorem (3.1), A_B is unitarily similar to a triangular bimatrix T_B .

$$\sum_{1 \le i, j \le n} \left| a_{i,j}^1 \cup a_{i,j}^2 \right|^2 = tr(A_B^* A_B)$$

$$= tr(D_B^* D_B)$$

$$= \sum_{1 \le i, j \le n} \left| t_{i,j}^1 \cup t_{i,j}^2 \right|^2 \qquad (8)$$

On the other hand , we have $\sum_{1 \le i \le n} \left| \lambda_i^1 \cup \lambda_i^2 \right|^2 = \sum_{1 \le i \le n} \left| t_{i,i}^1 \cup t_{i,i}^2 \right|^2$ (9)

because the diagonal entries of T_B are the eigen bivalues $\lambda_1^1 \cup \lambda_1^2, \ldots, \lambda_n^1 \cup \lambda_n^2$ of A_B .

Thus, the equality between (8) and (9) imply that $t_{i,j}^1 \cup t_{i,j}^2 = 0_B$ whenever $i \ne j$, that is, T_B is a diagonal bimatrix.

Hence, A_B is unitarily diagonalizable.

IV CONCLUSION

A spectral theory for unitary and normal bimatrices is studied. This can be used to study two system simultaneously.

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