International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 2 Issue 1 || January. 2014 || PP-49-55

Variational Iteration and Homotopy Perturbation Methods for Solving Fredholm-Volterra Integro-Differential Equations

[1,] O.A.Taiwo, [2,] A. K. ADIO
[1,] *Department of Mathematics; University of Ilorin, Ilorin, Nigeria.*
[2,] *Department of Basic Sciences; Babcock University, Illisan, Nigeria.*

ABSTRACT: *In this article, He's variational iteration and Homotopy Perturbation methods are modified by adding a perturbation term to the resulting system of first integro-differential equation obtained after reducing higher order Fredholm-Volterra integro-differential equations.*
The resulting revealed that both methods, are very effective and simple. We also observed that the higher the values of n (the degree of approximant), the closer the approximate solutions obtained to the exact solutions. Numerical examples are given to illustrate the applications of the methods.

KEYWORDS: *Integro-differential equations, Variational Iteration and Homotopy Methods, Perturbed, Error.*

1 INTRODUCTION

Mathematical modeling of real life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro-differential equations, stochastic equations.

Many Mathematical formulation of physical phenomena contain integro-differential equations, these equations arise in many fields like Fluid dynamics, Biological models and chemical kinetics.
Integro-differential equations are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution (Sweilam, 2007).

Variational iteration method (He, 1997, 199, 2007) is a powerful device for solving various kinds of equations, linear and nonlinear. The method has successfully been applied to many situations. for example, He (2007) used the method to solve some integro differential equations where he chose initial approximate solution in the form of exact solution with unknown constants.

Abbasbandy and Shivanion (2009) used VIM to solve systems of nonlinear Volterra's Integro-differential equations. Biazar et al (2010) employed VIM to solve linear and nonlinear system of IDEs.
A new perturbation method called Homotopy Perturbation Method (HPM) was proposed by He in 1997 and description in 2000, which is, in fact, coupling of the traditional perturbation method and homotopy in topology.

The method has equally been applied successfully to many situations. For example, Mirzaei (2011) employed HPM and VIM to solve Volterra integral equations.
Biazar and Eslami (2010) applied HPM to solve nonlinear Volterra-Fredholm integro-differential equations.

In this paper, we considered the Tau reduction of nth order Fredholm-Volterra integro-differential equation into systems of first order Fredholm-Volterra integro differential equations. The resulting systems are then perturbed and solved by Variational Iteration and Homotopy Perturbation Methods. The basic motivation here is to get a better approximation.

In case of nonlinear Fredholm-Volterra Integro differential equations; Newton's linearization scheme of appropriate order is used to linearize and hence leads to iterative procedure.

2. VARIATIONAL ITERATION METHOD:

Variational Iteration Method (VIM) is based on the general Langranges's multiplier method (Inokuti et al, 1978). The main feature of the method is that the solution of a Mathematical problem with linearization assumption is used as initial approximation. Then a more highly precise approximation at some special point can be obtained.

To illustrate the basic concepts of VIM, we consider the following nonlinear differential equation

$$Lu + Nu = g(x) \tag{1}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous term.

According to VIM (He 1999, 2000, 2006), we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_{\circ}^{x} \lambda \{ Lu_n(T) + N\bar{u}_n(T) - g(T) \} dT \tag{2}$$

λ is a general lagrangian multiplier (Inokuti et al, 1978) which can be identified optimally via Variational theory. Subscript n denotes the nth-order approximation, $\bar{u}_n$ is considered as a restricted variation (He, 1999, 2000) i.e $\delta\bar{u}_n = 0$

3. Homotopy Perturbation Method:

Consider the nonlinear algebraic equation

$$f(x) = 0, \quad x \in R \tag{3}$$

The basic idea of the homotopy perturbation method is to construct a homotopy $H(v, p) : \mathbf{R} \times [0,1] \to \mathbf{R}$ which satisfies

$$H(v, p) = pf(v) + (1 - p)(f(v) - f(x_\circ)) = 0, \ v \in R; \ p \in [0,1] \tag{4}$$

or

$$H(v, p) = f(v) - f(x_\circ) + pf(x_\circ) = 0, \ v \in R; \ p \in [0,1]$$

where p is an embedding parameter, and $x_\circ$ is an initial approximation of equation (3). (Usually $x_\circ$ is an initial guess close to α).
Obviously from equation (4), we have

$$H(v,0) = f(v) - f(x_\circ) = 0, \tag{5}$$

and

$$H(v,1) = f(v) = 0, \tag{6}$$

The embedding parameter p increases from 0 to 1 monotonically as trivial problem $H(v,0) = f(v) - f(x_\circ) = 0$ is continuously transformed to the original problem $H(v,1) = f(v) = 0$.
The HPM uses the embedding parameter p as a "small parameter" and writes the solution of equation (4) as a power series of p i.e

$$v = x_\circ + \tilde{x}_1 p + \tilde{x}_2 p^2 + \cdots \tag{7}$$

setting $p = 1$ results in the approximate solution of equation (3)

$$x = \lim_{p \to 1} = x_\circ + \tilde{x}_1 + \tilde{x}_2 + \cdots \tag{8}$$

where $x_\circ, \tilde{x}_1, \tilde{x}_2, \cdots$ are the coefficients of the power series (7).

If $v = v(p)$ in equation (7), then $x_\circ = v(0)$, $\tilde{x}_1 = v'(0)$,

$$\tilde{x}_2 = \frac{1}{2!} v''(0), \cdots, \tilde{x}_n = \frac{1}{n!} v^{(n)}(0), \cdots \tag{9}$$

and $v^{(n)}(0)$ can be determined from $f(x_\circ), f'(x_\circ), \cdots, f^{(n)}(x_\circ)$ by equation (4) or by the equation

$$F(v(p)) - f(x_\circ) + pf(x_\circ) = 0$$

recursively.

4. Solution Techniques:

Consider the nonlinear Fredholm-Volterra Integro differential equation of the form

$$\sum_{i=0}^{m} p_i y^i(x) + \lambda_1 \int_a^b f_i(x,t) \sum_{i=1}^{2} y^i(t)dt + \lambda_2 \int_a^x k_j(x,t) \sum_{j=1}^{2} y^j(t)dt = g(x) \qquad (10)$$

unde the mixed conditions

$$\sum_{j=0}^{m-1} \left[a_{ij} y_j(-1) + b_{ij} y^j(1) + c_{ij} y^j(c) \right] = \mu - i, \; i = 0,1,\cdots,m-1, \; -1 \le c \le 1 \qquad (11)$$

where $y(x)$ is an unknown function, the functions $g(x)$, $p_i(x)$, $f_i(x,t)$ and $k_j(x,t)$ are defined on the interval $a \le x,t \le b$ and $a_{ij}, b_{ij}, c_{ij}, \lambda_1, \lambda_2, \mu_i$ are constants.

Now the general nth order Fredholm-Volterra integro differential equation is of the form :

$$p_\circ y(x) + p_1 y'(x) + p_2 y''(x) + \cdots + p_m y^m(x) + \lambda_1 \int_a^b f_i(x,t) \sum_{i=1}^{2} y^i(t)dt$$
$$+ \lambda_2 \int_a^x k_j(x,t) \sum_{j=1}^{2} y^j(t)dt = g(x) \qquad , a \le x,t \le b \qquad (12)$$

under the mixed conditions stated in equation (11) using the transformation

$$y(x) = y_1(x) \Rightarrow y'(x) = y_1'(x) = y_2(x)$$
$$y'(x) = y_2(x) \to y''(x) = y_2'(x) = y_3(x) \qquad (13)$$
$$\vdots$$
$$y^{m-1}(x) = y_m(x) \Rightarrow y^m(x) = y_m'(x) = y_{m+1}(x)$$

Equation (13)is written as a system of differential equations as

$$\frac{dy_1}{dx} = y_2(x)$$
$$\frac{dy_2}{dx} = y_3(x) \qquad (14)$$
$$\vdots$$
$$\frac{d}{dx} \sum_{i=0}^{m} p_i y^i(x) = g(x) - \lambda_1 \int_a^b f_i(x,t) \sum_{i=1}^{2} y^i(t)dt - \lambda_2 \int_a^x k_j(x,t) \sum_{j=1}^{2} y^j(t)dt = g(x)$$

Thus, equation (14) is expressible in matrix form as

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ -p_1 & -p_2 & -p_3 & -p_4 & -p_5 & \cdots & -p_{m-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_m \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f(x) \end{pmatrix} \qquad (15)$$

where

$$f(x) = g(x) - p - \circ y_1 - \lambda_1 \int_a^b f_i(x,t) \sum_{i=1}^{2} y^i(t)dt - \lambda_2 \int_a^x k_j(x,t) \sum_{j=1}^{2} y^j(t)dt$$

In view of VIM, a correction functional is of the form

$$y'_{j,n+1} = y_{j,n} + \int_{\circ}^{x} \lambda(\varepsilon)\{y'_{j,n}(\varepsilon) - \bar{f}_j\} d\varepsilon \tag{16}$$

where

$$y'_j = g(x) - p_{j-1}y_j - p_{j-2}y_{j-1} - \cdots p_1 y_2 - p_{\circ} y_1$$

$$-\lambda_1 \int_a^b f_i(x,t) \sum_{i=1}^{2} y^i(t)dt - \lambda_2 \int_a^x k_j(x,t) \sum_{j=1}^{2} y^j(t)dt + \tau_1 T_N \quad (4.8)$$

$$j = 1,2,\cdots,m$$

$\bar{f}_j$ is considered as a restricted variation i.e $\delta\bar{f}_j = 0$, τ_1 is a constant parameter to be determined and $T_N(x)$ is a chebyshev polynomial of degree $N(=1,2)$ valid in the interval $-1 \le x \le 1$ given by

$$T_N(x) = \cos(N \cos^{-1} x) \tag{17}$$

which satisfy the recurrence relation given by

$$T_N(x) = 2xT_N(x) - T_{N-1}(x), \quad N \ge 1$$

For nonlinear (FVIDE) (10) and (11), in order to solve the equations by Tau-variational iteration method, equation (10) is linearized using Newton's linearization scheme of the form

$$\sum_{k=0}^{m} p_k(x)y^k(x) + \lambda_1 \int_a^b \sum_{j=1}^{2} \left[\Delta y_n^j \frac{\partial G}{\partial y_n^{(j)}} F_j(x,t) \right] dt + \lambda_2 \int_a^b \sum_{j=1}^{2} \left[\Delta y_n^j \frac{\partial G}{\partial y_n^{(j)}} K_j(x,t) \right] \tag{18}$$

where

$$\nabla y_n^{(j)} = y_{n+1}^{(j)} - y_n^{(j)} \tag{19}$$

The basic idea of Tau-HPM is the addition of perturbation terms to equation (10) and then writing the equation as a system of IDEs using the transformation in (13). Thus, we have a system of equations of the form

$$y_1(x) = \alpha_1 + p\int_{\circ}^{z} y_2^{(k)}(t)dt$$

$$y_2(x) = \alpha_2 + p\int_{\circ}^{z} y_3^{(k)}(t)dt \tag{20}$$

$$\vdots$$

$$y_m(x) = \alpha_m + p\int_{\circ}^{z} \{g(t) - p_{m-1}y_m(t) - p_{m-2}y_{m-1}(t) - \cdots - p_1 y_2(t) - p_{\circ} y_1(t)\} dt$$

$$\lambda_1 p \int_{\circ}^{x} \left\{ \int_a^b f_j(w,t) \sum_{i=1}^{2} y^i(t) \right\} dw - \lambda_2 p \int_{\circ}^{x} \left\{ \int_a^w k_j(w,t) \sum_{j=1}^{2} y^j(t) \right\} dw + p\int_{\circ}^{x} H_n(t)dt$$

$$H_n(x) = \sum_{k=1}^{m} \tau_k T_{n+k-2}(x), \quad n \ge \tag{21}$$

is the perturbation term.

Thus, by HPM, equation (20) is written in expansion form and coefficients of like powers of $p^i\ (i \ge 0)$ are compared to give the values of the constants $a_{i'}s, b_{i'}s$ and $c_{i'}s$. The unknown function function $y(x)$ is expressed as summation of $a_{i'}$'s i.e

$$y(x) = y_1(x) = \sum_{i=0}^{\infty} a_i = a_\circ + a_1 + a_2 + \cdots \tag{22}$$

using the initial/boundary conditions, the unknown constants can be evaluated, the values of which are then substituted back into the approximate solutions earlier obtained.

5. Applications

We illustrate the ability and reliability of the methods with the following examples.

Example 1:

Consider the Volterra-Fredholm integro differential equations (Biazer, J and Eslami, M. (2010)).

$$y''(x) - xy'(x) + xy(x) = f(x) + \int_{-1}^{x} (x-2t)(y(t))^2 dt + \int_{-1}^{1} xty(t)dt$$

where

$$f(x) = \frac{2}{25}x^6 - \frac{1}{3}x^4 + x^3 - 2x^2 - \frac{23}{15}x + \frac{5}{3}$$

with condition $y(0) = -1, y'(0) = 0$ and exact solution $y(x) = x^2 - 1$.

Now using the transformation in equation (4.4), we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -x & x \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) + \int_{-1}^{x} (x-2t) y^2(t)dt + \int_{-1}^{1} xty(t)dt \end{pmatrix}$$

and

$$y_{2,n+1}(x) = y_{2,n} + \frac{5}{3}x + \frac{2}{175}x^7 - \frac{1}{15}x^5 + \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}\left(-y_{1,n} + y_{2,n} - \frac{23}{15}\right)x^2$$

$$-y_{1,n}^2(x^2-1)x + \frac{1}{2}y_{1,n}^2(x+1)x^2 + \tau_1(2x-1)$$

Example 2:

$$y'''(x) - xy''(x) + \sin xy(x) = e^x(1 - x + \sin x) - 2 + \int_{-1}^{1} e^{-2t} y^2(t)dt$$

with conditions

$$y(0) = y'(0) = y''(0) = 1$$

Exact solution for the problem is $y(x) = e^x$

The Newton's linearization scheme of order three is given as

$$G + \frac{\partial G}{\partial y_k}\nabla y_k + \frac{\partial G}{\partial y'_k}\nabla y'_k + \frac{\partial G}{\partial y''_k}\nabla y''_k + \frac{\partial G}{\partial y'''_k}\nabla y'''_k = 0$$

where $\nabla y_k^i = y_{k+1}^i - y_k^i$

and

$$G = y'''(x) - xy''(x) + \sin xy(x) - e^x(1 - x + \sin x) - 2 + \int_{-1}^{1} e^{-2t} y^2(t)dt$$

Thus,

$$y'''(x) - xy''(x) + \sin xy(x) - 2yy' + \int_{-1}^{1} e^{-2t} y^2(t)dt = e^x(1 - x + \sin x) - 2 + \int_{-1}^{1} e^{-2t} y^2(1 - 2y')dt$$

$$+\tau_1(2x^2-1)+\tau_2(4x^3-3x)+\tau(8x^4-8x^2+1)$$

Thus following the procedure and comparing the coefficients of like powers of $P^i\ (i \geq 0)$, we have

Coefficients of p° : $a_\circ = 1$

$$b_\circ = 1$$

$$c_\circ = 1$$

Coefficients of p^4 : $a_4 = c_2 x^2$

$$b_4 = c_3 x$$

$c_4 = x^2 c_3 - c_1 x^3 \sin x + 7.25372081\ \ 6c_1 x^3 yy'$ and

$$y(x) = 1 + x + x^2 + (x^2 + x^4)\{x^2 - x\sin x + 7.25372081\ \ 6c_1 x^3 yy' + xe^x(1 - x + \sin x)$$

$$+7.25372081\ \ 6y^2(1-2y') - 2x + \tau_1(2x^3 - x) + \tau_2(4x^4 - 3x^2) + \tau_3(8x^5 - 8x^3 + x)$$

$$-x^4 \sin x + 7.25372081\ \ 6yy'x^4\}$$

Remark : The values of Tau-parameters are determined in each example using the conditions given

Table 1: Results obtained for example 1 and error

x	Exact Solution	Approx. Solution	Error
-1	0.000000	0.000041	4.00E-6
-0.8	-3.600000	-0.359997	3.00E-6
-0.6	-0.640000	-0.639902	9.80E-6
-0.4	-0.840000	-0.839004	9.96E-4
-0.2	-0.960000	-0.959793	2.07E-4
0.0	-1.000000	-0.999782	2.18E-4
0.2	-0.960000	-0.959793	2.07E-4
0.4	-0.840000	-0.839004	9.96E-4
0.6	-0.640000	-0.639902	9.80E-5
0.8	-0.360000	-0.359997	3.00E-6
1.0	0.000000	0.000004	4.00E-6

Table 2: Results obtained for example 2 and error

x	Exact Solution	Approx. Solution	Error
-1	0.3678794412	0.3678735712	5.87E-6
-0.8	0.4493289641	0.4493223041	6.66E-6
-0.6	0.5448811636	0.5448805126	6.51E-6
-0.4	0.6703200460	0.6903186060	1.44E-6
-0.2	0.8187307531	0.8187296431	1.11E-6
0.0	1.0000000000	0.9999968900	3.12E-6
0.2	1.2214027580	1.2213625580	4.02E-5
0.4	1.4918246980	1.4917423980	8.23E-5
0.6	1.8221188000	1.8221129000	7.51E-5
0.8	2.2255409280	2.2255084280	3.25E-5
1.0	2.7182818280	2.7182716280	1.02E-5

6. CONCLUSION

In this paper, Tau-VIM and Tau-HPM have been successfully applied to find solutions of Fredholm-Volterra integro differential equations.

The solutions are expressed as polynomials and the errors are also obtained. Both methods performed creditably well for the examples considered. It is observed that the higher the values of n (the degree of approximant), the closer the approximate solutions obtained to the exact solutions.

REFERENCES

[1] Abbasbandy, S., & Shivamoin, E.. ``Application of the Variational Iteration Method for System of Nonlinear Volterra Integro-Differential equations''. Mathematical and Computational Applications, (2009), 14(2), 147–158.

[2] Batiha, B., Noorani, M.S.M., & Hashim, I.. ``Numerical Solutions of the Nonlinear Integro-Differential Equation''. Int. J. Open problems Compt., (2008), 1(1).

[3] Biazar, J., Porshokouhi, M.G., & Ebrahimi, H.. ``Variational Iteration Method for Linear and Nonlinear Systems of Integro-Differential Equation''. Int. Mathematical forum, (2010), 5(64), 3185–3191.

[4] Biazar, J., & Eslami, M.. ``Exact Solutions for Nonlinear Volterra – Fredholm integro Differential Equation by He's HPM''. Int. J. of Nonlinear Sciences, (2010), 9(3), 285–289

[5] He, J.H.. ``Variational Iteration Method for Eight order initial Boundary Value Problem''. Phys. Sc., (2007), 76, 680–682

[6] He, J.H.. ``A coupling Method of a Homotopy technique and a Perturbation Technique for Nonlinear problems''. Int. J. of Nonlinear Mechanics, (2000), 35, 37–43.

[7] He, J.H.. ``Variational Iteration Method: A kind of Nonlinear Analytical Technique, Some examples'', Int. J. Nonlinear Mech, (1999), 34(4), 699–708.

[8] Inokuti, M. Sekine, H., & Mura, T. ``General use of the Lagrange Multiplier in Nonlinear Mathematical Physics In. S. Nemat – Nassed, ed., Variational Method in the Mechanics of Solid''. Pergamon Press, New York, (1978), 156–162

[9] Mirzaei, S.M.. ``Homotopy Perturbation Method and Variational Iteration Method for Volterra Integral Equations". Journal of Applied Maths and Bioinformatics, (2011) 1(1), 105–113

[10] Sweilam, N.H.. ``Fourth Order Integro Differential Equations using Variational Iteration Method". Comp & Math with Appl., (2007), 54(7–8), 1086–1091.