

## On The Value Distribution of Some Differential Polynomials

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**ABSTRACT:** We prove a value distribution theorem for meromorphic functions having few poles, which improves several results of C.C.Yang and others. Also we obtain a generalized form of Clunie’s result.

C. C. Yang [8] has stated the following.

**Theorem A** Let  $f(z)$  be a transcendental meromorphic function with  $N(r, f) = S(r, f)$ .

$$\text{If } P[f] = f^n + a_1 \pi_{n-1}(f) + a_2 \pi_{n-2}(f) + \dots + a_n \tag{1}$$

Where each  $\pi_i(f)$  is a homogeneous differential polynomial in  $f$  of degree  $i$ , then  $T(r, P[f]) = nT(r, f) + S(r, f)$ .

This result is required at a later stage and we prove the above theorem on the lines of G. P. Barker and A. P. Singh [1].

**Proof** we have,  $P[f] = f^n + a_1 \pi_{n-1}(f) + a_2 \pi_{n-2}(f) + \dots + a_n$

$$\begin{aligned} &= f^n \left\{ 1 + \frac{a_1 \pi_{n-1}(f)}{f^n} + \frac{a_2 \pi_{n-2}(f)}{f^n} + \dots + \frac{a_n}{f^n} \right\} \\ &= f^n \left\{ 1 + \frac{A_1}{f} + \frac{A_2}{f^2} + \dots + \frac{A_n}{f^n} \right\} \end{aligned} \tag{2}$$

$$\text{Where } A_i = \frac{a_i \pi_{n-i}(f)}{f^{n-i}}, i = 1, 2, \dots, n$$

Now, since each  $\pi_n(f)$  is a homogeneous differential polynomial in  $f$  of degree  $n$ ,

we have ,

$$\begin{aligned} m\left(r, \frac{\pi_n(f)}{f^n}\right) &= m\left(r, \frac{\sum f^{l_0} (f^{(1)})^{l_1} \dots (f^{(k)})^{l_k}}{f^n}\right) \\ &= m\left(r, \sum \left(\frac{f^{(1)}}{f}\right)^{l_1} \left(\frac{f^{(2)}}{f}\right)^{l_2} \dots \left(\frac{f^{(k)}}{f}\right)^{l_k}\right) \\ &\leq \sum m\left(r, \left(\frac{f^{(1)}}{f}\right)^{l_1} \left(\frac{f^{(2)}}{f}\right)^{l_2} \dots \left(\frac{f^{(k)}}{f}\right)^{l_k}\right) + O(1) \\ &= S(r, f), \text{ using Milloux’s theorem [10].} \end{aligned}$$

Hence  $m(r, A_i) = S(r, f)$ , for  $i = 1, 2, \dots, n$ .

Now on the circle  $|z| = r$ , let

$$A(re^{i\theta}) = \text{Max} \left\{ |A_1(re^{i\theta})|, |A_2(re^{i\theta})|^{1/2}, \dots, |A_n(re^{i\theta})|^{1/n} \right\}$$

Then,  $m(r, A(z)) = S(r, f)$  (3)

Let  $E_1 = \left\{ \theta \in [0, 2\pi] \mid |f(re^{i\theta})| > |2A(re^{i\theta})| \right\}$  and  $E_2$  be the complement of  $E_1$ .

Then, on  $E_1$ , we have

$$\begin{aligned} |P[f]| &= \left| f^n \left( 1 + \frac{A_1}{f} + \frac{A_2}{f^2} + \dots + \frac{A_n}{f^n} \right) \right| \\ &\geq |f|^n \left\{ 1 - \left| \frac{A_1}{f} \right| - \left| \frac{A_2}{f^2} \right| - \dots - \left| \frac{A_n}{f^n} \right| \right\} \\ &\geq |f|^n \left\{ 1 - \frac{1}{2} - \frac{1}{2^2} - \dots - \frac{1}{2^n} \right\} \\ &= \frac{1}{2^n} |f|^n \end{aligned}$$

Hence, on  $E_1$ ,  $\log^+ 2^n |P[f]| \geq \log^+ [ |f|^n ]$

Therefore,  $n \log^+ |f| \leq n \log 2 + \log^+ |P[f]|$

$$\begin{aligned} \text{Therefore, } n m(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} n \log^+ |f| \, d\theta \\ &= \frac{1}{2\pi} \int_{E_1} n \log^+ |f| \, d\theta + \frac{1}{2\pi} \int_{E_2} n \log^+ |f| \, d\theta \\ &\leq \frac{1}{2\pi} \int_{E_1} (n \log 2 + \log^+ |P[f]|) \, d\theta + \frac{n}{2\pi} \int_{E_2} \log^+ |2A| \, d\theta \\ &\leq \frac{1}{2\pi} \int_{E_1} \log^+ |P[f]| \, d\theta + \frac{n}{2\pi} \int_{E_2} \log^+ |A| \, d\theta + O(1) \\ &= m(r, P[f]) + n m(r, A) + O(1) \\ &= m(r, P[f]) + S(r, f). \end{aligned}$$

Thus,  $n m(r, f) \leq m(r, P[f]) + S(r, f)$ . (4)

Adding  $n N(r, f)$  both sides and noting  $N(r, f) = S(r, f)$ , we get,

$$nT(r, f) \leq m(r, P[f]) + S(r, f).$$

Or  $n T(r, f) \leq T(r, P[f]) + S(r, f)$  (5)

$$\begin{aligned} \text{Now, } m(r, P[f]) &= m(r, f^n + A_1 f^{n-1} + A_2 f^{n-2} + \dots + A_{n-1} f + A_n), \text{ by (2)} \\ &\leq m(r, f \{ f^{n-1} + A_1 f^{n-2} + \dots + A_{n-1} \}) + m(r, A_n) + O(1) \\ &\leq m(r, f) + m(r, f^{n-1} + A_1 f^{n-2} + \dots + A_{n-1}) + S(r, f) \end{aligned}$$

Proceeding on induction, we have

$$m(r, P[f]) \leq n m(r, f) + S(r, f)$$

Therefore,  $T(r, P[f]) \leq n m(r, f) + N(r, P[f]) + S(r, f)$

But,  $N(r, P[f]) = S(r, f)$ , since  $N(r, f) = S(r, f)$ .

Hence,  $T(r, P[f]) \leq n T(r, f) + S(r, f)$  (6)

From (5) and (6) we have the required result.

We wish to prove the following result

**Theorem 1** Let  $f(z)$  be a transcendental meromorphic function in the plane and  $Q_1[f], Q_2[f]$  be differential polynomials in  $f$  satisfying  $Q_1[f] \neq 0, Q_2[f] \neq 0$ . Let  $P(f)$  be as defined in (1).

$$\text{If } F = P[f] Q_1[f] + Q_2[f], \tag{7}$$

$$\text{Then } (n - \gamma_{Q_2})T(r, f) \leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{P(f)}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\bar{N}(r, f) + S(r, f)$$

To prove the above theorem, we require the following Lemmas.

**Lemma 1 [5]:** If  $Q[f]$  is a differential polynomial in  $f$  with arbitrary meromorphic co-efficients  $q_j, 1 \leq j \leq n$ , then

$$m(r, Q[f]) \leq \gamma_Q m(r, f) + \sum_{j=1}^n m(r, q_j) + S(r, f).$$

**Lemma 2 [5]:** Let  $Q^*[f]$  and  $Q[f]$  denote differential polynomials in  $f$  with arbitrary meromorphic co-efficients  $q_1^*, q_2^*, \dots, q_n^*$  and  $q_1, q_2, \dots, q_s$  respectively.

If  $P[f]$  is a homogeneous differential polynomials in  $f$  of degree  $n$  and  $P[f]Q^*[f] = Q[f]$  where  $\gamma_Q \leq n$ , then

$$m(r, Q^*[f]) \leq \sum_{j=1}^n m(r, q_j^*) + \sum_{j=1}^s m(r, q_j) + S(r, f).$$

This leads to a generalised form of Clunie's result.

**Lemma 3 [5]:** Suppose that  $M[f]$  is a monomial in  $f$ . If  $f$  has a pole at  $z = z_0$  of order  $m$ , then  $z_0$  is a pole of  $M[f]$  order  $(m-1) \gamma_m + \Gamma_M$ .

**Lemma 4 [5]:** Suppose that  $Q[f]$  is a differential polynomial in  $f$ . Let  $z_0$  be a pole of  $f$  of order  $m$  and not a zero or a pole of the co-efficients of  $Q[f]$ . Then  $z_0$  is a pole of  $Q[f]$  of order atmost  $m \gamma_Q + (\Gamma_Q - \gamma_Q)$

**Proof of Theorem 1**

Let us suppose that  $n > \gamma_{Q_2}$ .

We have  $F = P[f] Q_1[f] + Q_2[f]$

$$\text{Therefore, } F' = \frac{F'}{F} P[f] Q_1[f] + \frac{F'}{F} Q_2[f]$$

$$\text{and } F' = P[f]' Q_1[f] + P[f] (Q_1[f])' + (Q_2[f])'$$

Hence, it follows that  $P[f] Q^*[f] = Q[f]$  (8)

where  $Q^*[f] = \frac{F'}{F} Q_1[f] - \frac{(P[f])'}{P[f]} Q_1[f] - (Q_1[f])'$  (9)

and  $Q[f] = (Q_2[f])' - \frac{F'}{F} Q_2[f]$ . (10)

We have by Lemma 2 and (8),

$$m(r, Q^*[f]) = S(r, f) \tag{11}$$

Again from (8),  $P[f] = \frac{Q[f]}{Q^*[f]}$

Therefore,  $m(r, P[f]) \leq m(r, Q[f]) + m\left(r, \frac{1}{Q^*[f]}\right)$  (12)

From (10) and Lemma 1, we have

$$m(r, Q[f]) \leq \gamma_{Q_2} m(r, f) + S(r, f). \tag{13}$$

By the First Fundamental Theorem and (11), we get

$$m\left(r, \frac{1}{Q^*[f]}\right) = N(r, Q^*[f]) - N\left(r, \frac{1}{Q^*[f]}\right) + S(r, f). \tag{14}$$

Clearly, the poles of  $Q^*[f]$  occur only from the zeros and poles of  $F$  and  $P[f]$ , the poles of  $f$  and the zeros and poles of the co efficient. Suppose that  $z_0$  is a pole of  $f$  of order  $m$ , but not a zero or a pole of the co-efficient of  $P[f]$ ,  $Q_1[f]$  and  $Q_2[f]$ .

Hence, from Lemma 4,  $z_0$  is a pole of  $Q[f]$  of order atmost  $m\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)$

If  $z_0$  is a pole of  $Q^*[f]$ , we have from (8),  $Q^*[f] = \frac{Q[f]}{P[f]}$

and hence  $z_0$  is a pole of  $Q^*[f]$  of order atmost  $m\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1) - mn$   
 $= m(\gamma_{Q_2} - n) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)$

Also, from (8),  $\frac{1}{Q^*[f]} = \frac{P[f]}{Q[f]}$

Hence  $z_0$  is a zero of  $Q^*[f]$  of order atleast  $mn - \{m\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\}$

Thus, we have,

$$N(r, Q^*[f]) - N\left(r, \frac{1}{Q^*[f]}\right) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(\frac{1}{P[f]}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\bar{N}(r, f) + (\gamma_{Q_2} - n)N(r, f) + S(r, f) \tag{15}$$

we have from (8),  $P[f] = \frac{Q[f]}{Q^*[f]}$

Therefore,  $m(r, P[f]) = m\left(r, \frac{Q[f]}{Q^*[f]}\right)$

Therefore,  $m(r, P[f]) \leq m(r, Q[f]) + m\left(r, \frac{1}{Q^*[f]}\right)$

Using (14) we have,

$$m(r, P[f]) \leq m(r, Q[f]) + N(r, Q^*[f]) - N\left(r, \frac{1}{Q^*[f]}\right) + S(r, f)$$

Using (4), (13) and (15) we have

$$\begin{aligned} n m(r, f) \leq \gamma_{Q_2} m(r, f) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{P[f]}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\bar{N}(r, f) \\ + (\gamma_{Q_2} - n)N(r, f) + S(r, f). \end{aligned}$$

Hence, we get ,  $(n - \gamma_{Q_2})T(r, f) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{P[f]}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\bar{N}(r, f) + S(r, f),$

which is the required result.

**Remark (1)** Putting  $P[f] = f^n$  in the above result, we get

$$F = f^n Q_1[f] + Q_2[f] \quad \text{and}$$

$$(n - \gamma_{Q_2})T(r, f) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{f}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\bar{N}(r, f) + S(r, f),$$

which is the result of Zhan Xiaoping [9]

(2) Putting  $P[f] = a_n f^n + a_{n-1} f^{n-1} + \dots + a_0$  in the above result,

$$\text{We get, } F = P[f] Q_1[f] + Q_2[f] \quad \text{and}$$

$$(n - \gamma_{Q_2})T(r, f) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{P[f]}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\bar{N}(r, f) + S(r, f)$$

which is the result of Hong Xun Yi[4].

**Theorem 2** Let  $F = f^n Q[f]$ , where  $Q[f]$  is a differential polynomial in  $f$  and  $Q[f] \neq 0$ .

$$\text{If } \overline{\lim}_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f)}{T(r, f)} < n, \text{ then } \Theta(a, F) < 1 \quad \text{where } a \neq 0.$$

**Proof** Let  $F = f^n Q[f]$ . Putting  $Q_2[f] \equiv 0$  in the result of Zhan Xiaoping (remark 1) we have,

$$n T(r, f) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) + S(r, f)$$

Hence  $n - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r, f)} < n$  using hypothesis.

Therefore,  $\overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r, f)} > 0$

Hence,  $\overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{F-a}\right)}{T(r, F)} > 0$ , for  $a \neq 0$ .

Therefore,  $1 - \Theta(a, F) > 0$

Or  $\Theta(a, F) < 1$  for  $a \neq 0$ .

Hence the result.

**Theorem 3** Let  $F = f^n Q_1[f] + Q_2[f]$  where  $Q_1[f]$  and  $Q_2[f]$  are as defined in Theorem 1.

$$\text{If } \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{f}\right) + (\Gamma_{Q_2} - \gamma_{Q_1} + 1)\overline{N}(r, f)}{T(r, f)} < (n - \gamma_{Q_2}) \text{ and}$$

$$Q_2[f] \neq 0, \text{ then } \Theta(a, F) < 1 \text{ where } a \neq 0.$$

**Proof** Let  $F = f^n Q_1[f] + Q_2[f]$  where  $Q_1[f]$  and  $Q_2[f]$  are as defined in Theorem .

By the result of Zhan Xiaoping [Remark 1], we have

$$(n - \gamma_{Q_2}) + T(r, f) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{f}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r, f) + S(r, f).$$

$$\text{Or } (n - \gamma_{Q_2}) - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r, f)} \leq (n - \gamma_{Q_2}) \text{ using hypothesis}$$

Hence,  $\overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{F-a}\right)}{T(r, f)} > 0$

which implies  $1 - \Theta(a, F) > 0$ .

Or  $\Theta(a, F) < 1$  for  $a \neq 0$ .

Hence the result.

**Theorem 4** Let  $F = P[f]Q[f]$  where  $P[f]$  is as defined in (1) and  $Q[f]$  is a differential polynomial in  $f$  such that  $Q[f] \neq 0$ . If  $n > 1$ , then  $\rho_F = \rho_f$  and  $\lambda_F = \lambda_f$

**Proof** Let  $G = F - a$ , where  $a(\neq 0)$  is a finite complex number. Then by Theorem 1,

$$n T(r, f) \leq \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) + S(r, f)$$

Obviously, the zeros and poles of  $f$  are that of  $F$  respectively

$$\text{Therefore, } \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + S(r, f)$$

$$\begin{aligned} \text{Thus, } n T(r, f) &\leq \bar{N}\left(r, \frac{1}{F-a}\right) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + S(r, f) \\ &\leq 3T(r, F) + S(r, f) \end{aligned}$$

Therefore,  $T(r, f) = O\{T(r, F)\}$  as  $r \rightarrow \infty$

Also,  $T(r, F) = O\{T(r, f)\}$  as  $r \rightarrow \infty$

Hence the theorem follows.

As an application of Theorem 1, we observe the following .

**Theorem 5** No transcendental meromorphic function  $f$  can satisfy an equation of the form  $a_1 P[f] Q[f] + a_2 Q[f] + a_3 = 0$ , where  $a_1 \neq 0, a_3 \neq 0, P[f]$  is as in (1) and  $Q[f]$  is a differential polynomial in  $f$ .

Putting  $P[f] = f^n$  in the above theorem, we have the following.

**Theorem 6** No transcendental meromorphic function  $f$  can satisfy an equation of the form  $a_1 f^n Q[f] + a_2 Q[f] + a_3 = 0$ , where  $a_1 \neq 0, a_3 \neq 0, n$  is a positive integer and  $Q[f]$  is a differential polynomial in  $f$ .

This improves our earlier result namely ,

**Theorem 7:** No transcendental meromorphic function  $f$  with  $N(r, f) = S(r, f)$  can satisfy an equation of the form

$$a_1(z)[f(z)]^n \pi(f) + a_2(z)\pi(f) + a_3(z) = 0 \tag{1}$$

where  $n \geq 1, a_1(z) \neq 0$  and  $\pi(f) = M_i(f) + \sum_{j=1}^{i-1} a_j(z) M_j(f)$  is a differential polynomial in  $f$  of degree  $n$  and

each  $M_i(f)$  is a monomial in  $f$ .

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