International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 2 Issue 1 ‖ January. 2014‖ PP-30-40

On New Iterative Methods for Numerical Solution of Higher-Order Parametric Differetial Equations

1,O.A TAIWO and 2,A. O.ADEWUMI
1,2,Department of mathematics, university of Ilorin, Ilorin, Nigeria

ABSTRACT : *In this paper, we attempt to answer the question "Is it possible to reduce the order of the Homotopy Analysis Method (HAM) approximation to obtain the required approximation analytical solution to a given accuracy "? YES. Based on the Homotopy Analysis Method, we developed two iterative methods, namely; Integrated Chebyshev Homotopy Analysis Methods (HC-HAM) and Integrated Chebyshev-Tau Homotopy Analysis Method (HC-THAM) for solving higher-order parametric boundary-value problems. Homotopy Aalysis Method is blended with Integrated Chebyshev Polynomials and Tau Methods and this is done by using Integrated Chebyshev Polynomials to represent the initial approximation and the derivative corresponding to m=1 and also by introducing a perturbation terms in the deformation equation. The performance of the proposed methods is validated through examples from literature. Apart from ease of implementation, better accuracy is obtained. Comparison with existing methods such as Standard Homotopy Analysis Method, Adomian Decomposition Method, Extended Adomian Decomposition Method, Optimal Homotopy Asymptotic Method and Homotopy Perturbation Method are made to show the superiority and simple applicability of the proposed iterative methods.*

KEYWORDS*: Homotopy analysis method, integrated Chebyshev polynomial, Tau method, Parametric Differential equations, Perturbation term*

I. INTRODUCTION

Higher-order parametric differential equations [14] appear frequently in physical problems and there are numbers of real time phenomena which are modeled by such equations [13]. Since exact solutions to these differential equations are very rare, so researchers always look for the best approximation solution [1]. The recent literature for the solutions of differential equations includes: the Adomian Decomposition Method (ADM) [3], the Differential Transform Method (DTM) [5], the Variation Iteration Method (VIM) [9],the Homotopy Perturbation Method (HPM) [7,8], the Extended Adomian Decomposition Method (EADM) [4], Homotopy Analysis Method [2,13]. etc. The classical Perturbation Methods are restricted to small or large parameters and hence their use is confined to a limited class of problems. The HPM as well as HAM, which are the elegant combination of Homotopy from topology and perturbation techniques, overcomes the restrictions of small or large parameters in the problems [1]. Liao [11,12] developed Homotopy Analysis Method as this method has been applied on a wide class of initial and boundary value problems [2]. Also, Marinca and Herisanu [15,16] introduced the Optimal Homotopy Asymptotic Method (OHAM), which uses the more generalized auxiliary function (HCP). They reported different forms of auxiliary that can be expressed in a compact form as $H(P)=f(r)g(P, C_i)$ is the power series in P, and the unknown constants C_i, which control the convergence of the approximating series solution, are optimally determined [1]. G. Ebadietal [4] used Extended Adomian Decomposition Method for the solutions of fourth-order parametric boundary value problems, J. A li. et al [1] applied Optimal Homotopy Asymptotic Method for solving parameterized sixth-order boundary-value problem and S.T. Mohyus Din [13] solved higher-order parametric differential equations by Homotopy Analysis Method. In this paper, we solved higher-order parametric differential equations by IC-HAM and IC-THAM. The results are then compared with those of exact solution and the solution obtained by HAM, HPM, OHAM, ADM and EADM. The structure of this paper is organized as follows; brief discussion on Chebyshev Polynomials is presented in sections 2. Section 3 is devoted to the construction of the proposed methods. In sections 4, the new methods are applied to some numerical examples and finally, section 5 is devoted to conclude the paper.

II. CONSTRUCTION OF CHEBYSHEV POLYNOMIALS

The Chebyshev Polynomial of degree n over {-1,1} is defined by the relation

$$T_n(x) = \cos\left(n \cos^{-1} x\right) \tag{2.1}$$

And,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1 \tag{2.2}$$

Equation (2.2) is the recurrence relation of the Chebyshev Polynomials in the interval [-1, 1]. Few terms are: $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$

etc

These could be converted into any interval of consideration. For example, in [a , b],we have

$$T_n(x) = \cos\left[nCos^{-1}\left\{\frac{2x-a-b}{b-a}\right\}\right] \tag{2.3}$$

And the recurrence relation is given as

$$T_n(x) = 2\left\{\frac{2x-a-b}{b-a}\right\}T_n(x) - T_{n-1}(x), \quad n \geq 1 \tag{2.4}$$

Numerical Solution Techniques

It is necessary in the first instance to give a brief review of the Homotopy Analysis Method since our techniques build on this method and serve to improve the accuracy of the HAM. The new methods refined the HAM by using a more accurate initial approximation solution and other derivatives corresponding to m=1 by integrating truncation Chebyshev polynomial and solving the higher-order deformation equations using tau methods, known for better higher accuracy.

Basic idea of HAM

Consider the following differential equation

$$N[U(t)]=0 \tag{3.1}$$

Where N is a nonlinear operator, t denotes independent variable, U(t) is an unknown function respectively. By means of generalizing the traditional Homotopy Method, Liao [11,12] constructs the so-called zero-order deformation equation

$$(1-q)L[\phi(t,q) - U_0(t)] = qc_0H(t)N\ [\phi(t,q)] \tag{3.2}$$

Where $q\varepsilon[0,1]$ is the embedding parameter, $C_0 \neq 0$ is a non-zero auxiliary parameter, H(t) ≠ 0 is an auxiliary function, L is an auxiliary linear operator, $U_o(t)$ is an initial guess of U(t), $\phi(t,q)$ is an unknown function respectively. Obviously, when q=0 and q=1, it holds for

$$\phi(t,0) = U_o(t), \quad \phi(t,1) = U(t) \tag{3.3}$$

Thus, as q increase from 0 to 1, the solution $\phi(t,q)$ varies from initial guess $U_0(t)$ to the solution $U(t)$, $\exp anding\ U_0(t)$ in Taylor series with respect to q, we have

$$\phi(t,q) = U_0(t) + \sum_{m=1}^{\infty} U_m(t)q^m \tag{3.4}$$

Where

$$U_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t,q)}{\partial q^m}\Big|_{q=0} \tag{3.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary function and convergence-control parameter, are properly chosen, the series (3.4), converges at q=1, then we have

$$U(t) = U_0(t) + \sum_{m=1}^{\infty} U_m(t) \tag{3.6}$$

According to the definition (3.5), the governing equation can be deduced from the zero-order deformation equation (3.2)

Define the vector

$$U_m = \{U_0(t),\ U_1(t), \ldots\ldots\ldots\ ,\ U_n(t)\} \tag{3.7}$$

Differentiating equation (3.2) m times with respect to the embedding parameter q and then setting q=0 and finally dividing them by m!, we have the so-called mth-order deformation equation

$$L[U_m(t) - X_m U_{m-1}(t)] = C_o H(t) R_m\left(\vec{U}_{m-1}(t)\right) \tag{3.8}$$

Where,

$$R_m(U_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t,q)]}{\partial q^{m-1}} \tag{3.9}$$

And,

$$X_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

(3.10)

If we multiply with L^{-1} each side of the equation (3.8), we obtained the following mth-order deformation equation

$$U_m(t) = X_m U_{m-1}(t) + C_0 H(t) L^{-1}\left(R_m\left(\vec{U}_{m-1}\right)\right)$$

(3.11)

It should be emphasized that $U_m(t)$ *for* $m \ge 1$ is governed by the linear equation (3.8) with the boundary conditions, which is easily solved by symbolic composition software such as Maple or Mathematics.

3.2 Construct of the NHAMs Algorithms

We consider the general higher-order boundary-value problem of the form:

$$y^{(n)}(x) = f\ \left(x, y, y^1, \ldots\ldots, y^{(n-1)}\right), \quad a \le x \le b$$

(3.12)

Subject to the two-point boundary conditions

$$y(a) = \alpha_0\ , \quad y^1(a) = \alpha_1\ , \ldots\ldots,\ y^{(r)}(a) = \alpha_r$$

(3.13)

$$y(b) = \beta_0\ , \quad y^1(b) = \beta_1\ , \ldots\ldots,\ y^{(r)}(a) = \beta_r$$

Where $0 \le r \le n-2$ is an integer, f is a polynomial in
$x, y(x),\ y^1(x), \ldots\ldots y^{(n-1)}(x), and\ \ a, b, \alpha_0, \alpha_1, \ldots\ldots, \ \alpha_r, \beta_0, \beta_1, \ldots. \beta_{n-r-2}\ \ are\ real\ constants.$

The zeroth-order deformation equations are given as

$$(1-q)\ \ L[r(x,q) \quad - y_0(x)] = qc_0 H(\varepsilon)\,[N(Y(\varepsilon,q))] \tag{3.14}$$

And

$$(1-q)\ \ L[Y(x,q) \quad - y_0(x)] = qc_0 H(\varepsilon) \quad [N(Y(\varepsilon,q)) - H^{-1}(\varepsilon) DH_N(\varepsilon)],\ q\varepsilon(0,1) \tag{3.15}$$

Where,

$$DH_N(\varepsilon) = \frac{d}{d\varepsilon}\left(T_1 T_N(\varepsilon) + T_2 T_{N-1}(\varepsilon) + T_3 T_{N-2}(\varepsilon) + \ldots\ldots + T_N T_{N-1}(\varepsilon)\right),$$

$$H(\varepsilon) = \frac{(x-\varepsilon)^{n-1}}{(n-1)\,!}, \quad L^{-1} = \int_0^x (.) d\varepsilon$$

$$N[Y(\varepsilon,q)] = \frac{\partial^n y}{\partial \varepsilon^n} - f\left(\varepsilon, y, \frac{\partial y}{\partial \varepsilon}, \frac{\partial^2 y}{\partial \varepsilon^2}, \ldots\ldots\ldots, \frac{\partial^{n-1} y}{\partial \varepsilon^{n-1}}\right) \tag{3.16}$$

It should be emphasized that $y_o(x)$ of the solution $y(x)$ and other derivatives corresponding to $m = 1$ are determined as follows.

Following [10], we have

$$\frac{d^n y_0(x)}{dx^n} = \sum_{i=0}^{N} a_i\ T_i(x) \tag{3.17}$$

Integrating equation (3.17) successively, we obtain

$$\frac{d^{n-1} y_0(x)}{dx^{n-1}} = \sum_{i=0}^{N} a_i \int\ T_i(x) dx \ + c_1 = \sum_{i=1}^{N+1} Q_i\ \ \phi_i[n-1] \tag{3.18}$$

$$\frac{d^{n-2} y_0(x)}{dx^{n-2}} = \sum_{i=1}^{N+1} Q_i \int\ \phi_i^{[n-1]} dx \ + c_1 x + c_2 = \sum_{i=0}^{N+1} Q_i\ \ \phi_i^{[n-2]} \tag{3.19}$$

$$\frac{dy_0(x)}{dx} = \sum^{N} Q_i \int\ \phi_i^{[2]} dx \ + c_1 \frac{x^{n-2}}{(n-2)!} + c_2 \frac{x^{n-3}}{(n-3)!} + \ldots\ldots + c_{n-2} x + c_{n-1} = \sum_{i=0}^{N+n-1} Q_i\ \ \phi_i^{[0]} \tag{3.20}$$

$$y_0(x) = \sum_{i=0}^{N} Q_i \int \phi_i^{[1]} dx + c_1 \frac{x^{n-1}}{(n-1)!} + c_2 \frac{x^{n-2}}{(n-2)!} + \dots\dots + c_{n-1}x + c_n = \sum_{i=0}^{N+n} Q_i \phi_i^{[0]} \quad (3.21)$$

Unlike in the case of the HAM, the auxiliary function and convergence-control parameter are not necessary as there is no need for the solution of the higher-order deformation to confirm to some rules of solution expression.

METHOD

Following the HAM procedure, we formulate the higher-order deformation equation by differentiating the zero-order deformation equation m-times with respect to q and then dividing by m! to get

$$L[y_m(x) - X_m \, y_{m-1}(x)] = H(\varepsilon) R_m \left(y_{m-1}(\varepsilon)\right)$$

(3.22)

Operating the operator L^{-1}, the inverse of $\frac{d}{d\varepsilon}$ to both sides of (3.22), then the mth –order deformation have the following form:

$$y_m(x) = X_m \, y_{m-1}(x) + L^{-1} \left(H(\varepsilon) R_m \left(\vec{y}_{m-1}(\varepsilon)\right)\right)$$

(3.23)

Where

$$R_m(\vec{y}_{m-1}(\varepsilon)) = \frac{\partial^n y}{\partial \varepsilon^n} - f\left(\varepsilon, y, \frac{\partial y}{\partial \varepsilon}, \dots\dots, \frac{\partial^{n-1} y}{\partial \varepsilon^{n-1}}\right)$$

Thus, the recursive formula for the Integrated Chebysher Homotopy Analysis Method (IC-HAM) is formulated as:

$$y_0(x) = \sum_{i=0}^{N+n} Q_i \phi_i^{(0)}$$

$$y_1(x) = L^{-1} \left(H(\varepsilon) R_m \left(\vec{y}_{m-1}(\varepsilon)\right)\right)$$

.

.

(3.24)

$$y_m(x) = y_{m-1}(x) + L^{-1} \left(H(\varepsilon) R_m \left(\vec{y}_{m-1}(\varepsilon)\right)\right), \; m = 2,3\dots\dots\dots$$

Method :

Following the same procedure as discussed in method 1 (IC-HAM), the mth-order deformation of method 2 (IC-THAM) has the form:

$$y_m(x) = X_m y_{m-1}(x) + L^{-1} \left(H(\varepsilon) \, R_m \left(\vec{y}_{m-1}(\varepsilon)\right)\right)$$

(3.25)

Where,

$$R_m(\vec{y}_{m-1}(\varepsilon)) = \frac{\partial^n y}{\partial \varepsilon^n} - f\left(\varepsilon, y, \frac{\partial y}{\partial \varepsilon}, \dots\dots, \frac{\partial^{n-1} y}{\partial \varepsilon^{n-1}}\right) - (1 - X_m) H^{-1}(\varepsilon) D H_N(\varepsilon)$$

Also, the recursive formula for the IC-THAM is given as:

$$y_0(x) \quad = \sum_{i=0}^{N+n} Q_i \quad \phi_i^{(0)}$$

$$y_1\ (x) = \ L^{-1}\ (H\ \ (\varepsilon)\ R_m\ \ (\vec{y}_{m-1}(\varepsilon)))$$

.

(3.26)

$$y_m\ (x) = y_{m-1}(x) + \ L^{-1}\ (H\ \ (\varepsilon)\ R_m\ \ (\vec{y}_{m-1}(\varepsilon))),\ m = 2,3..........$$

Application of methods on some Examples

In this section, we apply the techniques described in section 3.To some illustrative example of fourth, sixth and eighth-order parameter boundary-value problems.

Example:

Consider the following linear problem [4]

$$y^{iv}(x) = (1+c)\ y^{ii}(x) - cy(x) + \frac{1}{2}\ cx^2\ \ -1\ ,0 < x < 1$$

Subject to

$$y(0) = 1\ , \qquad y^1(0) = 1$$

$$y(1) = 1.5 + smh\ (1)\ , \qquad y^1(1) = 1.5 + \cosh(1)$$

The exact solution for this problem is

$$y(x) = 1 + \frac{1}{2}x^2 + smh(x)$$

According to (3.24) the zeroth-order deformation is given by

$$(1-q)\ L[Y(x,q) - y_0(x)] = \ qH(\varepsilon)\left[\left(\frac{d^4 y(\varepsilon,q)}{d\varepsilon^4} - (1+c)\frac{d^2 y(\varepsilon,q)}{d\varepsilon^2}\ + cy(\varepsilon;q) - \left(\frac{1}{2}c\varepsilon^2 - 1\right)\right)\right]$$

Now, our initial approximation has the form of equation (3.21), we chose the auxiliary linear operation

$$L(y(\varepsilon;q)) = \frac{dy(\varepsilon;q)}{d\varepsilon}\ \ and\ \ H(\varepsilon) = \frac{(x-\varepsilon)^3}{6}$$

Hence, the mth-order deformation can be given by

$$L[y_m(x) - x_m y_{m-1}(x)] \ = H(\varepsilon)R_m(\vec{y}_{m-1}(\varepsilon))$$

Where

$$R_1(\vec{y}_0(\varepsilon)) = \frac{d^4 y_0(\varepsilon;y)}{d\varepsilon^4} - (1+c)\frac{d^2 y_0(\varepsilon;q)}{d\varepsilon^2} + cy_0(\varepsilon;q) - \left(\frac{1}{2}c\varepsilon^2 - 1\right)$$

And

$$R_m(\vec{y}_{m-1}(\varepsilon)) = \frac{d^4 y_{m-1}(\varepsilon;y)}{d\varepsilon^4} - (1+c)\frac{d^2 y_{m-1}(\varepsilon;q)}{d\varepsilon^2} + cy_{m-1}(\varepsilon;q) - (1-x_m)\left(\frac{1}{2}c\varepsilon^2 - 1\right);\ m \ge 2$$

Now, the solution of the m-th order deformation equation for

$m \ge 1$ becomes $y_m(x) = X_m y_{m-1}(x) + \int_0^x \left(\frac{1}{6}(x-\varepsilon)^3 R_m(\vec{y}_{m-1}(\varepsilon))\right) d\varepsilon$

Consequently, the first few terms of the IC-HAM series solution for N=4 and C=1 are as follows;

$$y_0(x) = c_4 + c_3 x + \frac{1}{2}c_2 x^2 + \frac{1}{6}c_1 x^3 + \frac{1}{24}a_0 x^4 + \left(-\frac{1}{2}x^4 + \frac{1}{60}x^5\right)a_1 + \left(\frac{1}{24}x^4 - \frac{1}{45}x^6\right)a_2$$

$$+\left(-\frac{1}{24}x^4 + \frac{3}{20}x^5 - \frac{2}{15}x^6 + \frac{4}{105}x^7\right)a_3 + \left(\frac{1}{24}x^4 - \frac{4}{15}x^5 + \frac{4}{a}x^6 - \frac{32}{105}x^7 + \frac{8}{105}x^8\right)a_4$$

$$y_1(x) = \left(\frac{1}{5040}x^7 - \frac{1}{60}x^5\right)c_1 + \left(\frac{1}{720}x^6 - \frac{1}{12}x^4\right)c_2 + \frac{1}{120}x^5 c_3 + \frac{1}{24}x^4 c_4 + \frac{1}{24}x^4 - \frac{1}{720}x^6 +$$

$$\left(\frac{1}{24}x^4 - \frac{1}{360}x^6 + \frac{1}{40320}x^8\right)a_0 + \left(-\frac{1}{24}x^4 + \frac{1}{60}x^5 + \frac{1}{360}x^6 - \frac{1}{1260}x^7 - \frac{1}{40320}x^8 + \frac{1}{181440}x^9\right)a_1$$

$$\left(\frac{1}{24}x^4 - \frac{1}{15}x^5 + \frac{7}{360}x^6 + \frac{1}{315}x^7 - \frac{31}{40320}x^8 - \frac{1}{45360}x^9 + \frac{1}{226800}x^{10}\right)a_2 +$$

$$\left(-\frac{1}{24}x^4 + \frac{3}{20}x^5 - \frac{47}{360}x^6 + \frac{13}{420}x^7 + \frac{191}{40320}x^8 - \frac{61}{60480}x^9 - \frac{1}{37800}x^{10} + \frac{1}{207900}x^{11}\right)a_3 +$$

$$\left(\frac{1}{24}x^4 - \frac{4}{15}x^5 + \frac{53}{120}x^6 - \frac{92}{315}x^7 + \frac{811}{13440}x^8 + \frac{19}{2268}x^9 - \frac{13}{8100}x^{10} - \frac{2}{51975}x^{11} + \frac{1}{155925}x^{12}\right)a_4$$

and so on.

The first order approximation solution by IC-HAM is

$$y(x) = y_0(x) + y_1(x)$$

And the residual of the solution is

$$R = y^{iv}(x) - (1+c)\ y^{ii}(x) + cy(x) - \left(\frac{1}{2}cx^2 - 1\right)$$

Using the boundary conditions, we obtained $c_1, c_2, \ldots\ c_4$ and minimizing the residual error by using Least Square Method, we obtained the following values of a_0 , a_1 , a_3 *and* a_4 *for* $c = 1$

$a_0 = 0.5541761136$; $a_1 = 0.5816075870$; $a_2 = 0.0332521244\ 2$; $a_3 = 0.0059547394\ 61$;

$a_4 = 0.0001715598\ 043$.

Thus, the approximation solution becomes:

$$y(x) = 1 + 0.4999999964\ x^2 + 0.1666666688\ x^3 + 0.0000031231\ 3x^4 + 0.0083148926\ 99x^5 +$$

$$0.0000423351\ 8090\,x^6 + 0.0001511510\ 114\,x^7 + 0.0000253855\ 3147\,x^8 + 1.100271312\ x10^{-19}x^{12}$$

$$+\ 2.204069758\ x10^{-8}x^{11} - 2.862613806\ x10^{-7}x^{10} - 0.0000020962\ 70555\,x^9$$

For c = 10, the following values a_0 , a_1 , a_3 *and* a_4 are obtained

$$a_0 = 0.5541759788; a_1 = 0.5816065302; a_2 = 0.03325208987\ ;\ a_3 = 0.005953393889\ ;$$
$$a_4 = 0.0001715410873$$

In this case, the approximate solution is

$$y(x) = 1 + x + 0.4999999868\ x^2 + 0.1666666666\ x^3 + 0.0000033190\ 90608\,x^4 + 0.0083144687\ 0x^5$$

$$+ 0.000042173 25x^6 + 0.0001530912179x^7 + 0.0000219478 3736x^8 + 1.100151273\,X\,10^{-8}x^{12} +$$
$$2.203494560X\,10^{-7}x^{11} - 0.000001554979248x^{10} + 8.63843375X10^{-7}x^9$$

Method 2:

Following the same procedures as discussed in method I (IC-HAM) and using equation (3.2.14), we obtained $C_1,.....C_4$ in terms of $a_0, a_1,...a_4, \tau_1,..., \tau_4$ and minimizing of the residual error, we obtained the following values of $a_0,...a_4, \tau_1...\tau_4$ *for* $C = 1$:

$$a_0 = 0.1480912750;\ a_1 = -1.677563749; a_2 = 0.004350300326; a_3 = -0.008606762145;$$
$$a_4 = 0.00008410994856; \tau_1 = -0.0003445697303; \tau_2 = -0.02176231068; \tau_3 = -0.008251027514;$$
$$\tau_4 = 1 - 0552333902.$$

Also, by substituting these values into first order approximation $(y_0 + y_1)$, the approximation solution becomes:

$$y(x) = 1 + 0.4999999979x^2 + 0.166666680x^3 + 0.0000015534x^4 + 0.008324195 51x^5 +$$
$$0.000020901367x^6 + 0.0001751188113x^7 + 1.000000000x + 0.0000126470 5882x^8$$
$$+ 5.394256762X^{-10}x^{12} - 4.463512138X10^{-8}x^{11} + 1.118820631X10^{-7}x^{10} + 4.36488315X10^{-8}x^9$$

Similarly, for C=10, the following values of $a_0\,; a_1 .... a_4\,;\ \tau_1\,;.........\,.\tau_4$ are obtained;

$$a_0 = 0.03731442487; a_1 = 0.4615749665; a_2 = -0.08481031518; a_3 = 0.003695192408;$$
$$a_4 = -0.002389928628;\ \tau_1 = 0.0000779304 3043047;\ \tau_2 = 0.0008494323998\ ;$$
$$\tau_3 = -0.0004018919498; \tau_4 = -0.05881308214\ .$$

Thus, we have the following approximate solution:

$$y(x)=1+0.4999999997x^2+0.1666666647x^3+0.0000042137x^4+0.008298404337x^5$$
$$+0.0001157971 8x^6+7.64551X10^{-7}x^7+1.000000000x+0.00018368070 83x^8$$
$$-1.53274246X10^{-7}x^{12}+0.0000010973 84388x^{11}+0.0000154309 0318x^{10}-0.0000847063 6057x^9$$

Table 1:
Absolute errors of the first-order approximate solution when C=1 and the error for third order approximation solution of HAM [13]

x	Analytical solution	[4] EHPM	[4] EADM	[4] EEADM	[13] EHAM	E_{IC} -HAM	E_{IC} -THAM
0.0	1.000000000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	1.105166750	7.4e-05	7.4e-05	1.1488e-06	4.8e-06	0.0000	0.0000
0.2	1.221336002	2.5e-04	2.5e-04	3.2027e-07	8.6e-05	2.000e-09	0.0000
0.3	1.349520293	4.6e-04	4.6e-04	1.1328e-05	3.2e-04	3.00e-09	2.00e-09
0.4	1.490752326	6.5e-04	6.5e-04	3.4636e-05	6.5e-04	2.00e-09	2.00e-09
0.5	1.646095306	7.6e-04	7.6e-04	6.6411e-05	9.6e-04	1.00e-09	0.0000
0.6	1.816653582	7.5e-04	7.5e-04	9.6330e-05	1.1e-03	3.00e-09	3.00e-09
0.7	2.003583702	6.1e-04	6.1e-04	1.1038e-04	1.00e-03	4.00e-09	1.00e-09
0.8	2.208105982	3.8e-04	3.8e-04	9.6471e-05	7.0e-04	1.00e-09	1.00e-09
0.9	2.431516726	1.3e-04	1.3e-04	5.2931e-05	2.5e-04	0.0000	1.00e-09
1.0	2.675201194	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2:
Absolute errors of the first-order approximate solution when C=10 and errors for third order approximation of HAM [13]

X	Analytical solution	[4] EHPM	[4] EADM	[4] EEADM	[13] EHAM	EIC-HAM	EIC-THAM
0.0	1.000000000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	1.105166750	1.7e-04	1.7e-04	4.50410e-06	2.9e-06	0.0000	0.0000
0.2	1.221336002	5.7e-04	5.7e-04	3.02581e-05	9.2e-05	1.00e-09	0.0000
0.3	1.349520293	1.0e-03	1.0e-03	8.72832e-05	3.1e-04	2.00e-09	0.0000
0.4	1.490752326	1.4e-03	1.4e-03	1.67419e-04	6.2e-04	2.00e-09	0.0000
0.5	1.646095306	1.6e-03	1.6e-03	2.44493e-04	9.2e-04	1.00e-09	1.00e-09
0.6	1.816653582	1.6e-03	1.6e-03	2.83793e-04	1.1e-03	0.0000	0.0000
0.7	2.003583702	1.2e-03	1.2e-03	2.58064e-04	1.0e-03	1.00e-09	2.00e-09
0.8	2.208105982	7.6e-03	7.6e-03	1.66169e-04	7.2e-04	1.00e-09	1.00e-09
0.9	2.431516726	2.5e-03	2.5e-03	4.94701e-05	2.7e-04	1.00e-09	1.00e-09
1.0	2.675201194	0.0000	0.0000	0.0000	0.0000	0.0000	1.30e-09

Example 2: Consider the following problem [1]

$$y^{(vi)}(x)=(1+c)y^{(iv)}(x)-Cy^{(ii)}(x)+Cx;\ 0<x<1$$

with the boundary conditions

$$y(0)=1\ ;\ y^{i}(0)=1\ ;\ y^{ii}(0)=0$$

$$y(1)=\frac{7}{6}+\sinh(1)\,;\ y^{i}(1)=\frac{1}{2}+\cosh(1)\,;\ y^{ii}(1)=1+\sinh(1)$$

The exact solution is given as:

$$y(x)=1+\frac{1}{6}x^3+\sinh(x)$$

This problem is solved by the method applied in Example 1 and for each test point, the absolute error between the analytical solution and the results obtained by the HAM [13], OHAM [1], and the IC-HAM and IC-THAM when N=10 are compared in Table3 for C=1000. With only one iteration, a better approximation is obtained.

Table 3: Absolute errors of the first –order approximate solution when C=1000 and errors of the third-order OHAM [1] and fifth-order HAM [1] are tabulated below for comparison.

x	Analytical solution	Error of IC-HAM	Error of IC-THAM	[1] EHAM	[1] EOHAM
0.1	1.100333417	1.0e-09	1.0e-09	9.1e-06	1.1e-05
0.2	1.202669336	1.0e-09	2.0e-09	1.6e-04	3.3e-06
0.3	1.309020293	0.0000	1.0e-09	4.4e-04	1.4e-05
0.4	1.421418993	1.0e-09	4.0e-09	6.8e-04	5.2e-06
0.5	1.541928638	2.0e-09	4.0e-09	7.3e-04	4.2e-05
0.6	1.672653587	1.0e-09	3.0e-09	5.8e-04	5.7e-05
0.7	1.815750369	2.0e-09	2.0e-09	3.2e-04	4.9e-05
0.8	1.973439315	2.0e-09	2.0e-09	9.8e-05	4.5e-05
0.9	2.148016726	1.0e-09	2.0e-09	4.7e-06	2.4e-05

Example 3:

Consider the following eighth-order parametric differential equation type [13]

$$\frac{d^8 y(x)}{dx^8} = (1+c)\frac{d^4 y(x)}{dx^4} - Cy(x) + \frac{c}{24}x^4 - 1$$

Subject to the boundary conditions

$$y(0) = 1; \quad \frac{dy(0)}{dx} = 1; \frac{d^2 y(0)}{dx^2} = 0 \quad \frac{d^3 y(0)}{24} = 1$$

$$y(1) = \frac{25}{24} + \sinh(1); \frac{dy(1)}{dx} = \frac{1}{6} + \cosh(1); \frac{d^2 y(1)}{dx^2} = \frac{1}{2} + \sinh(1); \frac{d^3 y(1)}{dx^3} = 1 + \cosh(1)$$

The exact solution for is given as:

$$y(x) = 1 + \frac{1}{24}x^4 + \sinh(x)$$

The numerical results obtained by IC-HAM and IC-THAM are compared with the analytical solution and the results obtained by HAM [13] are presented in Table 4. These results are evaluated at m=1and it is seen from the numerical results in the table that IC-HAM and IC-THAM are more accurate than the second order application of the HAM solution in [13]

Table 4:

Absolute errors of the first-order approximate solution obtained by IC-HAM, IC-THAM (N=10) and the second-order approximation of HAM [13] for example 3 when C=1000

X	Analytical solution	EHAM [13]	EIC-HAM	EIC-THAM
0	1.000000000	0.0000	0.0000	0.0000
0.1	1.100170917	8.7e-09	0.0000	0.0000
0.2	1.201402670	9.8e-08	0.0000	0.0000
0.3	1.304857793	3.3e-07	0.0000	0.0000
0.4	1.411818993	6.7e-07	0.0000	0.0000
0.5	1.523699472	9.4e-07	0.0000	0.0000
0.6	1.642053582	9.9e-07	0.0000	0.0000

0.7	1.768587869	7.8e-07	1.0e-9	1.0e-09
0.8	1.905172649	3.9e-07	0.0000	1.0e-09
0.9	2.053854226	7.8e-07	1.0e-09	2.0e-09
1.0	2.216867860	0.0000	0.0000	0.0000

- 0 Denotes less than 10^{-10}

III. CONCLUSION

The main concern of this work is to develop efficient algorithms for the numerical solution of higher-order parametric differential equation. The goal and the question raised in the beginning of the paper are achieved by blending integrated Chebyshev Polynomials and tau Methods with Homotopy Analysis Method to solve this class of problems in question. The proposed algorithm produced rapidly convergent series and the results obtained by the new methods agreed well with the analytical solutions with less computational work. These confirm the belief that the efficiency of the proposed methods give much wider applicability for general classes of parametric differential equations.

REFERENCES

[1] A. Jared, I. Saeed, K. Hamid and Z . Gul. The solution of a parameterized sixth-order boundary-value problem by the Optimal Homotopy Asymptotic Method, proceeding of the Romanian Academy, Series A, 12 (2011), 167-172

[2] B. Ghanbari and M.Ghaolani. A New Method for solving Nonlinear BVPs, Gen. Math. Notes, Vol.1, No 2, (2010), 143-152

[3] G. Adomian. Solving Frontier Problems of Physics; The Decompositions method, Vol.60 of Fundamental Theories of physics, kluwer Academic publishers, Boston, Mass, USA, 1994.

[4] G. Ebadi and S. Rashedi. The Extended Adomian decomposition method for fourth order boundary value problems, Acta Universitatis Apulensis, 22 (2010), 65-78

[5] J. Ali, siraju-Haq and Siraj-ul-islam. Numerical solution of special 12^{th}- order boundary value problems using differential transform method, comm. Noul. Sc. Nu. Sim. 14, 4(2009),1132-1138.

[6] J.C, Mason and D.C Handscomb. Chebyshev Polynomials Chapman and Hall-CRC, Boca Raton, London (2003)

[7] J-it. He, Homotopy Perturbation Technique, Computer Methods in Applied Mechanics and Engineering, 170, 3-4 (1999), 257-262.

[8] J-H. He. Homotopy Perturbation Method for solving boundary value problems. Phys. Lett .A, 350 (2006) 87-88

[9] J-H, He. Variation Approach to the sixth-order boundary value problems, Appl. Math. Comput , 143,2-3 (2003),537-538

[10] N.Mai- Duy and R.I.Tanner. Solving higher-order ordinary differential equations with radial basis Function networks, International Journal For Numerical Methods in Engineering, 62 (2005), 824-852

[11] S.J, Liao, Homotopy Analysis Method in Nonlinear Differential Equations, Higher Education Press, Beijing and Springer - Verlay Berlin Herdelberg, 2012.

[12] S.J. Liao. Beyond Perturbation: Introduction to Homotopy Analysis Method, Champman and Hall/CRC Press.

[13] S.T. Mohyud -Dm, A .Hussain and A, Yildrim. Homotopy Analysis Method for Parametric Differential Equations, World Applied Sciences Journal II (7), (2010), 851-856.

[14] S. Momani and Noor. M.A. Numerical comparison of methods for solving a special fourth-order boundary value problem, Appl. Math. Comput. 191 (2007), 218-224

[15] V. Marinca, N. Herisanu and I. Nemes. Optional Homotopy Asymptotic Method with application to thin film flow, Cent. Eur. J. phys, 6, 3 (2008), 648-653.

[16] V. Marince , N. Herisana. An Optimal Homotopy Asymptotic Method for solving nonlinear equations arising in heat transfer, Int. Comm. Heat and Mass Transfer, 35(2008), 710-715.