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Common fixed point theorem through weak compatibility in Menger space

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ABSTRACT: *The object of this paper is removing the condition of continuity in[6]and establish a unique common fixed point theorem for six mappings using the concept of weak compatibility in Menger space which is an alternative result of Chandel and Verma[1].*

KEYWORDS: *Continuous t-norm, PM-space, Menger space, compatible, weak compatible.*

I. INTRODUCTION

In 1942, Menger [4] has introduced the theory of probabilistic metric space in which a distribution function was used instead of non-negative real number as value of the metric. In 1962,Schweizer and Sklar [8] studied this concept and gave fundamental result on this space. In 1972,Sehgal and Bharucha–Reid[9] obtained a generalization of Banach Contraction principle on a complete metric space which is a milestone in developing fixed point theory in Menger space. In 1982,Sessa [10] introduced weakly commuting mappings in Menger space.In 1986,Jungck enlarged this concept to Compatible maps .in 1991, Mishra [5] has been introduced the notion of compatible maps in Memger space.In 1998 Jungck and Rhoades[3] introduced the concept of wealky compatibility and showed that each pair of compatible maps is wekly compatible but the converse need not to be true.In 2005 Singh and Jain [11]generalized the result of Mishra[5] using the concept of weak compatibility and compatibility of pair of self maps.In this paper prove a fixed point theorem for six weakly compatible mappings in Menger space. First recall some definitions and known result in Menger space.

II. PRELIMINARY NOTES

Definition 2.1*A mapping T: $[0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if T satisfied the following conditions:*

(1) $T(a, 1) = a$, $T(0,0) = 0$
(2) $T(a, b) = T(b, a)$,
(3) T is continuous
(4) $T(a, b) \le T(c, d)$, *whenever* $a \le c$ *and* $b \le d$, *and* $a, b, c, d \in [0,1]$
(5) $T(T(a, b), c) = T(a,T(b, c))$ *for all* $a,b,c \in [0,1]$

Definition 2.2 *A Mapping* $F:R \to R^+$ *is said to be a distribution function if it is non-decreasing and left continuous with*
$Inf\{F(t):t \in R\} = 0$ *and* $Sup\{F(t):t \in R\} = 1$
We will denote the Δ *the set of all distribution function defined on* $[-\infty,\infty]$ *while* $H(t)$ *will always denote the specific distribution function defined by*

$$H(t)=\begin{cases} 0, & if\ t \le 0 \\ 1, & if\ t > 0 \end{cases}$$

Definition 2.3*(Schweizer and Sklar[8])The ordered pair (X,F) is called a probabilistic metric space(shortly PM-space) if X is nonempty set and F is a probabilistic distance satisfyin the following condtions:*

PM-1 $F_{x,y}(t) = 1$ *if and only if* $x=y$
PM-2 $F_{x,y}(0) = 0$
PM-3 $F_{x,y}(t) = F_{y,x}(t)$
PM-4 *If* $F_{x,z}(t) = 1$ *and* $F_{z,y}(s) = 1$ *then* $F_{x,y}(t+s)=1$ *for all* $x,y,z \in X$ *andt,s> 0*

the ordered triple (X,F,T) is called Menger space if (X,F) is PM-space and T is a triangular norm such that for all $x,y,z \in X$ *and* $t,s> 0$

PM-5 $F_{x,y}(t+s) \ge F_{x,z}(t) + F_{z,y}(s)$

Definition 2.4*A Menger space (X, F, T) with the continuous t-norm Tis said to be complete iff every Cauchy sequence in X converges to a point in X.*

Definition 2.5 *The self maps A and B of a Menger Space (X,F,T) are said to be compatible if* $F_{ABxn,BAxn}(t) \to 1$ *for all t>0*
Whenever {xn} is a sequence in X such that Axn,Bxn→ x for some x∈X
asn → ∞

Definition 2.6 *Two self-maps A andB of a non-empty set X are saidto be weakly compatible (or coincidentally commuting) if they commute at theircoincidence points, i.e. if Ax= Bxfor some x ∈ X, then ABx= BAx.*

Lemma 2.7(Singh and Jain [11]) *Let* $\{x_n\}$ *be a sequence in a Mengerspace (X, F, T) with continuous t-norm T and T(a,a) ≥ a. If there exists a constant k ∈ (0, 1) such that*
$F_{xn,xn+1}(kt) \geq F_{xn-1,xn}(t)$ *for all t > 0 and n = 1, 2,..........*
then $\{x_n\}$ *is a Cauchy sequence in X.*

Lemma 2.8 (Singh and Jain [11]) *Let (X, F, ∗) be a Menger space. If there exists k ∈(0, 1) such that Fx,y(kt) ≥Fx,y(t) for all x, y ∈ X and t > 0, then x = y.*

III. MAIN RESULT

Theorem 3.1: *Let A, B, S, T, L And M be self maps on a Menger space (X, F, t) with continuous t-norm and defined by t(a, b)= min{a,b} for alla∈[0,1] and satisfying the following:*
(3.1.1) AB (X)⊆M(X) and ST(X) ⊆L(X)
(3.1.2) M(X) and L(X) are complete subspace of X
(3.1.3) (AB,L) and (ST,M) are weakly compatible
(3.1.4) For all x, y∈X ,k∈(0,1) , t>0
$F^3_{ABx,STy}(kt) \geq \min\{ F^3_{Lx,My}(t)\ F^3_{ABx,Lx}(t)\ F^3_{STy,Lx}(t)\ F_{ABx,My}(2t)$
$F_{STy,Lx}(t)\ F^3_{STy,My}(t)\}$
Then AB, ST, L and M have a unique common fixed point in X.
Proof: *Let* x_0*be any arbitrary pointof X.*
Since AB (X) ⊆ M(X) and ST(X) ⊆ L(X).there exist $x_1,x_2 \in X$ *such that*
$ABx_0 = Mx_1$, *and* $STx_1 = Lx_2$,
Inductively we construct the sequence $\{ x_n \}$*and* $\{y_n\}$*in X such that*
$Mx_{2n-1} = ABx_{2n-2} = y_{2n-1}$*and* $y_{2n} = Lx_{2n} = STx_{2n-1}$*for n=1,2,3,…………*
Step 1: *By taking* $x=x_{2n}$ *and* $y=x_{2n+1}$ *in (iv), we have from(3.1.4)*

$F^3_{ABx2n,STx2n+1}(kt) \geq \min\{ F^3_{Lx2n,Mx2n+1}(t)\ F^3_{ABx2n,Lx2n}(t) F^3_{STx2n+1,Lx2n}(t)$
$F_{ABx2n,Mx2n+1}(2t)\ F_{STx2n+1,Lx2n}(t)\ F^3_{STx2n+1,Mx2n+1}(t)\}$

$\Rightarrow F^3_{y2n+1,y2n+2}(kt) \geq \min\{ F^3_{y2n,y2n+1}(t)\ F^3_{y2n+1,y2n}(t)\ F^3_{y2n+1,y2n}(t)$
$F_{y2n+1,y2n+1}(2t)\ F_{y2n+1,y2n}(t)\ F^3_{y2n+1,y2n+1}(t)\}$
$\Rightarrow F_{y2n+1,y2n+2}(kt) \geq F_{y2n,y2n+1}(t)$
Similarly we can written as,
$F_{y2n,y2n+1}(kt) \geq F_{y2n-1,y2n}(t)$
In general, for all n even or odd ,we have
$F_{yn,yn+1}(kt) \geq F_{yn-1,yn}(t)$ *for k∈(0,1) and all t>0*
Thus by lemma 2.7 $\{y_n\}$*is a Cauchy sequence in X and subsequence are also Cauchy sequence in X.*
Step 2.*Since M(X) is a complete subspace of X.Therefore* $\{ y_{2n+1} \}$ *converges to z∈Xthen Mu=z*
Now taking $x=x_{2n-2}$, $y = u$*, we havefrom(3.1.4)*
$F^3_{ABx2n-2,STu}(kt) \geq \min\{ F^3_{Lx2n-2,Mu}(t)\ F^3_{ABx2n-2,Lx2n-2}(t)\ F^3_{STu,Lx2n-2}(t)$
$F_{ABx2n-2,Mu}(2t)\ F_{STu,Lx2n-2}(t)\ F^3_{STu,Mu}(t)\}$
taking limit n → ∞,we have
$F^3_{z,STu}(kt) \geq \min\{ F^3_{z,z}(t)\ F^3_{z,z}(t)\ F^3_{STu,z}(t) F_{z,z}(2t)\ F_{STu,z}(t)\ F^3_{STu,z}(t)\}$
This gives,$F^3_{z,STu}(kt) \geq F^3_{STu,z}(t)$
Hence by lemma 2.8 ,STu = z .Therefore STu=Mu=z
We can say,u is a coincidence point of ST and M.
Step 3. *Since L(X) is a complete subspace of X****.****Therefore* $\{y_{2n+1}\}$ *converges to*
z∈X then Lv=z.Nowusing x=v and $y=x_{2n-1}$*we have from(3.1.4)*

$\mathcal{F}^3_{ABv,STx_{2n-1}}(kt) \geq \min\{ \mathcal{F}^3_{Lv,Mx_{2n-1}}(t)\ \mathcal{F}^3_{ABv,Lv}(t)\ \mathcal{F}^3_{STx_{2n-1},Lv}(t)$
$\mathcal{F}_{ABv,Mx_{2n-1y}}(2t)\ \mathcal{F}_{STx_{2n-1},Lv}(t)\ \mathcal{F}^3_{STx_{2n-1},My}(t)\}$
taking limit $n \to \infty$, we have
$\mathcal{F}^3_{ABv,z}(kt) \geq \min\{ \mathcal{F}^3_{z,z}(t)\ \mathcal{F}^3_{ABv,z}(t) \mathcal{F}^3_{z,z}(t) \mathcal{F}_{ABv,z}(2t)\ \mathcal{F}_{z,z}(t)\ \mathcal{F}^3_{z,z}(t)\}$

$\Rightarrow \mathcal{F}^3_{ABv,z}(kt) \geq \mathcal{F}^3_{ABv,z}(t)$
Hence by lemma 2.8 ,ABv = z . Therefore ABv=Lv=z
We can say, v is a coincidence point of AB and L.

Step 4.*Since the pair{ST,M} is weakly compatible for some $u \in X$*
(ST)Mu=M(ST)u whenever STz=Mz
Now using $x = x_{2n-2}$ andy=z,we havefrom(3.1.4)
$\mathcal{F}^3_{ABx_{2n-1},STz}(kt) \geq \min\{ \mathcal{F}^3_{Lx_{2n-1},Mz}(t)\ \mathcal{F}^3_{ABx_{2n-1},Lx_{2n-1}}(t)\ \mathcal{F}^3_{STz,Lx_{2n-1}}(t)$
$\mathcal{F}_{ABx_{2n-1},Mz}(2t) \mathcal{F}_{STz,Lx_{2n-1}}(t)\ \mathcal{F}^3_{STz,Mz}(t)\}$
taking limit $n \to \infty$, we have
$\mathcal{F}^3_{z,STz}(kt) \geq \min\{ \mathcal{F}^3_{z,z}(t)\ \mathcal{F}^3_{z,z}(t)\ \mathcal{F}^3_{STz,z}(t)\ \mathcal{F}_{z,z}(2t)\ \mathcal{F}_{STz,z}(t)\ \mathcal{F}^3_{STz,z}(t)\}$
Thus, $\mathcal{F}^3_{z,STz}(kt) \geq \mathcal{F}^3_{STz,z}(t)$
Hence by lemma 2.8 ,STz = z . Since Mz=STz .Therefore STz=Mz=z
Step5:*Since the pair{ST,M} is weakly compatible for some $v \in X$*
(AB)Lv = L(AB)v whenever ABz=Lz
now x= z and y= x_{2n-1} , we havefrom (3.1.4)
$\mathcal{F}^3_{ABz,STx_{2n-1}}(kt) \geq \min\{ \mathcal{F}^3_{Lz,Mx_{2n-1}}(t)\ \mathcal{F}^3_{ABz,Lz}(t) \mathcal{F}^3_{STx_{2n-1},Lz}(t) \mathcal{F}_{ABz,Mx_{2n-1}}(2t)$
$\mathcal{F}_{STx_{2n-1},Lz}(t)\ \mathcal{F}^3_{STx_{2n-1},Mx_{2n-1}}(t)\}$
Taking limit $n \to \infty$, we have
$\mathcal{F}^3_{ABz,z}(kt) \geq \min\{ \mathcal{F}^3_{z,z}(t)\ \mathcal{F}^3_{ABz,z}(t)\ \mathcal{F}^3_{z,z}(t) \mathcal{F}_{ABz,z}(2t) \mathcal{F}_{z,z}(t)\ \mathcal{F}^3_{z,z}(t)\}$

$\Rightarrow \mathcal{F}^3_{ABz,z}(kt) \geq \mathcal{F}^3_{ABz,z}(t)$
Hence by lemma 2.8 ,ABz = z .
Since Lz=ABz ,Therefore ABz=Lz=z
Step6.Uniqueness *Let w (w≠z) be another common fixed point of AB, ST, L and M , then w=ABw =STw =Lw=Mw taking x=z and y=w thenfrom (3.1.4)*
$\mathcal{F}^3_{ABz,STw}(kt) \geq \min\{ \mathcal{F}^3_{Lz,Mw}(t)\ \mathcal{F}^3_{ABz,Lz}(t)\ \mathcal{F}^3_{STw,Lz}(t)\ \mathcal{F}_{ABz,Mw}(2t)$
$\mathcal{F}_{STw,Lz}(t)\ \mathcal{F}^3_{STw,Mw}(t)\}$

$\Rightarrow \mathcal{F}^3_{z,w}(kt) \geq \min\{ \mathcal{F}^3_{z,w}(t)\ \mathcal{F}^3_{z,z}(t)\ \mathcal{F}^3_{w,z}(t)$
$\mathcal{F}_{z,w}(2t)\ \mathcal{F}_{w,z}(t)\ \mathcal{F}^3_{w,w}(t)\}$
$\mathcal{F}^3_{z,w}(kt) \geq \mathcal{F}^3_{z,w}(t)$
Hence by lemma 2.8, z=w which is a contradiction of our hypothesis .
Therefore,z is a common fixed point of AB,ST,L and M.

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