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A comparative Study on Functions of Bounded Variation and Riemann Integral

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ABSTRACT: *the concept of bounded variation functions have many common characteristics with the notion of Riemann integral, however, in this paper, we examine the criteria for which a function* f *belong to the two notions. We proved that a continuous function is of bounded variation if and only if it can be expressed as a difference of two monotonic increasing functions, and we concluded that an R – integrable function* f *is also of bounded variation if and only if it can be expressed as a difference of two monotonic increasing functions.*

KEYWORDS: *Bounded Variation Function, Riemann Integral, Continuous Function, Monotonic Increasing Function.*

I. INTRODUCTION

Historically, the subject arose in an attempt to find lengths, areas and volumes of curve $y = f(x)$ enclosed within the ordinates $x = a, and\ x = b$ and the x – axis. This area can be viewed as made up of a very large number of thin strips of area and the limit of the sum of the area of these strips tends to infinity, represent the area in question. Thus, historically the subject is based on the notion of the limit of a type of sum when the number of terms in the sum tends to infinity, each term tending to zero.

The definition of integration from summation point of view is always associated with geometrical intuition and is applicable to those functions which may be represented graphically. As we know there are even many continuous functions which con not be represented graphically. Hence, it become all the more important and necessary to give a purely arithmetic treatment of the subject of integration which may be independent of all geometrical intuition and the notion of differentiation [3].

Riemann (1782-1867) was the first mathematician who give the satisfactory rigorous arithmetic treatment of integration and we shall follow the definition given by him. Riemann integration covers only bounded functions. Towards the end of the last century Thomas J. Stieltjes (1856-1894) introduce a broader concept of integration. Later on, early in the twentieth century (1902) Henry Lebesgue (1875-1941) developed the concept of measure of a set of real numbers and with it came a contribution of Lebesgue to the theory of integration [4].

The concept of functions of bounded variation has been well-known since Jordan gave the complete characterization of functions of bounded variation as a difference of two increasing functions in 1881. This class of functions immediately proved to be important in connection with the rectification of curves and with the Dirichlet's theorem on convergence of Fourier series. Functions of bounded variation exhibit so many interesting properties that make them a suitable class of functions in a variety of contexts with wide applications in pure and applied mathematics.

Functions of bounded variation are precisely those with respect to which one may find Riemann-stieltjes integrals of all continuous functions. Another characterization State that the function of bounded variation on a closed interval (as we shall prove letter) are exactly those f which can be written as a difference g-h, where g and h are bounded monotone.One of the most important aspects of functions of bounded variation is that they form algebra of discontinuous functions whose first derivatives exist almost everywhere. Due to this fact, they can and frequently are used to define generalized solutions of nonlinear problems involving functional, ordinary and partial differential equations in mathematics, physics and engineering [2].

BV functions were first introduced by Jordan (1881) in an attempt dealing with the convergence of Fourier series [1]. After him, several authors applied BV functions to study Fourier series in several variables, geometric measure theory, calculus of variations and mathematical physics. Caccipoli and Giorgi used them to define measure of non-smooth bodies of sets (cacciopoli set). Oleinik introduced her view of generalized solution of nonlinear partial differential equations as functions from the space BV in the paper [5], and was able to construct a generalized solution of bounded variation of a first order differential equation. Few years later Edward D Conway and Joel A. Smaller applied BV functions to study a single nonlinear hyperbolic partial differential equation of first order, providing that the solution of the Cauchy problem for such equations is a function of bounded variation.

Aizik developed extensively a calculus of BV functions in the paper [8]. He proved the chain rule for BV functions andhe jointly with his pupil Sergei explored extensively the properties of BV functions. His chain rule formula was later expanded by Ambrosio and Gianno. A real valued function f on the real line is said to be of bounded variation (BV function) on a chosen interval $[a,b] \subset \mathbf{R}$ if its total variation is finite. That is, $f \in BV(a,b) \Leftrightarrow V_b^a(f) < +\infty$ [6].

II. PRELIMINARY

Definition 1. *A partition p*: let $[a,b]$ be a closed bounded interval, by a partition p of $[a,b]$ we mean a finite subset $\{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that
$a = x_0, x_1, \dots, x_n = b$
Such a partition p is often denoted by, $p = \{a = x_0, x_1, \dots, x_n = b\}$

The points $x_0, x_1, \dots, x_n$ are called partitioning points of the partition p and the subsets $[x_{r-1}, x_r], r = 1,2,3, \dots n$ are called the sub-intervals (sometimes they are called cells) of the partition p. the length of the subintervals $[x_{r-1}, x_r]$ is $x_r - x_{r-1}$ and is denoted byδ_r.

The greatest of the length $\delta_1, \delta_2, \dots, \delta_n$ is defined as the norm of the partition *p* and is denoted by $\mu(p)$ or$\|p\|$. Thus $\mu(p) = Max\{\delta_1, \delta_2, \dots, \delta_n\}$ [4].

Definition 2. *Greatest lower bound (glb) or infimum*: given a subset S of real numbers, if there exists a finite real number m such that
$x \geq m \; for \; all \; x \in S$

Then, the set is said to be bounded below and the number *m* is called a lower bound of S [4].

Definition 3.*Least Upper Bound (lub) or suprimum*: Given a subset S of real numbers, if there exist a finite real number *M* such that
$x \geq M \; for \; all \; x \in S$

Then, the set S is said to be bounded above and the number M is called the upper bound of S. if no such number M can be found, we say that the set is not bounded above [4].

Definition 4.*Upper and Lower Riemann Sum*: Let M_r denote the $l.u.b.$ (suprimum) and m_r denote the $g.l.b.$ (infimum) of f in $[x_{r-1}, x_r], r = 1,2,3, \dots n$ and p be the partition of $[a,b]$, we form the sums

$$U(p,f) = \sum_{r=1}^{n} M_r \delta_r \quad and \quad L(p,f) = \sum_{r=1}^{n} m_r \delta_r$$

Then,$U(p,f)$ is called the upper Riemann sum of f on =$[a,b]$ corresponding to the partition p, and $L(p,f)$ is called the lower Riemann sum of f corresponding to the partition p.
Clearly, the $l.u.b.$ of the set U is $M(b-a)$ and the $g.l.b.$ of the set$L$ is $m(b-a)$. But since the set $U \; and \; L$ are both bounded U posses $g.l.b.$ and L posses $l.u.b.$

Definition 5. *Lower and upper Riemann Integral*: The least upper bound of the set $L = \{L(p)\}$ is called the least Riemann integral of f over $[a,b]$ and is denoted by

$$\int_{\underline{a}}^{b} f(x)\, dx, \qquad i.e. \; \int_{\underline{a}}^{b} f(x)\, dx = lub\{L(p)\}$$

Where lub is taken over all partitions of $[a,b]$.

The greatest lower bound of the set $U = \{U(p)\}$ is called the upper Riemann integral of f over $[a, b]$ and is denoted by

$$\int_a^{\overline{b}} f(x)dx, \quad i.e. \int_a^{\overline{b}} f(x)\,dx = glb\{U(p)\}$$

Where glb is taken over all partitions p of $[a, b]$ [4].

Definition 6. Let the lower Riemann integral be $\int_{\underline{a}}^{b} f$ and the upper Riemann integral be $\int_a^{\overline{b}} f$, then if $\int_{\underline{a}}^{b} f = \int_a^{\overline{b}} f$, that is the lower and upper Riemann integral are equal, then we say that the Riemann integral of f exist or that f is Riemann integrable (or simply R – integrable). The common value of these integrals is called the Riemann integral [4].

Definition 7. *Variation of a function f:* Let $f : [a, b] \to R$ be a function and let $[c,d]$ be any closed interval of $[a, b]$ if the set S = $\{|f(x_i) - f(x_{i-1})|\}$ such that $\{x_i: 1 \le i \le n\}$ *is a partition of* $[c, d]$, is bounded then, the variation of f on $[c, d]$ is defined to be $V(f,[c, d])$ = sup S. If S is unbounded then, the variation of f is said to be ∞. A function f is of bounded variation on $[c, d]$, if $V(f,[c, d])$ is finite [7]

Definition 8. Let $u : [a, b] \to \mathbb{R}$ be a function. For each partition $\pi : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of the interval $[a, b]$, we define

$$V(u; [a, b]) = \sup_{\pi} \sum_{i=1}^{n} |u(t_i) - u(t_{i-1})|,$$

Where the suprimum is taken over all partitions π of the interval $[a, b]$. If $V(u; [a, b]) < \infty$, we say that u has bounded variation. Denoted by $BV[a, b]$ the collection of all functions of bounded variation on $[a, b]$ [7].

Properties *BV[a,b]*: The following are some well – known properties of the space of functions of $BV[a, b]$.

(1) If the function u is monotone, then $V(u; [a, b]) = |u(b) - u(a)|$.

(2) If $u \in BV[a, b]$, then u is bounded on $[a, b]$.

(3) A function u is of bounded variation of an interval $[a, b]$ if and only if it can be decomposed as a difference of increasing functions.

(4) Every function of bounded variation has left- and right- hand limits at each point of its domain.

(5) $BV[a, b]$ is Banach space endowed with the norm

$$\|u\|_{BV} = |u(a)| + V(u; [a, b]), \qquad u \in BV[a, b].$$

III. BASIC THEORIES

Here we shall obtain the concept of Riemann integral from bounded variation function. Let $x \in [a, b]$ and $\omega \in BV[a, b]$, let p_n be any partition of $[a, b]$ and denote by $\eta(p_n)$ the length of the largest interval $[x_{i-1}, x_i]$ that is

$$\eta(p_n) = \max(t_1 - t_0, \dots, t_n - t_{n-1}).$$

For every partition p_n of $=[a, b]$ we consider the sum $S(p_n) = \sum_{i=1}^{n} x(t_i)[\omega(t_i) - \omega(t_{i-1})]$. There exist a number $\mathcal{J}$ with the property that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\eta(p_n) < 0$ implies $=|\mathcal{J} - S(p_n)| < \varepsilon$. $\mathcal{J}$ is called the Riemann Stieltjes integral of x *over* $[a, b]$ with respect to ω. Hence as a limit of the sums $S(p_n)$ for the sequence (p_n) of partition of $[a, b]$ satisfying $\eta(p_n) \to 0$ *as* $n \to \infty$ we get

$$\int_a^b x(t)\,dw(t).$$

Now, if for $w(t) = t$, the integral above is the familiar Riemann integral of x *over* $[a, b]$.

Remark 1. From the above by deductive reasoning, we assert that if f is a continuous function of bounded variation, then its Riemann integral necessarily exist. $i.e. \int_{\underline{a}}^{b} f = \int_a^{\overline{b}} f$.

Theorem 2. If f is continuous on $[a, b]$, then $f \in R[a, b]$ [4].

Proof: Let $p \equiv \{a = x_0, x_1, \dots, x_{r-1}, x_r, \dots, x_n = b\}$ be any partition of $[a,b]$ with the norm $\mu(p) < \delta. and\ let\ x_r - x_{r-1} = \delta_r,\ r = 1,2,\dots,n$. Since f is continuous on the closed interval $[a,b]$, therefore it is uniformly continuous on that interval. Hence (by definition) to every $\varepsilon > 0$, there exist $\delta > 0$ such that

$$|f(x') - f(x'')| < \varepsilon \qquad (1)$$

For all points $x', x'' \in [a,b]\ whenever\ |x' - x''| < \delta$. Again since f is continuous on $[a,b]$ it is bounded on $[a,b]$ and attains its infimum and suprimum on $[a,b]$ and so on every closed sub-intervals of $[a,b]$, let f attains its infimum m_r and suprimum M_r at some points $\alpha_r\ and\ \beta_r$ respectively of $|x_{r-1}, x_r|$. That is $f(\alpha_r) = m_r\ and\ f(\beta_r) = M_r$. Now $\alpha_r, \beta_r \in |x_{r-1}, x_r|$ where $x_r - x_{r-1} < \delta, therefore\ |\alpha_r - \beta_r| < \delta$. Hence it follows from (1) that $|f(\alpha_r) - f(\beta_r)| < \varepsilon', that\ is |m_r - M_r| < \varepsilon' m_r - M_r < \varepsilon'$. Now
$U(p) - L(p) = \sum_{r=1}^{n} M_r\delta_r - \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n}(M_r - m_r)\delta_r < \varepsilon' \sum \delta_r = \varepsilon'(b-a)$. Since $\sum_{r=1}^{n}\delta_r = (b-a), let\ \varepsilon' = \frac{\varepsilon}{b-a},\ we\ get$
$U(p) - L(p) < \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$. Hence f is R – integrable on $[a,b]$. Notice that the converse is not necessarily true.

Theorem 3. If f is monotone on $[a,b]$, then $f \in R[a,b]$ [4].
Proof: let us suppose that f is monotonically increasing on $[a,b]$. If $f(a) = f(b)$, then f would be a constant function, then f is R – integrable, therefore we assume that $f(a) \neq f(b)$.
Let p be a partition of $[a,b]$ and let $x_r - x_{r-1} = \delta_r,\ r = 1,2,3,\dots,n$. Let the norm of the partition $\mu(p_n) < \delta$. Consider the sub intervals $[x_{r-1}, x_r]$, let m_r, M_r be the inf. and sup. Of f respectively on $[x_{r-1}, x_r]$. Since f is monotonically increasing on $[x_{r-1}, x_r]$, therefore
$f(x_{r-1}) = m_r\ and\ f(x_r) = M_r$, Hence,

$$\begin{aligned} U(p) - L(p) &= \sum M_r\delta_r - \sum m_r\delta_r = \sum (M_r - m_r)\delta_r \\ &= \sum\{f(x_r) - f(x_{r-1})\}\delta_r \\ &< \delta \sum_{r=1}^{n}\{f(x_r) - f(x_{r-1})\} \\ &= \delta|\{f(x_1) - f(a)\} + \{fx_2) - f(x_1)\} + \cdots + \{f(b) - f(x_{n-1})\}| \\ &= \delta|f(b) - f(a)| \end{aligned}$$

Now, taking $\delta \le \frac{\varepsilon}{f(b)-f(a)}$ we get $U(p) - L(p) < \frac{\varepsilon}{f(b)-f(a)} \cdot f(b) - f(a) = \varepsilon$. It follows therefore that $f \in R[a,b]$. In a similar manner if f is monotonically decreasing on $R[a,b]$ it can be shown that $f \in R[a,b]$.

Theorem 4. A function of bounded variation is expressible as a difference of two monotonic increasing functions [4].
Proof: In other words, we need to show that if f is a function of bounded variation on [*a*, *b*], then there exist monotonic increasing functions *g* and *h* on [*a*, *b*] such that for $a \le x \le b$,

$$f(x) = g(x) - h(x) \qquad (1)$$

$$V_f(x) = g(x) + h(x) \qquad (2)$$

Proof: let us define g and h by $g = (V_f + f),\ h = (V_f - f)$ so that (1) and (2) holds.Now for $x_1 < x_2$, we have

$$\begin{aligned} g(x_2) - g(x_1) &= \tfrac{1}{2}[V_f(x_2) + f(x_2)] - \tfrac{1}{2}[V_f(x_1) + f(x_1)] \\ &= \tfrac{1}{2}[V_f(x_2) - V_f(x_1)] + \tfrac{1}{2}[f(x_2) - f(x_1)] \\ &= \tfrac{1}{2}[V(f, x_1, x_2) + \{f(x_2) - f(x_1)\}] \qquad (3) \end{aligned}$$

But since $V(f, x_1, x_2) \ge |f(x_2) - f(x_1)|$, therefore from (3), we have $g(x_2) - g(x_1) \ge |f(x_2) - f(x_1)| \ge 0$, which implies $g(x_2) \ge g(x_1)$Therefore g is monotonic increasing on [*a*, *b*]. Similarly, we have $h(x_2) - h(x_1) = \frac{1}{2}[V_f(x_2) + f(x_2)] - \frac{1}{2}[V_f(x_1) + f(x_1) = \frac{1}{2}[V(f, x_1, x_2) + \{f(x_2) - f(x_1)\}] \ge 0$, which implies $h(x_2) \ge h(x_1)$Therefore h is monotonic increasing on [*a*,*b*].
Hence f is expressible as a difference *g*-*h* of two monotonic increasing functions *g*, *h on* [*a*, *b*].

Theorem 5. If f is a continuous function of bounded variation on $[a,b]$, then its variation function is continuous on $[a,b]$.Proof: (see [4])

Theorem 6. Ifthe variation function of f is continuous on $[a, b]$, then f is a continuous function of bounded variation on $[a, b]$.

Proof: Let the variation function V_f of f be continuous at $c \in [a, b]$. Let $\varepsilon > 0$ be given, then there exist $\delta > 0$ such that

$|V_f(x) - V_f(c)| < \varepsilon, when\ |x - c| < \delta$

Suppose $x < c$, then$|f(x) - f(c)| \leq V_f(c) - V_f(x)$. And if $x > c$, then$|f(x) - f(c)| \leq V_f(x) - V_f(c)$. Hence, we have $|f(x) - f(c)| < \varepsilon\ whenever\ |x - c| < \delta$ and f is continuous at c.

IV. RESULT

In this section we present our result in the two remark below in form of deductive reasoning.

Remark 1. From theorem 4, 5 and 6 we deduced that "a function f is of bounded variation if and only if it can be expressed as a difference of two continuous monotonic increasing functions".

Proof: By theorem 4, if f is a function of bounded variation on $[a, b]$, then there exist monotone increasing functions $g\ and\ h$ on $[a, b]$ such that for $a \leq x \leq b$

$f(x) = g(x) - h(x), \qquad V_f(x) = g(x) + h(x).$

Where *g and h* are defined by $g = \frac{1}{2}(V_f + f),\ h = \frac{1}{2}(V_f - f).$

Now if f is continuous function of bounded variation then$V_f(x)$ is continuous by theorem 5 and consequently $g\ and\ h$ are continuous monotonic increasing functions. Conversely, if a continuous function $f\ on\ [a, b]$ can be expressed as a difference of two continuous monotonic increasing functions $g\ and\ h\ on\ [a, b]$ say $f = g - h$, then from page 421 of [4] $g\ and\ h$ are functions of bounded variation on $[a, b]$ and so is their difference $g - h$, that is f is of bounded variation.

Remark 2. Noticed that from theorem 2 we see that if f is continuous on $[a, b]$, then f is *R* – integrable on $[a, b]$. Hence, taking from the above remark, the following statement has sense; "an *R* – integrable function f is also of bounded variation if and only if it can be expressed as a difference of two monotonic increasing functions".

V. CONCLUSION

Functions of bounded variation and Riemann integrable functions have many interwoven concepts. Such properties as continuity and been monotonic are some example, however our study is concerned mainly on the relationship between the two notions. That is, an *R* – integrable function f is also of bounded variation if and only if it can be expressed as a difference of two monotonic increasing functions. Though more effort from any interesting reader can also yield more result.

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