International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 13 Issue 1 || Jan.-Feb. 2025 || PP: 23-32

The Raviart-Thomas Mixed Finite Element Method for Asymmetric Second-Order Elliptic Eigenvalue Problems

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ABSTRACT*:This paper studies the lowest-order Raviart-Thomas mixed finite element method for a more general class of second-order elliptic eigenvalue problems, and through analysis, obtains an a priori error estimate, which is optimal with respect to the mesh size*h*.First, based on the uniqueness of the solution to the corresponding steady-state problem, we define the fully continuous operator and derive an abstract error estimate. Then, on this basis, we obtain error estimates for the eigenvalues and eigenfunctions. Finally, we conduct corresponding numerical experiments, and the results are consistent with the theory.*
KEYWORDS*: Second-order elliptic eigenvalue,Raviart-Thomas mixed finite element method, a priori error.*

Date of Submission: 27-02-2025 Date of acceptance: 16-03-2025

I. INTRODUCTION

The second-order elliptic eigenvalue problem is one of the core issues in mathematical physics and engineering sciences, with widespread applications in fluid dynamics, electromagnetics, solid mechanics, quantum mechanics, and multiphysics coupling, among other fields.Reference[1]a)[1]discusses the mixed finite element formulation of the second-order elliptic equation and its solution estimates. Reference [1]a)[2]discusses the posterior error estimation and adaptive algorithms for the Stokes eigenvalue problem. Reference [1]a)[3]discusses the application of Richardson extrapolation in second-order elliptic eigenvalue problems. Reference [1]a)[4]uses the local discontinuous Galerkin method to solve the Steklov eigenvalue problem.The main feature of the Raviart-Thomas mixed finite element method is that it can approximate both the primal and auxiliary variables simultaneously. It has advantages such as local mass conservation, ease of combination with other methods, and the ability to handle polygonal meshes.Therefore, the Raviart-Thomas mixed finite element method has been developed for many problems, such as those in [1]a)[5]. In addition, the Raviart-Thomas mixed finite element method has also been used to solve various eigenvalue problems, such as the Laplace eigenvalue problem [1]a)[6]. This paper studies the a priori error estimation for the asymmetric second-order elliptic eigenvalue problem.

II. THEORETICAL PREPARATION

$\Omega \subset R^2$be a bounded domain with Lipshitz boundary$\partial\Omega$. and let**n**be the outward normal to$\partial\Omega$, consider the Dirichlet boundary condition eigenvalue problem

$$\begin{cases} -div(\nabla u + \mathbf{b}(x)u) + c(x)u = \lambda u, & in\ \Omega \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{2.1}$$

where $\mathbf{b}(x)$and $c(x)$are bounded positive functions on Ω.

Define the vector-valued function $\boldsymbol{\sigma} = \nabla u + \mathbf{b}(x)u$,then problem (2.1) can be equivalently written as

$$\begin{cases} -div\boldsymbol{\sigma} + cu = \lambda u, & in\ \Omega \\ \boldsymbol{\sigma} - \nabla u - \mathbf{b}\cdot u = 0, & in\ \Omega \\ u = 0, & on\ \partial\Omega. \end{cases} \tag{2.2}$$

Next, define the spaces

$$\mathbf{V} = H(div,\Omega), W = L^2(\Omega), G = L^2(\Omega), \mathbf{H} = \left(L^2(\Omega)\right)^2,$$

then, the weak form for the problem (2.1) can be defined as follows: Find$(\lambda,\boldsymbol{\sigma},u) \in C \times \mathbf{V} \times W$, with$\|u\|_0 = 1$,such that

$$\begin{cases} a(\boldsymbol{\sigma},\boldsymbol{\psi}) - b(\boldsymbol{\psi},u) + d(\boldsymbol{\psi},u) = 0, \forall\boldsymbol{\psi} \in \mathbf{V} \\ b(\boldsymbol{\sigma},v) + e(u,v) = \lambda r(u,v), \forall v \in W, \end{cases} \tag{2.3}$$

where the bilinear forms$a(\cdot,\cdot), b(\cdot,\cdot), d(\cdot,\cdot), e(\cdot,\cdot)andr(\cdot,\cdot)$are defined by

$$a(\boldsymbol{\sigma},\boldsymbol{\psi}) = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\psi}\,dx,\ \ b(\boldsymbol{\psi},v) = -\int_\Omega div\boldsymbol{\psi}\cdot v\,dx,\ \ d(\boldsymbol{\sigma},v) = -\int_\Omega \mathbf{b}\boldsymbol{\sigma}\cdot v\,dx,$$
$$e(u,v) = \int_\Omega cuv\,dx, \quad r(u,v) = \int_\Omega uv\,dx.$$

The bilinear forms $a(\cdot,\cdot)$, $e(\cdot,\cdot)$ and $r(\cdot,\cdot)$ are symmetric, and the bilinearforms defined above have the followingcharacteristics:

$$|a(\boldsymbol{\psi},\boldsymbol{\psi})| \gtrsim \|\boldsymbol{\psi}\|_{\mathbf{H}}^2, \quad |e(u,u)| \geqslant 0, \quad |r(u,u)| \geqslant 0,$$
$$|a(\boldsymbol{\sigma},\boldsymbol{\psi})| \lesssim \|\boldsymbol{\sigma}\|_{\mathbf{H}}\|\boldsymbol{\psi}\|_{\mathbf{H}}, \quad |b(\boldsymbol{\psi},v)| \lesssim \|\boldsymbol{\psi}\|_{\mathbf{V}}\|v\|_W,$$
$$|d(\boldsymbol{\psi},u)| \lesssim \|\boldsymbol{\psi}\|_{\mathbf{V}}\|u\|_W, \quad |e(u,v)| \lesssim \|u\|_W\|v\|_W, \quad |r(u,v)| \lesssim \|u\|_W\|v\|_W. \tag{2.4}$$

For the eigenvalue λ, the Rayleigh quotient can be expressed as

$$\lambda = \frac{a(\boldsymbol{\sigma},\boldsymbol{\sigma}) + d(\boldsymbol{\sigma},u) + e(u,u)}{r(u,u)}. \tag{2.5}$$

From[1]a)[7], the sequence of eigenvalues corresponding to the eigenvalue problem (2.3) is given by

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \lim_{k\to\infty}\lambda_k = \infty,$$

and the associated eigenfunctions

$$(\boldsymbol{\sigma}_1,u_1),(\boldsymbol{\sigma}_2,u_2),\cdots,(\boldsymbol{\sigma}_k,u_k),\cdots.$$

Theorem 2.1. Let (λ,u) be an eigenpair of (2.1), then $(\lambda,\boldsymbol{\sigma},u)$ satisfies (2.3); if $(\lambda,\boldsymbol{\sigma},u)$ satisfies (2.3), then (λ,u) is an eigenpair of (2.1), and $\boldsymbol{\sigma} = \nabla u$.

Proof. From the above derivation, it is known that the first half of the theorem holds, so it is sufficient to prove the second half of the theorem. Let$(\lambda,\boldsymbol{\sigma},u)$ satisfy (2.3), and consider the auxiliary problem

$$\begin{cases} -div(\nabla\bar{u} + \mathbf{b}(x)\bar{u}) + c(x)\bar{u} = \lambda u, & in\ \Omega \\ \bar{u} = 0, & on\ \partial\Omega, \end{cases} \tag{2.6}$$

let $\bar{\boldsymbol{\sigma}} = \nabla\bar{u} + \mathbf{b}(x)\bar{u}$, then the mixed variational form of (2.6) is: Find $(\lambda,\bar{\boldsymbol{\sigma}},\bar{u})$, such that

$$\begin{cases} a(\bar{\boldsymbol{\sigma}},\boldsymbol{\psi}) - b(\boldsymbol{\psi},\bar{u}) + d(\boldsymbol{\psi},\bar{u}) = 0, \forall\boldsymbol{\psi} \in \mathbf{V} \\ b(\bar{\boldsymbol{\sigma}},v) + e(\bar{u},v) = \lambda r(u,v), \forall v \in W. \end{cases} \tag{2.7}$$

Subtracting equation (2.7) from equation (2.3), we get: Find $(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}, u - \bar{u})$, such that

$$\begin{cases} a(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}},\boldsymbol{\psi}) - b(\boldsymbol{\psi},u - \bar{u}) + d(\boldsymbol{\psi},u - \bar{u}) = 0, & \forall\boldsymbol{\psi} \in \mathbf{V} \\ b(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}},v) + e(u - \bar{u},v) = 0. & \forall v \in W. \end{cases} \tag{2.8}$$

Taking $\boldsymbol{\psi} = \boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}$, v=$u - \bar{u}$ in equation (2.8), we get

$$\begin{cases} a(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}},\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}) - b(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}},u - \bar{u}) + d(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}},u - \bar{u}) = 0, \\ b(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}},u - \bar{u}) + e(u - \bar{u},u - \bar{u}) = 0, \end{cases} \tag{2.9}$$

adding all terms in equation (2.9) results in

$$a(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}},\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}) + d(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}},u - \bar{u}) + e(u - \bar{u},u - \bar{u}) = 0. \tag{2.10}$$

Since$\mathbf{b}, c$ is a bounded positive function on Ω, from equation (2.10), we can conclude that$\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}}, u = \bar{u}$, and $\boldsymbol{\sigma} = \nabla u$ is bounded.

The global stability result for problem (3.3) can be obtained from [1]a)[8]as follows.

Lemma2.1. For all $(\boldsymbol{\psi},v) \in \mathbf{V} \times W$, the following inf-sup condition holds

$$\sup_{0\neq(\boldsymbol{\sigma},u)\in\mathbf{V}\times W} \frac{a(\boldsymbol{\sigma},\boldsymbol{\psi}) - b(\boldsymbol{\psi},u) + d(\boldsymbol{\psi},u) + b(\boldsymbol{\sigma},v) + e(u,v)}{\|\boldsymbol{\sigma}\|_{\mathbf{V}} + \|u\|_W} \gtrsim \|\boldsymbol{\psi}\|_{\mathbf{V}} + \|v\|_W.$$

In this section, weexplore approximation methods for the eigenvalue problem (2.3) in mixed finite element method. To define the discrete approximation solution,we first define a shape regular mesh for the domain Ωdenoted by$\mathcal{T}_h = \{\kappa\}$, where each element κ has edge length h_E and the diameter h_κ. The mesh size is defined as $h = \max_{\kappa\in\mathcal{T}_h} h_\kappa$.This triangulation $\mathcal{T}_h$ satisfies the following conditions:

(i) Any two triangles share at most one edge or one vertex;
(ii) All triangles have a positive lower bound on their lowest interior angle;
(iii) There exists a constant γ^* such that for any element$\kappa \in \mathcal{T}_h$, the following holds

$$\frac{h_\kappa}{\rho_\kappa} \leqslant \gamma^*, \forall\kappa \in \bigcup_h \mathcal{T}_h,$$

where ρ_κ denotes the diameter of the largest inscribed circle of elementκ;
(iv) For any element $\kappa \in \mathcal{T}_h$, let the area of element κ be $|\kappa|$, we have

$$C_3h^2 \leqslant C_1h_\kappa^2 \leqslant |\kappa| \leqslant C_2h_\kappa^2 \leqslant C_4h^2,$$

where $C_i(i = 1,2,3,4)$are constants independent of the mesh size h, and h is a positive real number approaching zero.

Additionally, the boundary $\Gamma_h = \Gamma_h^0 \cup \Gamma_h^\partial$is divided into two parts: Γ_h^0represents the interior edges, and Γ_h^∂represents the edges on the boundary $\partial\Omega$.

Associated with the partition $\mathcal{T}_h$, we define the finite-dimensional spaces $\mathbf{V}_h$and $W_h$of the lowest order Raviart-Thomas mixed finite element spaces (see[1]a)[8]), where $P_m(\kappa)$denotes the spaces of a polynomial of degree $\leq m$ on κ.

Define

$$\mathbf{V}_h = \{\boldsymbol{\psi} \in \mathbf{V}: \boldsymbol{\psi}|_\kappa \in P_0(\kappa)^2 \oplus (x_1,x_2)^T P_0(\kappa), \forall\kappa \in \mathcal{T}_h\},$$

which clearly implies $\mathbf{V}_h \subset \mathbf{V}$.

Afterward, define

$$W_h = \{v \in W: v|_\kappa \in P_0(\kappa), \forall \kappa \in \mathcal{T}_h\},$$

likewise, we have$W_h \subset W$, and the connection $div\mathbf{V}_h = W_h$is valid.

With the discrete spaces defined above,we are in position to introduce the discretization of problem(2.3): Find$(\lambda_h, \boldsymbol{\sigma}_h, u_h) \in C \times \mathbf{V}_h \times W_h$,with $\|u_h\|_0 = 1$, such that

$$\begin{cases} a(\boldsymbol{\sigma}_h, \boldsymbol{\psi}) - b(\boldsymbol{\psi}, u_h) + d(\boldsymbol{\psi}, u_h) = 0, \forall \boldsymbol{\psi} \in \mathbf{V}_h \\ b(\boldsymbol{\sigma}_h, v) + e(u_h, v) = \lambda_h r(u_h, v), \forall v \in W_h. \end{cases} \tag{2.11}$$

From (2.11), the following Rayleigh quotient expression for $\lambda_h$also holds

$$\lambda_h = \frac{a(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) + d(\boldsymbol{\sigma}_h, u_h) + e(u_h, u_h)}{r(u_h, u_h)}. \tag{2.12}$$

And from [1]a)[7]the eigenvalue problem (2.11) has eigenvalues

$$0 \leqslant \lambda_{1,h} \leqslant \lambda_{2,h} \leqslant \cdots \leqslant \lambda_{k,h} \leqslant \cdots \leqslant \lambda_{N,h},$$

and the corresponding eigenfunctions

$$(\boldsymbol{\sigma}_{1,h}, u_{1,h}), (\boldsymbol{\sigma}_{2,h}, u_{2,h}), \cdots, (\boldsymbol{\sigma}_{k,h}, u_{k,h}), \cdots, (\boldsymbol{\sigma}_{N,h}, u_{N,h}),$$

where N is the dimension of the mixed finite element space$\mathbf{V}_h \times W_h$.

Consider the source problem associated with the general second-order elliptic eigenvalue problem (2.3) and its discrete mixed finite element formulation:
Find $(\mathbf{w}, \varphi) \in \mathbf{V} \times W$, such that

$$\begin{cases} a(\mathbf{w}, \boldsymbol{\psi}) - b(\boldsymbol{\psi}, \varphi) + d(\boldsymbol{\psi}, \varphi) = 0, & \forall \boldsymbol{\psi} \in \mathbf{V} \\ b(\mathbf{w}, v) + e(\varphi, v) = r(f, v), & \forall v \in W. \end{cases} \tag{2.13}$$

Fin$(\mathbf{w}_h, \varphi_h) \in \mathbf{V}_h \times W_h$d, such that

$$\begin{cases} a(\mathbf{w}_h, \boldsymbol{\psi}) - b(\boldsymbol{\psi}, \varphi_h) + d(\boldsymbol{\psi}, \varphi_h) = 0, & \forall \boldsymbol{\psi} \in \mathbf{V}_h \\ b(\mathbf{w}_h, v) + e(\varphi_h, v) = r(f, v), & \forall v \in W_h. \end{cases} \tag{2.14}$$

Problem (2.13) has a unique solution $(\mathbf{w}, \varphi) \in \mathbf{V} \times W$, and satisfies the following error estimate (see [1]a)[1])

$$\|\mathbf{w}\|_{\mathbf{V}} + \|\varphi\|_0 \lesssim \|f\|_0. \tag{2.15}$$

Assume that the mixed finite element spaces $\mathbf{V}_h$ and W_h satisfy the inf-sup condition, i.e.

$$\sup_{\boldsymbol{\psi}_h \in \mathbf{V}_h} \frac{b(\boldsymbol{\psi}_h, v_h)}{\|\boldsymbol{\psi}_h\|_{\mathbf{V}}} \gtrsim \|v_h\|_0, \forall v_h \in W_h. \tag{2.16}$$

Then, problem (2.14) also has a unique solution $(\mathbf{w}_h, \varphi_h) \in \mathbf{V}_h \times W_h$, and satisfies the following error estimate

$$\|\mathbf{w} - \mathbf{w}_h\|_{\mathbf{V}} + \|\varphi - \varphi_h\|_0 \lesssim \inf_{\boldsymbol{\psi} \in \mathbf{V}_h} \|\mathbf{w} - \boldsymbol{\psi}\|_{\mathbf{V}} + \inf_{v \in W_h} \|\varphi - v\|_0 \tag{2.17}$$

We assume that the following regularity estimate conditions hold: For any $f \in L^2(\Omega)$, $(\mathbf{w}, \varphi) \in H^\mu(\Omega)^2 \times H^{1+\mu}(\Omega)$, the following inequality holds

$$\|\mathbf{w}\|_\mu + \|\varphi\|_{1+\mu} \lesssim \|f\|_0, \tag{2.18}$$

where$\mu = \pi/\omega - \varepsilon$, $\omega < 2\pi$being the maximum interior angle of Ω.

Next, define the linear bounded operators $T$and$\mathbf{S}$, as well as their discrete versionsT_h and $\mathbf{S}_h$:

$$\begin{aligned} &T: G \to W \subset G, && Tg = u, \\ &T_h: G \to W_h \subset G, && T_h g = u, \\ &\mathbf{S}: G \to \mathbf{V} \subset G, && \mathbf{S}g = \boldsymbol{\sigma}, \\ &\mathbf{S}_h: G \to \mathbf{V}_h \subset G, && \mathbf{S}_h g = \boldsymbol{\sigma}_h. \end{aligned} \tag{2.19}$$

As a result, the operator form of the eigenvalue problems (2.3) and (2.11) can be equivalently converted as

$$\lambda T u = u, \mathbf{S}(\lambda u) = \boldsymbol{\sigma}, \tag{2.20}$$

$$\lambda_h T_h u_h = u_h, \mathbf{S}_h(\lambda_h u_h) = \boldsymbol{\sigma}_h. \tag{2.21}$$

In (2.13) and (2.14), take $f = \lambda u$, then by the definitions of $T, \mathbf{S}$ and $T_{\square}, \mathbf{S}_{\square}$, it is easy to see that$T(\lambda u), S(\lambda u)$are the solutions of (2.13); $T_{\square}(\lambda u), \mathbf{S}_{\square}(\lambda u)$are the solutions of (2.14).

Thus, solving the eigenvalue problem (2.3) for the eigenpair $(\lambda, \boldsymbol{\sigma}, u)$ is equivalent to solving the eigenpair of the operator$T$for $(\lambda^{-1}, u)$and $\boldsymbol{\sigma}$=$\mathbf{S}(\lambda u)$. Similarly,solving the eigenvalue problem (2.11) for the eigenpair $(\lambda_h, \boldsymbol{\sigma}_h, u_h)$is equivalent to solving for the eigenpair of the operator$T_h$for $(\lambda_h^{-1}, u_h)$and $\boldsymbol{\sigma}_h$=$\mathbf{S}_h(\lambda_h u_h)$.

Lemma 2.2. Both T and T_h are self-adjoint operators.

Proof. For $\forall f \in L^2(\Omega)$, (2.13) can be written in operator form as

$$\begin{cases} a(\mathbf{S}f, \boldsymbol{\psi}) - b(\boldsymbol{\psi}, Tf) + d(\boldsymbol{\psi}, Tf) = 0, & \forall \boldsymbol{\psi} \in \mathbf{V} \\ b(\mathbf{S}f, v) + e(Tf, v) = r(f, v), & \forall v \in W. \end{cases} \tag{2.22}$$

For $\forall g \in L^2(\Omega)$, we similarly have.

$$\begin{cases} a(\mathbf{S}g, \boldsymbol{\psi}) - b(\boldsymbol{\psi}, Tg) + d(\boldsymbol{\psi}, Tg) = 0, & \forall \boldsymbol{\psi} \in \mathbf{V} \\ b(\mathbf{S}g, v) + e(Tg, v) = r(g, v), & \forall v \in W. \end{cases} \tag{2.23}$$

Taking $\boldsymbol{\psi} = \mathbf{S}g, v = Tg$ in (2.22), we get

$$\begin{cases} a(\mathbf{S}f, \mathbf{S}g) - b(\mathbf{S}g, Tf) + d(\mathbf{S}g, Tf) = 0, \\ b(\mathbf{S}f, Tg) + e(Tf, Tg) = r(f, Tg). \end{cases} \tag{2.24}$$

Taking $\boldsymbol{\psi} = \mathbf{S}f, v = Tf$ in (2.23), we get

$$\begin{cases} a(\mathbf{S}g, \mathbf{S}f) - b(\mathbf{S}f, Tg) + d(\mathbf{S}f, Tg) = 0, \\ b(\mathbf{S}g, Tf) + e(Tg, Tf) = r(g, Tf). \end{cases} \tag{2.25}$$

From (2.24), (2.25), and the symmetry of $a(\cdot,\cdot)$,$e(\cdot,\cdot)$, we have

$$\begin{aligned} r(f, Tg) &= b(\mathbf{S}f, Tg) + e(Tf, Tg) \\ &= a(\mathbf{S}g, \mathbf{S}f) + d(\mathbf{S}f, Tg) + e(Tf, Tg) \\ &= a(\mathbf{S}f, \mathbf{S}g) + d(\mathbf{S}g, Tf) + e(Tg, Tf) \\ &= b(\mathbf{S}g, Tf) + e(Tg, Tf) \\ &= r(g, Tf). \end{aligned} \tag{2.26}$$

Thus, T is self-adjoint. The self-adjointness for$T_h$can be proven in a similar way.

Theorem 2.2. For the operators T,T_h defined above, when$h \to 0$, we have $\|T - T_h\|_0 \to 0$.

Proof. It has been proved in Reference [1]a)[1] that

$$\|u - u_h\|_0 \lesssim h^\alpha \|u\|_{\alpha+1},$$

where $\alpha = min\{1, \mu\}$, $\alpha = 1$ if Ω is convex and $\alpha = \mu$ for a nonconvex Ω, due to regularity result (2.18).
From the regularity estimate (2.18), since$\|u\|_{\alpha+1} \lesssim \|f\|_0$, we can obtain that

$$\|Tf - T_h f\|_0 \lesssim h^\alpha \|f\|_0.$$

Thus, when $h \to 0$, we get$\|T - T_h\|_0 \to 0$.

III. A PRIORI ERROR ESTIMATION

Assume that λ and λ_h are the k-th eigenvalues of (2.20) and (2.21), respectively; in this paper, we will also consider$\frac{1}{\lambda}$ as the k-th eigenvalue of T. The algebraic multiplicity of this eigenvalue is q, i.e. $\lambda = \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+q-1}$. Let$M(\lambda) = \{(\boldsymbol{\sigma}, u) \in \mathbf{V} \times W : \|u\|_0 = 1\}$ be the generalized eigenvector space of equation (2.3) related to λ, and $M_{□}(\lambda) = \{(\boldsymbol{\sigma}_{□}, u_{□}) \in \mathbf{V}_{□} \times W_{□} : \|u_{□}\|_0 = 1\}$ be the direct sum of the generalized eigenvector spaces of equation (2.11) related to λ_h. $\lambda_h$converges toλ.

Let $(\lambda_h, \boldsymbol{\sigma}_h, u_h)$ be the$k$-th eigenpair of (2.11), and $(\lambda, \boldsymbol{\sigma}, u)$ be the k-th eigenpair of (2.3), satisfying that$(\lambda_h, \boldsymbol{\sigma}_h, u_h)$ approximates $(\lambda, \boldsymbol{\sigma}, u)$. Then the following relationships hold.

Lemma 3.1. Let the multiplicity of λ be m, then the following estimate holds

$$\begin{aligned} |\lambda - \lambda_{\ell,h}| \lesssim{} & \|(\mathbf{S} - \mathbf{S}_h)|_{M(\lambda)}\|_0 \cdot \|(\mathbf{S}^* - \mathbf{S}_h^*)|_{M(\lambda^*)}\|_0 + \|(\mathbf{S}^* - \mathbf{S}_h^*)|_{M(\lambda^*)}\|_{\mathbf{V}} \cdot \|(T - T_h)|_{M(\lambda)}\|_0 \\ & + \|(\mathbf{S} - \mathbf{S}_h)|_{M(\lambda)}\|_{\mathbf{V}} \cdot \|(T^* - T_h^*)|_{M(\lambda^*)}\|_0 \\ & + \|(T - T_h)|_{M(\lambda)}\|_0 \cdot \|(T^* - T_h^*)|_{M(\lambda^*)}\|_0, \ell=1, 2, \cdots, m. \end{aligned} \tag{3.1}$$

Proof. Since the multiplicity of the eigenvalues and the dimension of the eigenspace of a self-adjoint operator are the same, let $\{\phi_i\}_1^m$ be a standard orthogonal basis of $M(\lambda)$, and $\{\phi_i^*\}_1^m$ be the dual basis. From Theorem 7.3 in [1]a)[7] and the steepness of the self-adjoint operator being $\alpha = 1$, it follows that

$$|\lambda^{-1} - \lambda_{\ell,h}^{-1}| \lesssim \left\{ \sum_{i,j=1}^{m} |r(T - T_h)\phi_i, \phi_j^*| + \|(T - T_h)|_{M(\lambda)}\|_0 \, \|(T^* - T_h^*)|_{M(\lambda^*)}\|_0 \right\}, \ell=1, 2, \cdots, m, \tag{3.2}$$

according to (3.2), for $\forall f, g \in L^2(\Omega)$, we only need to estimate $|r((T - T_h)f, g)|$.

Define the bilinear form

$$A((\mathbf{w}, \varphi), (\boldsymbol{\psi}, v)) = a(\mathbf{w}, \boldsymbol{\psi}) - b(\boldsymbol{\psi}, \varphi) + d(\boldsymbol{\psi}, \varphi) + b(\mathbf{w}, v) + e(\varphi, v),$$
$$A((\mathbf{w}_h, \varphi_h), (\boldsymbol{\psi}, v)) = a(\mathbf{w}_h, \boldsymbol{\psi}) - b(\boldsymbol{\psi}, \varphi_h) + d(\boldsymbol{\psi}, \varphi_h) + b(\mathbf{w}_h, v) + e(\varphi_h, v).$$

Then the source problems (2.13) and (2.14) can be rewritten as

$$A((\mathbf{w}, \varphi), (\boldsymbol{\psi}, v)) = r(f, v), \qquad \forall (\boldsymbol{\psi}, v) \in \mathbf{V} \times W \tag{3.3}$$
$$A((\mathbf{w}_h, \varphi_h), (\boldsymbol{\psi}, v)) = r(f, v), \qquad \forall (\boldsymbol{\psi}, v) \in \mathbf{V}_h \times W_h. \tag{3.4}$$

For $\forall f \in L^2(\Omega)$, write equations (3.3) and (3.4) in the operator forms

$$A((\mathbf{S}f, Tf), (\boldsymbol{\psi}, v)) = r(f, v), \qquad \forall (\boldsymbol{\psi}, v) \in \mathbf{V} \times W \tag{3.5}$$
$$A((\mathbf{S}_h f, T_h f), (\boldsymbol{\psi}, v)) = r(f, v), \qquad \forall (\boldsymbol{\psi}, v) \in \mathbf{V}_h \times W_h, \tag{3.6}$$

writethe dual problems of (3.5) and (3.6) respectively, which are

$$A((\boldsymbol{\psi}, v), (\mathbf{S}^* g, T^* g)) = r(v, g), \qquad \forall (\boldsymbol{\psi}, v) \in \mathbf{V} \times W \tag{3.7}$$
$$A((\boldsymbol{\psi}, v), (\mathbf{S}_h^* g, T_h^* g)) = r(v, g), \forall (\boldsymbol{\psi}, v) \in \mathbf{V}_h \times W_h. \tag{3.8}$$

Subtracting (3.5) and (3.6) gives

$$-A\left(((\mathbf{S} - \mathbf{S}_h)f, (T - T_h)f), (\boldsymbol{\psi}, v)\right) = 0. \tag{3.9}$$

In equation (3.7), take $\boldsymbol{\psi} = (\mathbf{S} - \mathbf{S}_h)f, v = (T - T_h)f$, we obtain

$$r((T - T_h)f, g) = A\left(((\mathbf{S} - \mathbf{S}_h)f, (T - T_h)f), (\mathbf{S}^* g, T^* g)\right). \tag{3.10}$$

Adding equation (3.9) and equation (3.10) gives

$$r\big((T-T_h)f,g\big)=A\Big(\big((\mathbf{S}-\mathbf{S}_h)f,(T-T_h)f\big),(\mathbf{S}^*g-\boldsymbol{\psi},T^*g-v)\Big),\tag{3.11}$$

in equation (3.11), take $\boldsymbol{\psi}=\mathbf{S}_h^*g, v=T_h^*g$, and obtain

$$r\big((T-T_h)f,g\big)=A\Big(\big((\mathbf{S}-\mathbf{S}_h)f,(T-T_h)f\big),\big((\mathbf{S}^*-\mathbf{S}_h^*)g,(T^*-T_h^*)g\big)\Big),\tag{3.12}$$

according to equation (2.4), for$\forall\boldsymbol{\psi}\in\mathbf{V}_h, v\in W_h$, we have

$$\begin{aligned}\big|r\big((T-T_h)f,g\big)\big|&=\Big|A\Big(\big((\mathbf{S}-\mathbf{S}_h)f,(T-T_h)f\big),\big((\mathbf{S}^*-\mathbf{S}_h^*)g,(T^*-T_h^*)g\big)\Big)\Big|\\&\le\big|a\big((\mathbf{S}-\mathbf{S}_h)f,(\mathbf{S}^*-\mathbf{S}_h^*)g\big)\big|+\big|b\big((\mathbf{S}^*-\mathbf{S}_h^*)g,(T-T_h)f\big)\big|+\big|d\big((\mathbf{S}^*-\mathbf{S}_h^*)g,(T-T_h)f\big)\big|\\&\quad+\big|b\big((\mathbf{S}-\mathbf{S}_h)f,(T^*-T_h^*)g\big)\big|+\big|e\big((T-T_h)f,(T^*-T_h^*)g\big)\big|\\&\lesssim\|(\mathbf{S}-\mathbf{S}_h)f\|_0\cdot\|(\mathbf{S}^*-\mathbf{S}_h^*)g\|_0+\|(\mathbf{S}^*-\mathbf{S}_h^*)g\|_{\mathbf{V}}\cdot\|(T-T_h)f\|_0\\&\quad+\|(\mathbf{S}-\mathbf{S}_h)f\|_{\mathbf{V}}\cdot\|(T^*-T_h^*)g\|_0+\|(T-T_h)f\|_0\cdot\|(T^*-T_h^*)g\|_0.\end{aligned}\tag{3.13}$$

In (3.13), substitute ϕ_i for f and ϕ_j^* for g,we get

$$\begin{aligned}\Big|r\Big((T-T_h)\phi_i,\phi_j^*\Big)\Big|&\lesssim\|(\mathbf{S}-\mathbf{S}_h)|_{M(\lambda)}\|_0\cdot\|(\mathbf{S}^*-\mathbf{S}_h^*)|_{M(\lambda^*)}\|_0\\&\quad+\|(\mathbf{S}^*-\mathbf{S}_h^*)|_{M(\lambda^*)}\|_{\mathbf{V}}\cdot\|(T-T_h)|_{M(\lambda)}\|_0\\&\quad+\|(\mathbf{S}-\mathbf{S}_h)|_{M(\lambda)}\|_{\mathbf{V}}\cdot\|(T^*-T_h^*)|_{M(\lambda^*)}\|_0\\&\quad+\|(T-T_h)|_{M(\lambda)}\|_0\cdot\|(T^*-T_h^*)|_{M(\lambda^*)}\|_0,\end{aligned}\tag{3.14}$$

substitute (3.14) into (3.2), and we can obtain (3.1).

Lemma 3.2. Let$(\lambda_h,\boldsymbol{\sigma}_h,u_h)$ be a mixed finite element eigenpair that satisfies (2.11), and $\|u_h\|_0=1$. Then, there exists an eigenpair $(\lambda,\boldsymbol{\sigma},u)$ of (2.3) such that the following holds

$$\begin{aligned}\|u-u_h\|_0&\lesssim\|(T-T_h)|_{M(\lambda)}\|_0,\\|\lambda-\lambda_h|&\lesssim\|(T-T_h)|_{M(\lambda)}\|_0.\end{aligned}\tag{3.15}$$

Proof. According to Theorem 7.3 and Theorem 7.4in [1]a)[7], the two estimation formulas of Lemma 3.2 hold.

Lemma 3.3. Let $(\lambda_h,\boldsymbol{\sigma}_h,u_h)$ be a discrete eigenpair that satisfies (2.11), then there exist eigenpairs $\lambda,u,\boldsymbol{\sigma}=\mathbf{S}(\lambda u)$ of (2.3) such that

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\lesssim\|(T-T_h)|_{M(\lambda)}\|_0+\|\mathbf{S}_h(\lambda u)-\mathbf{S}(\lambda u)\|_0.\tag{3.16}$$

Proof. Since

$$\|\mathbf{S}(\lambda u)-\mathbf{S}_h(\lambda_h u_h)\|_0\leqslant\|\mathbf{S}(\lambda u)-\mathbf{S}_h(\lambda u)\|_0+\|\mathbf{S}_h(\lambda u)-\mathbf{S}_h(\lambda_h u_h)\|_0,\tag{3.17}$$

therefore, it is only necessary to prove

$$\|\mathbf{S}_h(\lambda u)-\mathbf{S}_h(\lambda_h u_h)\|_0\lesssim\|(T-T_h)|_{M(\lambda)}\|_0.\tag{3.18}$$

Take $f=\lambda u$ in (2.14) and write it in the operator form

$$\begin{cases}a(\mathbf{S}_h(\lambda u),\boldsymbol{\psi})-b\big(\boldsymbol{\psi},T_h(\lambda u)\big)+d\big(\boldsymbol{\psi},T_h(\lambda u)\big)=0,\forall\boldsymbol{\psi}\in\mathbf{V}_h\\b(\mathbf{S}_h(\lambda u),v)+e(T_h(\lambda u),v)=r(\lambda u,v),\qquad\forall v\in W_h.\end{cases}\tag{3.19}$$

Write (2.11) in the operator form

$$\begin{cases}a(\mathbf{S}_h(\lambda_h u_h),\boldsymbol{\psi})-b\big(\boldsymbol{\psi},T_h(\lambda_h u_h)\big)+d\big(\boldsymbol{\psi},T_h(\lambda_h u_h)\big)=0,\forall\boldsymbol{\psi}\in\mathbf{V}_h\\b(\mathbf{S}_h(\lambda_h u_h),v)+e(T_h(\lambda_h u_h),v)=r(\lambda_h u_h,v),\forall v\in W_h.\end{cases}\tag{3.20}$$

Subtract (3.20) from (3.19) to obtain

$$\begin{cases}a(\mathbf{S}_h(\lambda_h u_h-\lambda u),\boldsymbol{\psi})-b\big(\boldsymbol{\psi},T_h(\lambda_h u_h-\lambda u)\big)+d\big(\boldsymbol{\psi},T_h(\lambda_h u_h-\lambda u)\big)=0,\forall\boldsymbol{\psi}\in\mathbf{V}_h\\b(\mathbf{S}_h(\lambda_h u_h-\lambda u),v)+e(T_h(\lambda_h u_h-\lambda u),v)=r(\lambda_h u_h-\lambda u,v),\qquad\forall v\in W_h.\end{cases}\tag{3.21}$$

Take $\boldsymbol{\psi}=\mathbf{S}_h(\lambda_h u_h-\lambda u), v=T_h(\lambda_h u_h-\lambda u)$ in (3.21) to get

$$\begin{cases}a\big(\mathbf{S}_h(\lambda_h u_h-\lambda u),\mathbf{S}_h(\lambda_h u_h-\lambda u)\big)-b\big(\mathbf{S}_h(\lambda_h u_h-\lambda u),T_h(\lambda_h u_h-\lambda u)\big)\\+d\big(\mathbf{S}_h(\lambda_h u_h-\lambda u),T_h(\lambda_h u_h-\lambda u)\big)=0,\\b\big(\mathbf{S}_h(\lambda_h u_h-\lambda u),T_h(\lambda_h u_h-\lambda u)\big)+e\big(T_h(\lambda_h u_h-\lambda u),T_h(\lambda_h u_h-\lambda u)\big)\\=r\big(\lambda_h u_h-\lambda u,T_h(\lambda_h u_h-\lambda u)\big),\end{cases}\tag{3.22}$$

add the two equations of (3.22) to obtain

$$\begin{aligned}&a\big(\mathbf{S}_h(\lambda_h u_h-\lambda u),\mathbf{S}_h(\lambda_h u_h-\lambda u)\big)+d\big(\mathbf{S}_h(\lambda_h u_h-\lambda u),T_h(\lambda_h u_h-\lambda u)\big)\\&+e\big(T_h(\lambda_h u_h-\lambda u),T_h(\lambda_h u_h-\lambda u)\big)=r\big(\lambda_h u_h-\lambda u,T_h(\lambda_h u_h-\lambda u)\big).\end{aligned}\tag{3.23}$$

From (3.23) and (2.4), weget

$$\begin{aligned}\|\mathbf{S}_h(\lambda_h u_h-\lambda u)\|_0^2&\lesssim a\big(\mathbf{S}_h(\lambda_h u_h-\lambda u),\mathbf{S}_h(\lambda_h u_h-\lambda u)\big)\\&\lesssim r\big(\lambda_h u_h-\lambda u,T_h(\lambda_h u_h-\lambda u)\big)\\&\lesssim\|\lambda_h u_h-\lambda u\|_0\cdot\|T_h(\lambda_h u_h-\lambda u)\|_0,\end{aligned}$$

since $\|T-T_h\|_0\to0\ (h\to0)$, we know thatT_h is uniformly bounded with respect to h. So from the above formula, we know

$$\|\mathbf{S}_h(\lambda_h u_h-\lambda u)\|_0^2\lesssim\|\lambda_h u_h-\lambda u\|_0^2.\tag{3.24}$$

Thus, it is deduced from Lemma 3.2 that

$$\|\mathbf{S}_h(\lambda u) - \mathbf{S}_h(\lambda_h u_h)\|_0 \lesssim \|(T - T_h)|_{M(\lambda)}\|_0,$$

the proof is completed.

We know that Lemma 3.3 transforms the error estimation of the mixed finite element eigenfunction $\boldsymbol{\sigma}_h$ into the error estimation of the mixed finite element solution of the corresponding steady - state problem. Next, we use Theorem 3.1 to discuss the error estimation of the Raviart - Thomas mixed finite element approximation for the eigenvalue problem.

Theorem 3.1. Let $(\boldsymbol{\sigma}_h, u_h) \in M_h(\lambda)$ be the direct sum of the generalized eigenspace of (2.11), then there exist eigenvalues $(\boldsymbol{\sigma}, u)$ of (2.3) such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{V}} + \|u - u_h\|_0 \lesssim h^{\alpha}. \tag{3.25}$$

Let$M(\lambda) \in H^{\alpha+2}(\Omega)$,$\|u_h\|_{\alpha+2} = 1$,then the following inequality holds

$$|\lambda - \lambda_h| \lesssim h^{2\alpha}, \tag{3.26}$$

where $\alpha = min\{1, \mu\}$, $\alpha = 1$ if Ω is convex, and$\alpha = \mu$ for a non-convexΩ, due to regularity result (2.18).

Proof. According to [1]a)[1]Theorem 4, we have

$$\|\mathbf{S}(\lambda u) - \boldsymbol{S}_h(\lambda u)\|_0 \lesssim h^{\alpha}\|T(\lambda u)\|_{\alpha+1}, \tag{3.27}$$

$$\left\|div\big(\mathbf{S}(\lambda u) - \boldsymbol{S}_h(\lambda u)\big)\right\|_0 \lesssim h^{\alpha}\|T(\lambda u)\|_{\alpha+2}, \tag{3.28}$$

$$\|T(\lambda u) - T_h(\lambda u)\|_0 \lesssim h^{\alpha}\|T(\lambda u)\|_{\alpha+1}. \tag{3.29}$$

From (3.27)-(3.29) and $M(\lambda) \in H^{\alpha+2}(\Omega)$, we get

$$\|(\mathbf{S} - \mathbf{S}_h)|_{M(\lambda)}\|_0 \lesssim h^{\alpha}, \tag{3.30}$$

$$\|div(\mathbf{S} - \mathbf{S}_h)|_{M(\lambda)}\|_0 \lesssim h^{\alpha}, \tag{3.31}$$

$$\|(T - T_h)|_{M(\lambda)}\|_0 \lesssim h^{\alpha}. \tag{3.32}$$

Substituting (3.27) and (3.32) into (3.16), we get

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{0}} \lesssim h^{\alpha}. \tag{3.33}$$

In the second equation of (3.21), taking $v = div\big(\mathbf{S}_h(\lambda_h u_h - \lambda u)\big)$, we get

$$\left\|div\big(\mathbf{S}_h(\lambda_h u_h - \lambda u)\big)\right\|_0 \lesssim \|\lambda u - \lambda_h u_h\|_0 \lesssim h^{\alpha}. \tag{3.34}$$

Noting that $\boldsymbol{\sigma} = \mathbf{S}(\lambda u), \boldsymbol{\sigma}_h = \mathbf{S}_h(\lambda_h u_h)$, by the triangle inequality and (3.28), (3.34), we obtain

$$\|div(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0 \lesssim h^{\alpha}. \tag{3.35}$$

From $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{V}} = (\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 + \|div(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0^2)^{\frac{1}{2}}$ and using Lemma 3.3, along with (3.32), (3.33), and (3.35), we obtain (3.25).

From[1]a)[1]Theorem 4, we deduce

$$\|(\mathbf{S}^* - \mathbf{S}_h^*)|_{M(\lambda^*)}\|_0 \lesssim h^{\alpha}, \tag{3.36}$$

$$\|div(\mathbf{S}^* - \mathbf{S}_h^*)|_{M(\lambda^*)}\|_0 \lesssim h^{\alpha}, \tag{3.37}$$

$$\|(T^* - T_h^*)|_{M(\lambda^*)}\|_0 \lesssim h^{\alpha}. \tag{3.38}$$

Combining Lemma 3.1 and equations (3.30)-(3.32), (3.36)-(3.38), we obtain equation (3.26).

IV. Numerical experiments

In this section, some numerical experiments will be presented to demonstrate the effectiveness of our method.Consider the problem (2.1), where$\mathbf{b} = (0,0)^T, (3,0)^T, \big(1 + (x - 1/2)^2, (x - 1/2)(y - 1/2)\big)^T$, and $c = 0,0, e^{(x-1/2)(y-1/2)}$. Under the iFEM software package, the effectiveness of solving the general (non-symmetric) second-order elliptic eigenvalue problem using the lowest-order Raviart-Thomas mixed finite element method is verified by compiling the code.

We consider the following three test domains: an L-shaped domain$\Omega_L = (-1,1)^2 \backslash ([0,1) \times (-1,0])$, a square domain $\Omega_S$with vertices at $(0,1), (1,0), (0,-1), (-1,0)$, and a slit structure domain$\Omega_{SL} = (-1,1)^2 \backslash \{0 \leqslant x \leqslant 1, y = 0\}$. Since the exact eigenvalues are unknown, we select nine sufficiently accurate approximate values as reference eigenvalues for the numerical tests. These reference eigenvalues are obtained through adaptive computation to achieve the highest possible accuracy. The numerical results obtained through calculations are presented in Tables 1, 2, and 3. From Tables 1, 2, and 3, we can observe that the algorithm achieves optimal convergence rates.

TABLE I. When $\mathbf{b} = (0,0)^T$and$c = 0$, the eigenvalue numerical solution results for regionsΩ_L , Ω_{SL}, Ω_S

Domain	ref	h	λ_1	dof	Error	rate
Ω_L	9.63972384402 19	1/4	9.25186639028311	256	0.387857453738789	1.39343561784591
		1/8	9.49208310037427	992	0.147640743647630	1.372061799459300
		1/16	9.58268450733096	3904	0.057039336690940	1.35768332876240
		1/32	9.61746661677905	15488	0.022257227242850	1.348597668329300

Domain	ref	h	λ_1	dof	Error	rate
		1/64	9.63098401956659	61696	0.008739824455310	1.34292091123768
		1/128	9.63627841558727	246272	0.003445428434629	

Domain	ref	h	λ_1	dof	Error	rate
Ω_{SL}	8.3713297112	1/4	7.7428300904913	340	0.628499620708699	0.951393084121251
		1/8	8.04631189799779	1320	0.325017813202210	0.976581406758179
		1/16	8.20616134712378	5200	0.165168364076219	0.988629380555353
		1/32	8.28809206912915	20640	0.083237642070850	0.994415340660239
		1/64	8.32954947186527	82240	0.041780239334729	0.997234871316316
		1/128	8.35039951427504	328320	0.020930196924960	

Domain	ref	h	λ_1	dof	Error	rate
Ω_S	9.86960440108935	1/4	10.0292586936151	88	0.15965429252575	1.93708128679546
		1/8	9.91129719781299	336	0.04169279672364	1.98484685192227
		1/16	9.88013765586817	1312	0.01053325477882	1.99627197125496
		1/32	9.87224452825732	5184	0.00264012716797	1.99907240446987
		1/64	9.87026485739199	20608	0.00066045630264	1.99976840636148
		1/128	9.86976954167265	82176	0.00016514058330	

TABLE II. When $\mathbf{b} = (3,0)^T$ and $c = 0$, the eigenvalue numerical solution results for regionsΩ_L , Ω_{SL}, Ω_S.

Domain	ref	h	λ_1	dof	Error	rate
Ω_L	11.8897238447 2	1/4	11.4571951982342	256	0.432528646485801	1.41827025408868
		1/8	11.7278886451348	992	0.161835199585200	1.40330923153320
		1/16	11.8285402744618	3904	0.061183570258201	1.38610366357691
		1/32	11.8663152126254	15488	0.023408632094601	1.37105064300429
		1/64	11.8806738428923	61696	0.009050001827701	1.35934405499700
		1/128	11.8861965208938	246272	0.003527323826201	

Domain	ref	h	λ_1	dof	Error	rate
Ω_{SL}	10.6213297112	1/4	9.99982641313843	340	0.621503298061569	0.946909894352465
		1/8	10.2989296198302	1320	0.322400091369799	0.971656731162318
		1/16	10.4569314065271	5200	0.164398304672899	0.985481971516401
		1/32	10.5382992020554	20640	0.083030509144599	0.992673128494214
		1/64	10.5796030809503	82240	0.041726630249700	0.996322338788220
		1/128	10.6004131442915	328320	0.020916566908500	

Domain	ref	h	λ_1	dof	Error	rate
Ω_S	12.1196044010894	1/4	12.01443059907640	88	0.10517380201300	1.93186963647970
		1/8	12.09203947289010	336	0.02756492819930	1.97456194677753
		1/16	12.11259058310530	1312	0.00701381798410	1.99308107917803
		1/32	12.11784251712590	5184	0.00176188396350	1.99823465700823
		1/64	12.11916339078960	20608	0.00044101029980	1.99955639128538
		1/128	12.11949411460810	82176	0.00011028648130	

TABLE III. When $\mathbf{b} = \left(1 + (x - 1/2)^2, (x - 1/2)(y - 1/2)\right)^T$ and $c = e^{(x-1/2)(y-1/2)}$, the eigenvalue numerical solution results for regionsΩ_L , Ω_{SL}, Ω_S.

Domain	ref	h	λ_1	dof	Error	rate
Ω_L	11.43251825487 38	1/4	10.0036912872557	256	1.428826967618100	3.26247596400255
		1/8	11.2836244894135	992	0.148893765460301	1.36103770249336
		1/16	11.3745535886398	3904	0.057964666234000	1.36756856379865

Domain	ref	h	λ_1	dof	Error	rate
		1/32	11.4100544053825	15488	0.022463849491301	1.39772618203114
		1/64	11.4239926210814	61696	0.008525633792400	1.49037895611129
		1/128	11.4294838194234	246272	0.003034435450401	

Domain	ref	h	λ_1	dof	Error	rate
Ω_{SL}	10.0036912872557	1/4	9.39392426719569	340	0.609767020060010	0.91657004681461
		1/8	9.6806568268324	1320	0.323034460423271	0.97044003336355
		1/16	9.8388305281997	5200	0.164860759055969	1.00994604254506
		1/32	9.9218272342546	20640	0.081864053001070	1.05615897299642
		1/64	9.9643219855985	82240	0.039369301657249	1.13944958789309
		1/128	9.9858202727748	328320	0.017871014480949	

Domain	ref	h	λ_1	dof	Error	rate
Ω_S	10.7481152407113	1/4	10.88564959754360	88	0.13753435683230	1.94755707626164
		1/8	10.78377169075460	336	0.03565645004330	1.95689157841548
		1/16	10.75729973057650	1312	0.00918448986520	1.86545821273055
		1/32	10.75063579561340	5184	0.00252055490210	1.56478785606960
		1/64	10.74896725735790	20608	0.00085201664660	0.97076966004875
		1/128	10.74854996839130	82176	0.00043472768000	

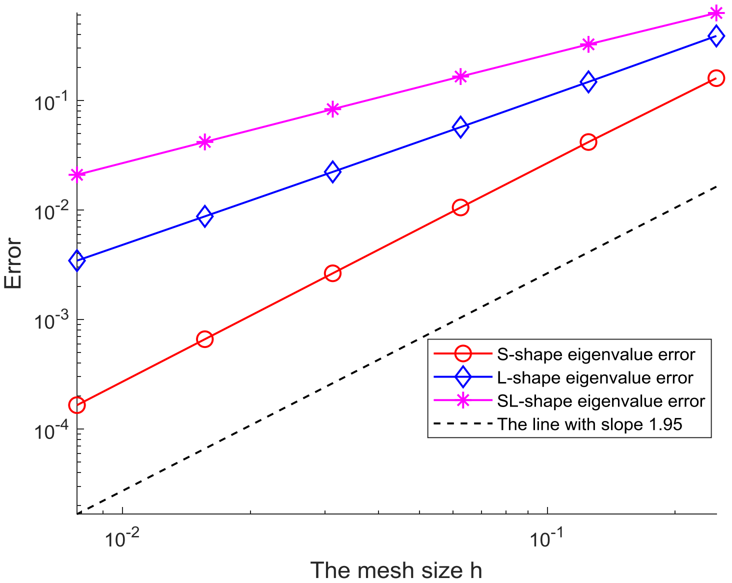

Figure I. When $\mathbf{b} = (0,0)^T$ and $c = 0$,the error curve of the first eigenvalue in the regions Ω_L , Ω_{SL}, Ω_S.

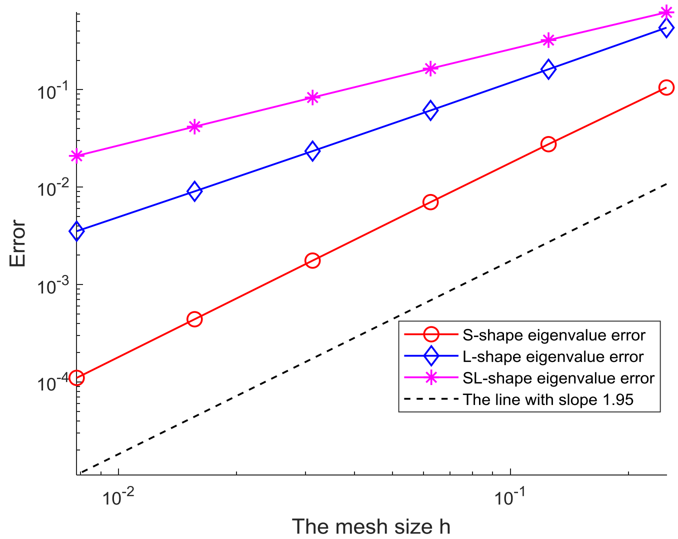

Figure II. When $\mathbf{b} = (3,0)^T$ and $c = 0$, the error curve of the first eigenvalue in the regions Ω_L, Ω_{SL}, Ω_S.

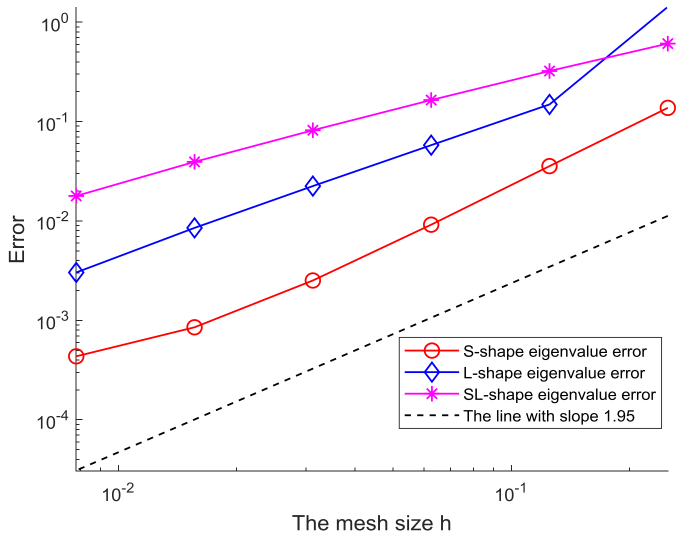

Figure III. When $\mathbf{b} = \left(1 + (x - 1/2)^2, (x - 1/2)(y - 1/2)\right)^T$ and $c = e^{(x-1/2)(y-1/2)}$, the error curve of the first eigenvalue in the regions Ω_L, Ω_{SL}, Ω_S.

V. CONCLUSION

The second-order elliptic equation has wide applications in real-world problems. This paper presents the Raviart-Thomas mixed finite element method for solving the asymmetric second-order elliptic eigenvalue problem. To derive the a priori error estimate, it is necessary to obtain the equivalent mixed variational form of the eigenvalue problem, then study the complete continuity of the operators T, S, T_h and S_h, and most importantly, derive the abstract error estimation formula.We conducted numerical experiments on three test domains $\Omega_L, \Omega_S, \Omega_{SL}$ and obtained numerical solutions for the eigenvalues. From the error curve and numerical results in the tables, it can be seen that our method achieves the optimal convergence order for the eigenvalues and provides the optimal-order error estimate for the eigenvalue functions. This numerical experiment demonstrates the effectiveness of the algorithm. Therefore, for practical engineering problems, this method has significant application value.

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