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On warranty of a multistate system under Phase Type Life

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ABSTRACT:*A multi-state system subjected to homogenous Poisson shocks is considered. The systemgetsdeterioratedaccordingtothemagnitudeoftheshocksactedonit.With the known probability distribution of the shock magnitude, the life distribution of the system is modeled under Phase Type distribution assumption. Assuming the life distribution of the system exhibits different life pattern in various places of installation according to the environmental conditions of the place. An optimal warranty period is obtained.*

KEYWORDS:-*Multi-state system, Poisson shock, Warranty, Reliability, Phase Type distribution.*
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I. INTRODUCTION

A fundamental concept in reliability analysis is that of a binary system, in which the system or component are considered to be in one of the two states- functioning or failed. But it is natural to think that the dichotomized modeling of the system over simplifies the reality. In many real-life situations, however, the systems are better modeled by allowing a range of levels of performance, from perfect functioning down to complete failure. In these situations, some kind of multi-state model is essential. Very early the topic was discussed by Barlow and Wu (1978).And a discussion on multi state reliability is performed by William Grifith (1980).

In real life situations we can see that many systems reach the end of its life span by moving through different states of their performance level, from best to worse. There may be some factors which affect the system adversely (called shocks) and lead the system to failure. In such situations the question is about the life of the system.

Consider a multi-state system which is subjected to Poisson shocks at its working environment. The distribution of the magnitude of shocks that would act on the system is known and the system goes from one working state to the next deteriorated state and finally to the failure state due to the shocks. In such cases the proposed method can be used to find the life distribution of the system. We consider the shock probabilities as having discrete phase type (DPH) distribution and obtain the life distribution of the system as a continuous phase type distribution. To face the competition in the field of marketing, it is better for a manufacturer to introduce an attractive warranty policy. By preventive maintenances, one can offer long warranty periods for the component with a desired level of reliability. Designing a proper warranty program has become an important marketing tool now, especially to advertise the quality of the product. Of course, offering warranty usually results in additional costs to the manufacturers. The warranty servicing cost depends on the warranty length, item reliability and other costs. By properly designing the warranty, the manufacturer can increase the sales and also the market share.

Maintenance policies during warranty are analyzed by several authors (see Nguyen and Murthy (1986), Jack and Dagpunar (1994) among all) in which a one-dimensional warranty characterized by an interval called the warranty period is considered. The case of two-dimensional warranty is discussed in several papers. The two-dimensional warranty is characterized by a region in two-dimensional plane with one axis representing age and the other one usage. Murthy et.al.(1990) deals with two-dimensional renewal processes to model the item failure behavior under the free replacement and no repair assumption. Chen and Popova (2002) propose a new maintenance policy which minimizes the total expected servicing cost for an item with two-dimensional warranty.

We consider the system with the underlying life distributions are phase-type and suggest a method of finding the optimal free replacement warranty period with $'n'$ number of preventative maintenances under the given cost restriction, ensuring a specified reliability for the product during the warranty period.

Suppose that the system is having random life time. The system seems to have different life time behavior over different places where they are used. The manufacturer can amply offer free maintenances (service) while assuring a reliability of at least a pre-decided 'p' percentage. The problem is to obtain the optimal warranty time for the system so as ensuring a reliability of at least 'p' percentage during the warranty period with $'n'$ free maintenances under the given cost restriction, that the expected cost of warranty should not exceed an allotted amount C. Let X denote the life time of the product under normal (factory) condition. When it is used at a different place the life time becomes $\beta X,\ \ 0<\beta<1$ where β refers to the degree of reduction in the lifetime owing to that particular place.

II. PHASE TYPE DISTRIBUTION

Definition2.1 *A probability density* $\{p_k\}$ *on the set of nonnegative integers is called a discrete phase type (DPH) distribution if it is the density of the time until absorption in a finite state Markov chain with transition probability matrix given by*

$$P=\begin{bmatrix} T & \underline{T}^{0} \\ \underline{0} & 1 \end{bmatrix} \tag{2.1}$$

And initial probability vector given by$(\underline{\alpha},\alpha_{m+1})$ *.Here T is an m×m sub stochastic matrix such that* $T\underline{e}+\underline{T}^{0}=e$ *and*$(I-T)$*is nonsingular.*

The DPH density is given by

$$p_0=\alpha_{m+1},\quad p_n=\underline{\alpha}T^{n-1}\underline{T}^{0},n\geq 1. \tag{2.2}$$

The pair $(\underline{\alpha},T)$is the representation of DPH and m is the order of the phase type distribution.

Definition 2.2 *A distribution F on* $[0,\infty)$ *is a continuous phase type (CPH) if it is the distribution of time until absorption in a finite state Markov process with generator*

$$P=\begin{bmatrix} T & \underline{T}^{0} \\ \underline{0} & 1 \end{bmatrix}$$

And initial probability vector $\left(\underline{\alpha},\alpha_{m+1}\right)$.

$T=(T_{ij})$*isanonsingularmatrixofordermandsatisfies*$T_{ii}\leq 0,1\leq i\leq m,T_{ij}\geq 0$

For $i\neq j$*. The distribution F is given by*

$$F(x)=1-\underline{\alpha}\exp(Tx)\underline{e},\qquad x\geq 0 \tag{2.3}$$

F is having the representation $(\alpha,T)_m$*.*

A finite mixture of CPH distributions is again a CPH distribution. If($p_1, p_2, \ldots, p_k$) is a mixing density and $F_i(.)$ has the representation CPH ($\underline{\alpha}_i, T_i$)oforder$m_i$,$1\le i\le k$, then the mixture distribution $M(x)=\sum_{i=1}^{k} p_i F_i(x)$ has the representation CPH ($\underline{\alpha}, T$);where $\underline{\alpha} = [p_1\underline{\alpha}_1, p_2\underline{\alpha}_2, \ldots, p_k\underline{\alpha}_k]$ (2.4)

$$T=\begin{bmatrix} T_1 & 0 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & T_k \end{bmatrix} \qquad (2.5)$$

For more details on Phase type distribution, one may refer to Neuts (1981).

III. MODEL

Consider a multi- state system. The system consists of n different working states. A new system which is functioning perfectly is said to be in state 1. State 2 is the next deteriorated functioning state of the system and the n^{th} state is the complete failure state of the system. The system is subjected to homogenous Poisson shocks with parameter λ. The system is getting deteriorated and reaches its failure state due to the shocks acted on the system. The magnitude of the shock acted on the system determines the severity of the deterioration towards the failure of the system. Any shock can deteriorate the condition of the performance level of the system. If there are no shocks acted on the system, it is assumed to continue in its current working state. Let Z denote the magnitude of the shock acted on the system. When one designs a system, he knows the capacity of the system to suffer shocks. The quantum of shocks that will shift the performance level of the system is as follows:

For the system working in i^{th} state where ($1 \le i\le n-1$), if the magnitude of the shock Z, acted on the system is such that, $0 < Z \le k$, system get deteriorated to $(i+1)^{th}$ state. If $k < Z \le 2k$, it gets deteriorated to $(i+2)^{th}$ state.

In general, if $(r-1)\, k<Z \le rk$, for $r=1,2,\ldots,(n-i-1)$ system gets deteriorated to $(i+r)^{th}$ state and if $Z\ge(n-i-1)k$, the system directly jumps to the failure state from the i^{th} state. There is no maintenance facility available to improve the current working state of the system. So, if the system is in the i^{th} state, according to the shock magnitude Z, it only moves to a state j, where $i<j\le n$.

Let p_i denote $P[(i-1)\; k<Z\le ik]$ for $i=1,2,\ldots,(n-2)$. And $\beta_i = P[Z>(n-i-1)k]$, for $i=1,2,\ldots,n-2$. Note that $\beta_{n-1} = P[Z>0]=1$.

Define Y_n=The working state of the system after the n^{th} shock acted on it, where $n=1,2,\ldots$.Then Y_n is a Markov chain with initial probability vector $\underline{\alpha}=(1,0,0,\ldots,0)$ and the transition probability matrix

$$T=\begin{bmatrix} 0 & p_1 & p_2 & p_3 & . & . & p_{n-2} & \beta_1 \\ 0 & 0 & p_1 & p_2 & . & . & p_{n-3} & \beta_2 \\ 0 & 0 & 0 & p_1 & . & . & p_{n-4} & \beta_3 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 0 & \beta_{n-1} \\ 0 & 0 & 0 & 0 & . & . & 0 & 1 \end{bmatrix}$$

This gives, $P=\begin{bmatrix} T & \underline{\beta} \\ \underline{0} & 1 \end{bmatrix}$

where,

$$T = \begin{bmatrix} 0 & p_1 & p_2 & p_3 & . & . & p_{n-2} \\ 0 & 0 & p_1 & p_2 & . & . & p_{n-3} \\ 0 & 0 & 0 & p_1 & . & . & p_{n-4} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And $\underline{0}$=(0,0,0,...0), $\beta = \underline{e} - T\underline{e}$, where $\underline{e}$=(1,1,1,...1)$_{(n-1)\times 1}$

Let X denotes the number of transitions required for this Markov chain to get in to the absorbing state n. It is to be noted that the transition in the working phase of the system is solely due to the shocks acting on it and each shock will result in a state change however small its magnitude be. Therefore, the number of transitions required for the Markov chain Y_n to get in to the absorbing state is equal to the total number of shocks acted on the system which leads the system to failure. This implies P(X=k)=P(Number of shocks acted on the system leading the system to failure=k) Notice that, the probability distribution of the number of transitions required for an absorbing Markov chain to get in to its failure state is a Discrete Phase type(DPH) distribution with parameters $\underline{\alpha}$ and T. Obviously, we get, the number of shocks to be acted on the system to its failure follows the same DPH. That is the shock probabilities a_k, the probability of the system fails by k^{th} shock, follows DPH ($\underline{\alpha}$,T).

In our model the arrival of the shocks is considered as a counting process, where, $N(t)$,the number of shocks in the time interval $\overline{(0,t)}$ is taken as a Poisson process with parameter λ.
Let A_k be the probability of surviving k shocks by the system, fork=0,1,2,... where $1 = A_0 \geq A_1 \geq A_2$...

Here, $$\overline{A_k} = \sum_{n=k+1}^{\infty} \alpha T^{n-1} \beta \quad (\text{Since} X \sim DPH\ (\underline{\alpha}, T)) \tag{3.1}$$

$$= \underline{\alpha} T^k \underline{e}_{(n-1)} \ (\text{Since}\ T\underline{e}_{(n-1)} + \beta = \underline{e}_{(n-1)} \tag{3.2}$$

Now from Manoharan et al(1992), the survival function $H(t)$of the system can be obtained as

$$H(t) = \sum_{k=0}^{\infty} P(N(t) = k)\overline{A_k} \tag{3.3}$$
$$= \underline{\alpha} e^{\lambda(T-I)t} \underline{e}_{(n-1)} \tag{3.4}$$

which is the survival function of a random variable following continuous phase type distribution with parameters ($\underline{\alpha}$,λ ($T - I$)), where I is the identity matrix. So, the lifetime of the system (let it be denoted by Y) follow CPH ($\underline{\alpha}$,λ ($T - I$)). That is $Y \sim CPH$ ($\underline{\alpha}$,λ ($T - I$)).This implies that the arrival rate of Poisson shocks and distribution of the shock magnitude 'Z' completely specifies the lifetime of the multi-state system considered above.

IV. DETERMINATIONOFWARRANTYPERIODOFTHESYSTEM

The manufacturer supplies the systems to 'r' different places including the place with factory conditions. Let us consider X be the life time of the system at the factory condition and $X \sim CPH$ $(\underline{\alpha}, T)_m$. Let $Y_1, Y_2, \ldots, Y_{r-1}$ respectively be the lifetimes at (r-1) different places where $Y_i = \beta_i X$, for i=1,2, . . ., r-1;0< β_i <1. Here $\beta_1, \beta_2, \ldots, \beta_{r-1}$ refer to the factors denoting the effect of the different places on the life of the system. Since X is assumed to have CPH, by the properties of phase type distribution, $Y's$ are CPH variates with respective parameters $(\underline{\alpha}, \beta^{-1}T)_m$. Let $(\delta_0, \delta_1, \delta_2, \ldots, \delta_{r-1})$ be the sale distribution of the system at the above r places, where δ_0 corresponding to the place of factory, and in general δ_i

corresponds to the place with component life Y_i. Now the lifetime distribution of the system is given by

$$F_Z(t) = P(Z \le t) = \delta_0 F_X(t) + \sum_{i=1}^{r-1} \delta_i F_{Y_i}(t).$$

It follow from the property that $Z \sim CPH(\underline{\gamma}, L)$, where

$\underline{\gamma} = \left(\delta_0\alpha, \delta_1\alpha, ..., \delta_{r-1}\alpha\right)$ and

$$L = \begin{bmatrix} T & 0 & 0 & . & . & 0 \\ 0 & \beta_1^{-1}T & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \beta_{r-1}^{-1}T \end{bmatrix}$$

Suppose that the manufacturer decides to offer a warranty with $'n'$ free maintenances so as to keep the total warranty cost not exceeding a pre-fixed amount, say $'C'$ and ensuring are liability of at least $'p'$ percentage during the warranty period.

To compute the warranty period, we shall proceed as follows: From the reliability function $R(t)$ of the lifetime Z following $CPH(\underline{\gamma}, L)$, where,

$$R(t) = P[Z > t] = \underline{\gamma} e^{Lx} \underline{e}$$

We find a time point t_1, such that $t_1 = Sup\{t/R(t) \ge p\}$. At this time point t_1 the manufacturer provides the first maintenance service to the system which boosts its life. But because of ageing it cannot act as a new system. The maintenance time is small relative to the mean working time of the component and so can be ignored. After the first maintenance, the lifetime of the component at the place of factory condition is denoted by X_1,where$X_1 = q_1 X$, and the life at the other $r-1$different places as Y_{i1}, where $Y_{i1} = q_1 Y_i$; i=1,2, . . ., (r-1). where q_1 represent the joint effect of ageing and maintenance on the lifetime of the component after the first maintenance. In general, it is represented as q_j for the system after the j^{th} maintenance. The lifetimes of the system after the j^{th} maintenance at the place of factory condition and at the remaining $(r-1)$ places are respectively denoted by X_j and$\boldsymbol{Y_{ij}}$, for $\boldsymbol{i}=1,2,...,(r-1)$; $\boldsymbol{j}=1,2,...,$ n. As in the last case, we can find the probability distribution of the lifetime Z_1 of the system after the first maintenance as a mixture by,

$$F_{Z_1}(t) = P(Z_1 \le t) = \delta_0 F_{X_1}(t) + \sum_{i=1}^{r-1} \delta_i F_{Y_{i1}}(t).$$

Here $X_1 \sim CPH(\underline{\alpha}, q_1^{-1}T)$ and $Y_{i1} \sim CPH(\underline{\alpha}, \left(q_1\beta_i\right)^{-1} T)$ for i= 1,2, . . ., (r-1).So, we have $Z_1 \sim CPH(\underline{\gamma}, L_1)$ where

$$L_1 = \begin{bmatrix} q_1^{-1}T & 0 & 0 & . & . & 0 \\ 0 & \left(q_1\beta_1\right)^{-1} T & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \left(q_1\beta_{r-1}\right)^{-1} T \end{bmatrix}$$

Using the life distribution of the system after the first maintenance, plot its reliability function and find the next time point t_2 for giving the second free maintenance, where $t_2 = Sup\{t/R_1(t) \ge p\}$,$R_1(t)$ being the reliability function associated with the life time Z_1.As before we can find the life time of the component, after the second maintenance by considering the joint effect of the ageing and maintenance on the life time of the component as q_2. Thus, we can derive the probability distribution of the lifetime Z_2 of the component after the second maintenance as $CPH(\underline{\gamma}, L_2)$ where,

$$L_2 = \begin{bmatrix} q_2^{-1}T & 0 & 0 & . & . & 0 \\ 0 & (q_2\beta_1)^{-1}T & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & (q_2\beta_{r-1})^{-1}T \end{bmatrix}$$

Next, we find the time point t_3,where $t_3=Sup\{t/R_2(t)\geq p\}$;$R_2(t)$ is the reliability function of the system after the second maintenance. The system works up to this time point $t_3$with at least p percentage of reliability and this is the time for the third maintenance. Continuing like this one can find the time points for the successive maintenances so as to ensure the system is working with at least p percentage of reliability. Then the system is considered working up to a time $t_{(n+1)}$ with $'p'$ percentage of reliability after the n^{th} maintenance. Hence the total time the system is expected to work with at least 'p' percentage of the reliability is $t= t_1+t_2+\ldots +t_{(n+1)}$.The manufacturer restricts that the total warranty cost with maintenances not to exceed a pre fixed amount $'C'$. So, it is to find the optimal number of maintenances and optimal warranty period within the restriction that the expected warranty cost $E(C_w) \leq C$. Expected cost of warranty with $'n'$ maintenances can be expressed as

$$E(C_w)=C_s(M_1(t_1)+M_2(t_2)+\ldots+M_{(tn+1}(t_{n+1})))+nC_m \qquad (4.1)$$

where,C_s is the system cost, $M_i(t_i)$ is the expected number of failures in the duration t_i after the $(i-1)^{th}$ maintenance. C_m is the cost per maintenance which is assumed uniform for all of the maintenances. Expected number of failures up to time t of a system can be expressed as

$$M(t) = \int_0^t h(t)dt = -\log \overline{F}(t) \qquad (4.2)$$

where, h(t) is the hazard function and F is the life distribution of the system. The expected number of failures between each of the maintenances represented by $M_1(t_1)$,$M_2(t_2)$,...,$M_{(tn+1)}$ can be found using the corresponding estimated CPH life distribution. The restriction $E(C_w)\leq C$, is verified for n=1,2,...,n. The highest value of n satisfying the restriction $E(C_w)\leq C$ is the optimal number of maintenances and with respect to the total number of maintenances $'n'$, the optimal warranty period that can offer is $t = t_1+t_2+\ldots +t_{(n+1)}$.

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