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A Review of the Homotopy Analysis Method with Various Hybrid Techniques

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ABSTRACT: *The Homotopy Analysis Method (HAM) has emerged as a powerful analytical technique for solving nonlinear ordinary and partial differential equations across diverse scientific and engineering fields. Unlike traditional perturbation methods, HAM offers flexibility in handling strongly nonlinear problems without requiring small parameters. This paper provides a comprehensive review of the theoretical foundation, computational implementations, and applications of HAM, its development, and its wide range of applications. The discussion includes the mathematical framework, convergence control, comparison with other methods, and a detailed survey of its applications in fluid mechanics, heat transfer, quantum mechanics, and beyond. Furthermore, we highlight recent advancements and potential future directions for HAM.*

KEYWORDS: *Homotopy analysis method, Partial Differential Equation, Ordinary differential equation, Various Transforms and their application.*

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I. INTRODUCTION

The Homotopy Analysis Method (HAM) is a powerful analytical technique employed to tackle nonlinear differential equations [43], which often arise in complex scientific and engineering problems. Unlike traditional methods, HAM offers a unique approach by constructing a homotopy, or a continuous transformation, that interpolates between a simple initial guess and the true solution. This flexibility allows HAM to provide accurate approximations even in cases where other perturbation methods or numerical techniques might struggle [39].

The strength of HAM lies in its ability to control the convergence of the solution through the introduction of an auxiliary parameter, providing a robust mechanism for improving the accuracy of approximations. Its versatility extends across a wide range of applications, from fluid dynamics and heat transfer to nonlinear oscillations and electromagnetic fields. The method is particularly valuable in situations where the nonlinearity of the problem is pronounced and traditional methods may fail to converge or provide viable solutions [[53],[54]].

This review aims to provide an in-depth overview of the Homotopy Analysis Method, discussing its theoretical foundations, key features, and various applications across different fields. By examining both the strengths and limitations of HAM, we will explore how this method has evolved over time and how it continues to offer a promising approach to solving complex nonlinear problems.

The remaining parts of this research paper are presented in the following sections: In Section 2, presents the theoretical foundation of HAM. Section 3, explains the application of HAM. Section 4, presents the numerical formulation of HAM. In Section 5, we review papers on linear and Nonlinear ordinary and Partial Differential Equations. In Section 6, there has been a discussion of the conclusion

II. THEORETICAL FOUNDATION OF HAM

2.1 Basic Principles:

The Homotopy Analysis Method is grounded in the concept of homotopy from topology. For a given nonlinear problem: where is a nonlinear operator, HAM constructs a homotopy: where is the homotopy parameter, is a linear operator, is the convergence control parameter, and is the homotopy function. The solution transitions from the initial guess to the actual solution by varying.

2.2 Convergence Control:

A significant advantage of HAM is the introduction of the convergence control parameter, which allows the user to ensure the series solution converges. The auxiliary function, auxiliary linear operator, and convergence control parameter together provide unparalleled flexibility in tailoring the method to specific problems.

2.3 Advantages of Traditional Methods:

1. **Parameter Independence:** HAM does not rely on small or large parameters, unlike perturbation methods.
2. **Flexibility:** The method offers adjustable auxiliary parameters to improve convergence.
3. **General Applicability:** HAM applies to both weakly and strongly nonlinear problems.

III. APPLICATIONS OF HAM

3.1 Fluid Mechanics:

HAM has been extensively used in fluid dynamics to solve problems such as boundary layer flows, non-Newtonian fluid dynamics, and stability analysis. For example:

- Liao (1999) applied HAM to solve the Blasius equation, demonstrating its superiority over perturbation techniques.
- Nonlinear convective heat transfer problems in nanofluids have also been effectively analyzed using HAM.

3.2 Heat and Mass Transfer:

HAM has proven instrumental in solving nonlinear equations governing conductive and convective processes in heat transfer. Key applications include:

- Solving nonlinear heat conduction equations in composite materials.
- Analyzing thermal radiation effects in boundary layer flows.

3.3 Quantum Mechanics:

HAM has been applied to nonlinear Schrödinger equations and quantum potential problems. Its ability to handle complex boundary conditions makes it suitable for quantum field theory and wave mechanics.

3.4 Engineering and Applied Sciences:

In engineering, HAM is used to model nonlinear oscillations, elastic stability, and wave propagation in complex structures. Its adaptability to diverse boundary and initial conditions ensures accurate solutions for real-world problems.

3.5 Other Applications:

- **Biological Systems:** Modeling population dynamics and epidemic spread.
- **Chemical Reactions:** Analyzing reaction-diffusion systems in catalysis and combustion.
- **Financial Mathematics:** Solving nonlinear differential equations in option pricing and market modelling.

3.6 Recent Advances:

Recent research has focused on enhancing HAM's computational efficiency and broadening its applicability:

1. **Hybrid Methods:** Combining HAM with numerical techniques for high-precision solutions.
2. **Optimization Algorithms:** Using machine learning to optimize the selection of auxiliary parameters.
3. **Software Development:** Automated tools and packages for implementing HAM in various programming environments.

3.7 Challenges and Future Directions:

While HAM is a robust method, certain challenges remain:

- **Convergence Issues:** Despite -control, improper choices of auxiliary parameters can hinder convergence.
- **Complexity:** Setting up the method for highly nonlinear problems can be intricate.

3.8 Future research could focus on:

- Developing systematic approaches for selecting optimal auxiliary parameters.
- Extending HAM to fractional differential equations and stochastic systems.
- Creating user-friendly software to simplify its application.

IV. NUMERICAL FORMULATION OF HOMOTOPY ANALYSIS METHOD [[40],[41],[43]]

Consider a particular nonlinear differential equation that follows

$$\mathcal{N}[\mathfrak{H}(\Omega,t)] = 0 \tag{1}$$

Where $\mathcal{N}$ is a nonlinear operator, $\mathfrak{H}(\Omega,t)$ is an unknown function, and Ω might be either $\{\mathfrak{x},\mathfrak{y}\}$ or$\{\mathfrak{x},\mathfrak{y},\mathfrak{z}\}$. Time and space are the two independent variables as represented by $\mathfrak{x},\mathfrak{y},\mathfrak{z}$, and t respectively. Making use of the traditional Homotopy technique, which Liao devised.

$$(1-p)L\,[\,\varphi(\Omega,t;p) - \mathfrak{H}_0(\Omega,t)] = p\hbar\mathcal{N}[\varphi(\Omega,t;p)] \tag{2}$$

Where L acts as an auxiliary linear operator, $\varphi(\Omega,t;p)$ indicates an unknown function, $\mathfrak{H}_0(\Omega,t)$ is the starting guess for$\mathfrak{H}(\Omega,t)$, $\hbar$ serves as a nonzero auxiliary parameter, and $p \in [0,1]$represents the embedding parameter. It is applicable if $p = 0$and $p = 1$.

$$\varphi(\,\Omega,t;0) = \mathfrak{H}_0(\Omega,t),$$

And

$$\varphi(\,\Omega,t;1) = \mathfrak{H}(\Omega,\mathrm{t}),$$

Thus, solution $\varphi(\Omega,t;p)$ deviates from the original guess $\mathfrak{H}_0(\Omega,t)$ to solution $\mathfrak{H}(\Omega,t)$ as p approaches from 0 to 1. When we extend the Taylor series $\varphi(\Omega,t;p)$concerning p, we obtain

$$\varphi(\,\Omega,t;p) = \mathfrak{H}_0(\Omega,t) + \sum_{m=1}^{\infty} \mathfrak{H}_m(\Omega,t)p^m \tag{3}$$

Where,

$$w_m(\Omega,t) = \frac{1}{m!}\frac{\partial^m \varphi(\Omega,t;p)}{\partial p^m}\bigg|_{p=0}$$

The sequence *(3)* convergence occurs at $p = 1$whenever the auxiliary linear operator, auxiliary parameter $\hbar$, initial guess, and auxiliary function have all been chosen correctly.

$$\mathfrak{H}(\Omega,t) = \mathfrak{H}_0(\Omega,t) + \sum_{m=1}^{\infty} \mathfrak{H}_m\,(\Omega,t), \tag{4}$$

The original nonlinear equation ought to have a legitimate solution in this instance. According to expression *(4)*, the governing equation might be obtained from the 0-order deformation equation *(2)* as well. Describe the function of the vector.

$$\overrightarrow{\mathfrak{H}_n} = \{\mathfrak{H}_0(\Omega,t), \mathfrak{H}_1(\Omega,t), \mathfrak{H}_2(\Omega,t) \dots \dots .. \mathfrak{H}_n(\Omega,t)\}$$

The m-times concerning the embedding parameter p, differentiating the zero-ordered deformation equation *(2)*. Once $p = 0$ has been entered and divided by $m!$the mth-order deformation equation is as follows:

$$L[\mathfrak{H}_m(\Omega,t) - \chi_m \mathfrak{H}_{m-1}(\Omega,t)] = \hbar \mathcal{R}_m[\mathfrak{H}_{m-1}(\Omega,t)] \tag{5}$$
$$[\mathfrak{H}_m(\Omega,t) - \chi_m \mathfrak{H}_{m-1}(\Omega,t)] = \mathrm{L}^{-1}\{\hbar \mathcal{R}_m[\mathfrak{H}_{m-1}(\Omega,t)]\}$$
$$\mathfrak{H}_m(\Omega,t) = \chi_m \mathfrak{H}_{m-1}(\Omega,t) + \mathrm{L}^{-1}\{\hbar \mathcal{R}_m[\mathfrak{H}_{m-1}(\Omega,t)] \tag{6}$$

Where

$$\mathcal{R}_m(\overrightarrow{\mathfrak{H}_{m-1}}) = \frac{1}{m-1!}\frac{\partial^{m-1}\mathcal{N}[\varphi(\Omega,t;p)}{\partial p^{m-1}}\bigg|_{p=0} \tag{7}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{8}$$

V. THE REVIEW OF LINEAR AND NONLINEAR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS.

The Homotopy Analysis method is vital in obtaining the solution of linear and Non-linear ordinary and partial differential equations. We presented the contribution of several authors to obtain the solution of Linear and non-linear ordinary and partial differential equations as below:-

1. **He J. H. (2000). [17]** In an article, the author introduces a combination of homotopy and perturbation methods called hybrid. As this creative idea is labelled, the Homotopy Perturbation Method (HPM) considers the benefits of both techniques in addressing complex nonlinear equations. He provides an overview of HPM and shows how it can be applied to solve complicated scenarios using a few examples with non-linear causes. By comparison with traditional perturbation techniques, the outcomes indicate that HPM can give up the exact solutions acquired. Being adaptive and having convergence characteristics in nature makes this method unique, especially for mechanics engineers who deal with nonlinearities.

2. **He, J. H. (2004)**. **[18]** The author introduces the homotopy analysis method (HAM) and the Homotopy Perturbation Method (HPM) in the paper. This study considers the theoretical basis of HAM and HPM and also looks at their applicability to nonlinear equations. Although both techniques are valid, HPM falls short in one area compared to HAM. Various examples show that HAM can handle nonlinear problems more robustly than its counterpart because it has a broader range of applications.

3. **Liao, S. (2004). [38]** In the paper, Liao investigates the HAM in solving nonlinear problems. This is a research focusing on the ability of HAM in dealing with complex non-linear differential equation models. The theoretical foundation of HAM is introduced by Liao, starting with fundamental concepts and further focusing on the theoretical aspect of HAM concerning its capability to handle convergence through auxiliary-parameter h. It is necessary to mention the facilities that can be provided by HAM, which can adjust and control the solution convergence, which is not typical for the majority of the traditional perturbation methods the mentioned parameter.

4. **Liao, S. (2005). [39]** In this paper, The author introduce the draws a detailed comparison between HAM (Homotopy Analysis Method) and HPM (Homotopy Perturbation Method). The research investigates both methods' theoretical differences as well as their practical uses, strengths and weaknesses. Liao points out that it is possible to modify regions of convergence in HAM through an additional parameter unlike HPM which does not have this capability but he also notes that it can be used flexibly more than any other method.

5. **Abbasbandy, S. (2007). [2]** In this paper, the author seeks to analyze the application of the Homotopy Analysis Method (HAM) in the case of heat radiation equations. This should demonstrate to the readers that HAM is stable and versatile enough to provide solutions for non-linear problems and with the appearance of small parameters, as is the case with traditional perturbation methods. Thus, Abbasbandy applies mathematical techniques and HAM to obtain heat radiation equations, which are accurate and convergent. Finally, this research proves, in Abbasbandy's (2007) view, that HAM has more advantages over other conventional methods in terms of convergence control and accuracy of solution and hence, it is quite effective in the analysis of heat transfer problems.

6. **Sajid and Hayat (2008). [65]** In this paper, the author introduces The Nonlinear Analysis: Real World Applications, which is devoted to the comparison of two methods, the Homotopy Analysis Method (HAM)

and the Homotopy Perturbation Method (HPM), while solving nonlinear heat conduction and convection equations. Thus, in this study, efficiency and convergence of both methods have been discussed critically with its strength and weaknesses. Peculiar to their research, the authors discovered that HPM can be effective at times but is generally ineffective especially when larger parameter values are involved leading into diversion. On a different note, however, while having some similarities to HPM we find that HAM has got adjust auxiliaries parameters thus giving more control over converging points making it strong enough for a wider class of problems than any of the methods considered here On a different note however while having some similarities to HPM we find that HAM has got adjust auxiliaries parameters Thus; HAM is stronger enough to solve a wide range of problems other numerical instances investigated by researchers include the following where they discovered that in most cases or often with all the tested numbers up to the present most people of all professions can confidently state that the HAM solution is always the best possible solution except that occasionally using the complex nonlinear formulation, the HPM yields a better result than any method before it or afterwards.

7. **Khan et.al (2009). [31]** The article must be examined by analyzing the Navier-Stokes equation with fractional orders using He's homotopy perturbation and variational iteration methods. This study illustrates how to solve a differential equation of a fractional order using different methods, such as mathematical formulation or comparison. Besides, numerical examples show that both ways converge and give accurate answers only when applied to complex nonlinear fractions systems. More importantly, they can do this many times over, proving that these techniques work well for one and many equations at once. Hence, it shows their solving abilities towards non-linear equations with a fraction, thus making it easier for those who deal with such types of problems.

8. **Li, J. L. (2009). [37]** The author of this paper studies which employs Adomian's Decomposition Method (ADM) and the Homotopy Perturbation Method (HPM) for solving nonlinear equations. By giving a detailed comparison between these two ways in terms of theoretical foundation as well as practical implementation, he hopes to help people have a better understanding about them. In fact, Li shows through many examples that both methods are effective in dealing with nonlinear problems but they have different advantages in view of convergence or simplicity. The reason why ADM is superior lies in its step-by-step breaking down of non-linear terms, while HPM wins a reputation for being adaptable and easily applicable too. The researcher concludes that either ADM or HPM should be selected depending on what kind of problem we encounter because it has strengths under various circumstances otherwise unbeatable by any other means. This research makes a great contribution to this field by figuring out how useful such methods can be for complex nonlinear equation solving, therefore providing an opportunity for scientists to choose the best technique according to their requirements.

9. **Liao, S. (2009). [40]** The study by Liao the presents definitions and theorems to develop the theory pertaining to Homotopy Analysis Method (HAM). In the following paper, some basics of HAM are described, and clear principles as well as strict mathematical statements. Also, some theorems are provided which substantiate this approach's effectiveness and efficiency in managing non-linear challenges/difficulties. The author also gives clear and concrete examples with proof that HAM is stable and can be adjusted to fit the situation of need; the author also explains that this method is useful in controlling convergence in nonlinear differential equations and in reaching accurate solutions.

10. **Liu, C, Shi. (2010). [44]** In this article Liu simplified the key aspects of HAMs for presentation in the article. Here, the topic of the work is defined more narrowly and strictly, as it is to cover the theoretical background for HAM and the areas of its application, with a focus on the distinctive features that distinguish it from other analytic tools. The author also shows how open this method is due to describing that convergence may vary using h, which is an auxiliary parameter and, thus, provides accurate control over the solution search process. The efficiency and accuracy of the method to solve the nonlinear differential equations through examples are presented in the paper which also underlines the relevance towards different fields of maths and engineering problems.

11. **Shateyi et.al (2010)**. **[71]** This article aims to analyse an improved Homotopy Analysis Method (HAM) described in the Mathematical Problems in Engineering for solving the Jeffery–Hamel flow problem. The authors optimize some of these parameters to enhance the convergence of HAM, where it seize to guess faster and provide accurate solutions. They confirm this by presenting a large number of numerical experiments where they compare their optimized version against standard HAM and other methods traditionally employed for such purposes and emphasise how significantly better results are obtained

concerning computation efficiency and precision when using it rather than any known technique. This kind of research provides a good background for solving non-linear fluid dynamics problems with the help of this method while stressing the fact that there is a flexibility in this method and its stronger performance in comparison with other methods when it is used in engineering complexities.

12. **Turkyilmazoglu (2010). [77]** The present paper witnesses the specifically selected method known as the homotopy analysis method. The details/discussion/conclusion – related to one of the aspects or parts that has relevance to issues within the direct sense – involves the matter regarding the homotopy analysis method convergence. If the author has intended to illustrate how, from an initial conjecture and some specific circumstances, the method of homotopy analysis becomes the solution of non-linear ordinary or partial differential equations. The residue error of the square is used to give the maximum value to the convergence control parameter. To this end, an estimate of the error is also produced. The additional formulas for estimating the error, along with the above ones, are as follows This.

13. **Ghazanfari and Veisi (2011). [14]** In this work, the HAM for a fractional nonlinear wave equation is to present numerical solution of the problem. Nevertheless, this generality is precisely what we want; thus, for different 'convergence rates also,' as it is mentioned in the work, the numerical outcomes are a perfect match with that of the homotopy analysis method. Last but not the least, the accuracy properties have been explained through some examples.

14. **Liao, S. (2011). [41]** In his book Homotopy Analysis Method in Nonlinear Differential Equations, Liao thoroughly examines the Homotopy Analysis Method (HAM) for solving nonlinear differential equations. The three parts of the book are basic concepts and theoretical development in HAM, applications with Mathematica package BVPh 1.0, and real-life examples from finance or fluid mechanics, among other fields dealing with non-linear partial differential equations. One crucial characteristic highlighted by Liao about this method is its flexibility and ability to ensure solution convergence without requiring small parameters. With many instances being given together with their practical uses, such an extensive piece demonstrates how well HAM deals with highly nonlinear problems, which should serve as an irreplaceable reference material for researchers and practitioners within applied mathematics, physics, and engineering.

15. **Chen and Cui (2012). [10]** With regard to this, the authors of this work employed the homotopy analysis method (HAM) in a way that they would be in a position to give singularly analytic solution for the time fractional Klein-Gordon equation with variable coefficients in terms of the Caputo fractional derivatives. Besides, it used HAM applications to developed series solutions that contain simpler mathematical terms for the generalized fractional equations.

16. **Matinfar et.al (2012). [46]** The author of this paper introduced the Homotopy Analysis Method (HAM) that is used in solving the system of partial differential equations in a new approach to get the exact solutions. The article analyses the theoretical backgrounds of HAM, focusing on the applicability and effectiveness of the method in complex non-linear systems. The authors provide several examples to demonstrate their points on the application of HAM and how effective this tool is in arriving at good and reliable solutions. They also claim that this method also has better control of convergence via an auxiliary parameter that makes it useful for mathematical and engineering problems that demand precision. This research is beneficial to this area of study since it takes the reader through various procedures of applying HAM to general PDEs and other forms of PDEs to display the versatility of this tool.

17. **El-Tawil & Huseen (2013). [12]** This paper seeks to focus on the merging of q- Homotopy Analysis Method (q-HAM) in solving non-linear differential equations. The research formatted and published in the International Journal of Contemporary Mathematical Sciences aims to discuss the theory of q-HAM alongside its application with a focus on the possibility of convergence if the appropriate choices of the parameters are to be made. It is utilized in specific numerical examples to demonstrate its effectiveness in solving such challenging problems having nonlinearities. At the same time, this proves that it is stable and computationally very efficient as well.

18. **Raftari et.al (2013). [61]** In the present study, the authors explains in detail how approximate analytical solutions are derived for the one-dimensional hyperbolic telegraph equation with initial conditions using the Homotopy Analysis Method (HAM). This method is shown to work well with series solutions convergence via a parameter that acts as an auxiliary. In addition to this, different examples prove its effectiveness and accuracy when applied to such types of hyperbolic partial differential equations. Such a conclusion

emphasizes the wider use of HAM in mathematical as well as engineering problems which involve complicated differential equations.

19. **Sakar and Erdogan (2013). [66]** In the present study, the authors have used two of the most recently developed techniques that are named Adomian's decomposition method (ADM) and Homotopy analysis method (HAM) while explaining the time fractional Fornberg–Whitham equation. This auxiliary parameter used in the homotopy analysis is believed to be another type of handle through which one may, in a manner, modify or determine the convergence circumference of the series of the solution. All the given fractional derivatives belong to Caputo's type. Piggybacking on what has been said, HAM has been compared to ADM based on the outcome obtained from it. As found by the current technique, the efficiency is proved to be higher and accompanied with ease compared to the previous methods. The stated issue was investigated for certain particular examples and, thus, yields numerical solutions.

20. **Baskonus *et.al* (2014). [5]** This paper aims to apply Homotopy Analysis Method & the novel equation trial modification scheme which was recently introduced to the literature to solve various differential equations, being a field of Physics chemistry, applied sciences, applied maths especially modeling of gas dynamics & traffic flow while addressing this paper and investigating the solution of the nonlinear time fractional generalized Burgers equation. In this perspective, a table is created to indicate numerical solutions coupled with findings pertaining to the time-fractional generalized Burgers equation related to the aforementioned conclusions. Finally, using programming language Mathematica 9, the 2D as well as 3D surfaces have been plotted to give physical significance to the figures representing analytical solution along with approximate solution and in the last part, what all these are containing has been looked for.

21. **Khan and Akbar (2014). [29]** The modified simple equation method was applied to the (2+1)-dimensional cubic Klein-gordon and the (3+1)-dimensional zakharov-Kuznetsov equations. The two scholars, Khan and Akbar, described in their paper on the Journal of the Association of Arab Universities for Basic and Applied Sciences how they were able to provide systematically the exact traveling wave solutions for these nonlinear equations. Apart from solitons and periodic solutions described above, many other solutions have been found by the authors with this method, therefore confirming its effectiveness for the solutions of more complicated, higher-order, non-linear differential equations also. The present investigation clearly substantiates that this modified simple equation method is very effective in solving as well as simplifying many of the problems from mathematical physics if the difficulties are non-linear in nature.

22. **Kumar *et.al* (2015). [32]** In as much as the key objectives of this paper are to find a numerical algorithm of the modified homotopy perturbation method in association with the Sumudu transform for simulating and to solve a fractional multi-dimensional diffusion equation, which describes the density changes of a material as it diffuses. It will be observed that there is no constraint of small parameters in the modified homotopy perturbation method as against the classical perturbation method. The mentioned technique provides an analytical solution in the form of the converting series, where every component which is a part of this series can be calculated without using linearization or small perturbation approximations. The outcomes of the offered approach demonstrate that the suggested methodology is very basic and time-efficient.

23. **Alkan, (2016). [3]** This work is extending HAM to the time-fractional Burgers equation through justifying the optimal value at which the convergence of the method is enhanced in the KMU Journal of Engineering and Natural Sciences. The residual error function is used by the researchers in the reduction of absolute error to search for the best arbitrary value of h. Several numerical examples are given in this paper, which shows that this improved HAM proves to be better than the earlier HAM specifically; considerable accuracy enhancements have been made on time-fractional partial differential equations.

24. **Maitama and Abdullahi (2016). [48]** The present work employs a newly developed analytic method called Natural Homotopy Perturbation Method (NHPM), to find related solution of linear and non-linear fractional partial differential equations (FPDEs). The analytical method proposed here is based on HPM in combination with NTM, which is one of the most famous methods. According to Caputo sense, the fractional derivative is given and in the context of this newly proposed analytical technique, the nonlinear term has been approximated through He's polynomials. Appreciated in this analytical system is the higher order derivation approach which applies to find equipped fractional linear and nonlinear PDEs yielding the correct solution and it matches to other such methods.

25. **Sakar *et.al* (2016). [67]** This paper focuses on finding the numerical approximations for fractional PDEs in connection to proportional delay in time t and shrinking in the space variable x without the need to linearize or assume small perturbation. The fractional derivatives used in this work are in the context of Caputo's. The mentioned evaluation and integration method seems to be efficient and logical as far as its application is concerned, as mentioned at the end of the method. Hence, graphical representation is provided concerning the numerical results for values of α, which have been given and ascertained above.

26. **Mohammed and Abbas (2017). [1]** In this paper, some general physics equations like the korteweg-devries (kdv) problem, the Non-homogeneous Boussinesq problem, and the Non-homogeneous system's Hirota-Satsuma problem are solved using the Homotopy analysis method of the partial differential equations. This method has been compared with numerical solutions where it is found to be very effective, and it can also be easily computed as the convergent series whose elements can solve any of the various non-linear problems related to partial differential equations.

27. **Singh and Kumar (2017). [72]** In view of this, the paper in this respect is the solution for TFPDEs with proportional delay in terms of the fractional variation iteration method where the fractional derivative is in the Caputo sense. In analysis of the results extracted from the computation of the values of the suggested series solutions, it is understood that they can arrive at a near solution more efficiently. So that FRDTM becomes efficient and credible, three sample test problems of TFPDEs with proportional delay were computed. Therefore, this appears to be a very robust, efficient and powerful scheme for analysing various forms of physical models, used in sciences and engineering.

28. **Ozpınar (2018). [55**] In this paper, The author have developed a new version of the homotopy analysis method for approximations to linear and nonlinear fractional PDEs for the time derivative $\alpha(0 < \alpha \leq 1)$. They named it the discrete homotopy analysis method (DHAM). Unlike the previous method, DHAM has ħ as an extra parameter that offers an easy means of ensuring the coverage region of the solution series is attained. To demonstrate the efficiency and accuracy of the described approach, the authors apply their method to initial value problems. They compare their results with exact solutions at $\alpha = 1$. From the above figures, it is also quietly evident that these last two methods are also in a reasonable agreement.

29. **Rathore and Saraswat (2018)**. **[62]** The present article recognizes the application of what is commonly referred to as the Homotopy Perturbation Method (HPM) in solving a nonlinear Burgers' equation based on information from the International Journal of Mathematics and its Applications. The paper also demonstrates how proficiently HPM can resolve them while also revealing that HPM interacts with all nonlinear partial differential equations by decomposing them into additive linear sub-equations. These are then solved in an iteration manner so that their solutions have the form of an infinite series which is rapidly convergent to the actual one. Researchers have compared numbers obtained with the numerical outcomes with those of simulations done in MATLAB, and discrepancies are observed to be small over varied space steps and kinematic viscosity. This is what it means that while it is extremely fast and accurate from the computational point of view when compared with other approaches used for similar purposes in other fields, and thus demonstrating its capabilities in those cases where other approaches most often fail – dealing with intricate nonlinear systems that can be derived from various scientific or engineering issues.

30. **Rehman *et.al* (2018). [63]** Regarding the novel technique employed in this study, the homotopy perturbation double Sumudu transform method (HPDSTM) is suggested and derived from the homotopy perturbation method, He's polynomials, and double Sumudu transforms. They used it to obtain a numerical solution of linear fractional one- and two-dimensional dispersive KdV and nonlinear fractional KdV equations for the purpose of checking the effectiveness of the method. Thus, from this technique, it becomes possible to appreciate how quickly the solutions of the given technique converge towards the true solution. It is a highly active method and functions in the solving of various linear and non-linear differential equations of the fractional orders.

31. **Chang et.al (2019). [8]** This paper focuses on the analysis of the application of the Homotopy method of fundamental solution (HMFS) to nonlinear heat conduction problems. In order to enhance the accuracy and convergence results in solving non-linear heat conduction equations, the authors have proposed a new method, which is the synthesis of perturbation homotopy and the method of fundamental solutions. Several numerical experiments have been carried out in this work, and those experiments indicated that HMFS provides a high accuracy in solving heat conduction equations because it is able to control the nonlinearity that exists in those equations, and this is more efficient than the traditional methods. The findings affirm

that HMFS user potentiality has the possibility to be applied to the engineering aspects of nonlinear heat transfer processes.

32. **Dinesha and Gurrala (2019). [11]** In the paper, the authors demonstrate how to apply the Multi-Stage Homotopy Analysis Method (MS-HAM) to the dynamic simulation in the power system. This paper is the work of the authors here, who have pictured a multistage version of HAM in a bid to improve the accuracy and the convergence rate of the non–linear dynamics of power systems. This technique offers a realistic solution to various numerical problems related to large power systems, particularly transient and voltage stability issues. Besides, using other test cases, a methodology known as MS-HAM is checked. The outcomes reveal that it provides better efficiency regarding the convergence rate and the numerical and computational complexity than conventional numerical integration techniques. Therefore, with respect to these outcomes, it is possible to recommend the use of the developed MS-HAM as a potentially effective means for solving numerous problems in the above-mentioned field of electrical engineering with high accuracy.

33. **Jena and Chakraverty (2019). [24]** In this paper, a new composite method referred to as q-Homotopy Analysis Aboodh Transform Method (q-HAATM) is proposed for fractional PDEs with a proportional delay of Caputo derivative. The new suggested technique termed as q-HAATM is developed from the Homotopy Analysis Method (HAM) and Aboodh Transform Method (ATM). Thus, a series of solutions were given for three problems in examples. Therefore, from these output results, it is evident that this method is relatively easy and, more importantly, provides an appropriate mechanism for selecting the relevant parameters, ħ & n, that will enable the provision of an approximate solution.

34. **Mesloub and Obaidat (2019). [50]** This article is devoted to the application of homotopy analysis techniques for the investigation of initial-boundary value problems related to time-fractional order diffusion equations while concerning non-local constraints of the integral type. It also gives a few illustrations, which is sufficient to indicate that HAM is indeed efficient in solving such problems.

35. **Shah, K. et.*al* (2020). [70]** Here in this paper, an algorithm is provided to find the actual solution of the two-dimensional fuzzy heat equation with the external source term of fractional order. The method used in this algorithm is the Homotopy Perturbation Method (HPM): The required result is found to be a series, and the convergence of the series is rather very fast in several cases in comparison with the actual solutions of the problems. To facilitate an understanding of the results of the discussed numerical solutions, they are provided below; these are then compared to the accurate answer to prove the effectiveness and possibility of the suggested technique.

36. **Baleanu and Jassim (2020). [4]** In this paper, dealing with the solution of 2D fractional partial differential equations and for proposing the numerical scheme for the solution of the fractional Riccati equation, fundamental knowledge of Sumudu decomposition method (SDM) and its modifications have been considered. Indeed, the above-mentioned procedure consists of sumudu transform technique in combination with the decomposition method. We recall the definition of the fractional derivative in the Caputo's sense. Hence, it shall be noted that the sequences in hand, as well as the sequences yielded by this study and discerned employing this technique, demonstrate a correct approach to a variety of equations involving fractional derivatives.

37. **Georgieva and Hristova (2020). [13]** This Paper explored the application of the Homotopy Analysis Method (HAM) solving two-dimensional non-linear Volterra-Fredholm fuzzy integral equations. It was published in Fractal and Fractional Journal; HAM was used to solve challenge integration and fuzziness of integral equations; mathematical expressions and examples were described. The outcomes also state that HAM has high performance when it comes to obtaining accurate and reliable solutions, which proves its applicability for expanding the utilization in the fuzzy mathematical modeling.

38. **Hasan and Sulaiman (2020). [16]** This review work focus on the analysis of HPM in solving a linear system of mixed Volterra-Fredholm integral equations with the approach of convergency analysis. This paper was published in the Baghdad Science Journal and underwent analysis of its theoretical framework and practicality of employing the HPM for such equations with regard to its reliability and efficiency as a method. Convergence criteria are stated in formulae from which the authors give actual calculation examples of how solid the method is. With their results it can be understood that with HPM is a powerful tool to solve the integral equations of efficient solutions and fast convergence. Therefore, the present

research is relevant since it aids numerical analysis and computational mathematics besides offering an efficient strategy for solving complex forms of such equations.

39. **Kapoor, M. (2020). [26]** To obtain accurate solutions of the coupled 1D nonlinear Burgers' equations, the Journal of Physics Communications in this paper uses the Homotopy Perturbation Method (HPM). In greater detail, this article further illustrates and proves the application of HPM in solving what may be considered as non solvable analytically nonlinear differential equations. The solutions are given in the form of power series and this confirms the convergence of HPM and its effectiveness. Kapoor finally, through two numerical examples, demonstrates that HPM is a suitable technique for solving coupled 1D nonlinear Burgers' equations and could be used in mathematical modeling and applied physics.

40. **Khan et.al (2020). [28]** In the paper, the authors propose a method called Laplace-Adomian Decomposition Method (LADM) for the non-linear systems of fractional order partial differential equations. This technique uses Laplace transform and ADM, in order to solve problems subjected to Caputo operator representing fractional derivatives, conveniently. It is demonstrated by more examples to prove that it has high efficiency and can approximate solutions very close to exact solutions. As per the results derived from this work, it reveals that LADM holds a great promise in simplification as well as in the solution-searching process connected with fractional order problems those are so precise in nature and used as a useful tool in several branches of science and engineering.

41. **Khan, A. A., and Akter, M. T. (2020). [27]** Talking about the applicability of Homotopy Perturbation Method (HPM) in solving highly nonlinear partial differential equations in this paper, the author discusses about how one can use HPM in solving such nonlinear partial differential equations. The authors found that the paper appeared in the American Journal of Applied Mathematics. That shows HPM is a combination of conventional perturbation methods and in topology, homotopy can solve these equations without involving any small parameters. The authors implemented HPM on burger's equation and compared it with Adomian Decomposition Method (ADM) where they dichotomy it is more efficient and easier to get solution.

42. **Qu and Liu (2020). [60]** In the paper, they use Homotopy Analysis (HAM) method on three types of fractional partial differential equations. These are the fractional Cauchy-Riemann equation, the fractional acoustic wave equation, and a two dimensional space-time fractional PDE. Regarding these equations, series solutions are determined using HAM by the authors and the authors demonstrate that it is possible to handle with the fractional derivatives in a flexible and efficient manner based on the Caputo sense fractional derivatives definition. In this article, in order to check the correctness and convergence of the suggested answers, some numerical problems have also been solved. By their work, it is clear that the wider applicability of HAM within the fields of mathematical physics as well as the engineering sciences has been highlighted based on the present work, which offers efficient analytic solutions of the tough differential equations coupled with fractions

43. **Singh and Sharma (2020). [74]** They offer a comparison of two approaches, namely HPTM and HPETM, that can be employed to address fractional nonlinear PDEs. As stated in this paper,the researcher employed both methods to solve Fractional Fisher's equation, time-fractional Fornberg-Whitham equation and time-fractional Inviscid Burgers' equation. They used Caputo sense for the fractional derivatives and realized that these two methods bring out series solutions that are precise, and converging faster, hence, fewer iterations would be required. Additionally, they mentioned that the consideration of Elzaki transform with HPM improves the rate of convergence and numerical effectiveness. The authors of the paper in consideration then conclude their paper by pointing out that while both the paradigms are enormously beneficial in solving a wide range of practical and theoretical problems, HPETM excels over HPTM for complex nonlinear PDEs with fractions to solve conjoined complicated nonlinear PDEs hence providing a clue as to useful paradigms of engineering as well as the science-related issue .

44. **Kumar et.al (2021). [34]** In this paper, the author inroduce about the theoretical basis for new Homotopy Perturbation Method (NHPM) and show how it is applied to linear as well as nonlinear forms of the Telegraph equation. We point out that this approach converts difficult differential equations into simpler ones which can either be solved analytically or semi-analytically. The authors demonstrate numerical reliability and accuracy of NHPM through several examples thereby indicating its potentiality as an efficient tool in mathematical physics and engineering for solving different types of differential equations too. They conclude that compared with conventional methods NHPM offers significant computational

savings while ensuring higher precision thus becoming an essential numerical technique for solving such equations.

45. **Kumar *et.al* (2021). [33]** In this research paper, the "time fractional nonlinear Schrödinger equation" was solved using Homotopy Analysis Method to find an analytical periodic and solitary wave solution. The reason why this approach is known to have a rate of quick convergence is that it uses a control parameter for convergence. Therefore, what we get is a series which converges rapidly towards the solution of time fractional differential equation . Then, the present obtained analytic solution has been established against previous outcomes where they are found to match well with each other. HAM can be easily employed and shows potential for efficient analysis of time-fractional PDEs as presented in this work.

46. **Mousa and Alsharari (2021). [52]** In this paper, the author presenting the HPM as the main concept, then showed how it can be used to solve certain types of nonlinear, integral, integro-differential and differential equations through convergence theorems. I also gave an error bound for approximate solutions and proved a theorem about it. So far we have only proven these HPM convergence theorems but also tried out different ways where this algorithm could be effective in solving various classes of non-linear integral/integro differential equations.

47. **Salih et.al (2021). [68]** The authors of this article provided accurate iterative methods to find the solutions of 1D, 2D and 3D Fisher's equation in the IIUM Engineering Journal. DJM, TAM and BCM were employed to obtain the solutions of these equations. They talked about such methods mentioning that they are easy and fast, and do not involve derivative calculation, or Lagrange multiplier. In this regard, they endeavored to give several examples for these approaches and ascertain the correctness and efficiency of them through error analysis and numerical result check. And thus, the present study provides robust solutions for mathematical physics nonlinear complicated issues.

48. **Jena, R.M et.al (2022). [25]** The present work is concerned with development of Homotopy Perturbation Elzaki Transform Method (HPETM) for solving random nonlinear partial differential equations. In this research, the authors concentrate on two type of distribution; normal distribution and uniform distribution as a way of illustrating that the method works under a variety of circumstances. The above-mentioned hybrid scheme is tested by some initial-value problems whose results are obtained using the MAPLE software with graphics view for them. From the results one can see that the number of steps for HPETM balances and gives highly accurate approximate solutions of random nonlinear PDEs.

49. **Ibraheem and Mahmood (2022). [21]** In recent years, nonlinear problems have various ways of being solved and this can be done using a well-known analytical tool commonly referred to as homotopy analysis method. In particular, there has been a lot of research on the homotopy analysis method for usual nonlinear problems. So, in order to improve the efficiency and solve partial differential equations with the help of homotopy method of analysis; this paper seeks to achieve two goals – firstly it presents an idea based on Pade approximations which can be used so as to make the necessary improvements in terms of efficiency according to this article's proposed method Secondly these improvements were verified after solving two cases each followed by calculating mean squared errors between HAM results (homotopy analysis) against those obtained when applying modified version where improvement is done through Pade approximation.

50. **Imran and Mehmood (2022). [22]** In this paper the author introduce the applies of the Homotopy Analysis Method (HAM) to solve systems of non-linear partial differential equations. This work shows that HAM is a good way to find exact solutions for complicated nonlinear PDE systems.The writers gave some cases in point to show how effective and simple HAM can be. They also tested its compatibility and accuracy with numerical results. In this context, the examples illustrate the capacity of HAM in giving exact answers which brings out its usefulness in solving difficult mathematical problems.

51. **Jasrotia and Singh (2022). [23]** In this paper ,the authors introduce the Accelerated Homotopy Perturbation Elzaki Transformation Method (AHPETM) for solving nonlinear partial differential equations. This method combines the accelerated homotopy perturbation method with the Elzaki transform in order to decrease computational effort and increase efficiency. By applying it to nonlinear PDEs, the authors show its effectiveness and demonstrate that computational speed has been significantly improved as well as accuracy. What they found out emphasizes AHPETM as a promising solution for difficult mathematical problems.

52. **Samajdar et.al (2022). [69]** In this paper the work that examines how a variety of the two-dimensional Laplace equation can be solved with the help of the HPM is presented. The study also concludes that HPM can solve Dirichlet or Neumann problems of Laplace's equation that relates to the steady state temperature distributions. The authors have shown the results of the comparison of HPM with Variable Separation Method revealing the coincident figures and higher efficiency and accuracy of HPM which suggests for its ability to solve complicated partial differential equations with high convergence.

53. **Buhe et.al (2023). [7]** As for the solution of this non-integer mathematical model, the paper utilizes the tool called Homotopy Perturbation Method (HPM). Finally, the research has offered an approximate solution based on HPM that is computationally effective in terms of a number of perturbation terms. Parameters which are taken into account relate to such aspects as the population density and the pressure on forest resources. This comparison are also made with numerical solutions performed by the Runge-Kutta 4th-order Method (RK-4). This therefore means, HPM can actually be applied suitably and competently in the sense of incorporating the essence of depletion of forest resources based on given parametric assumptions

54. **Chauhan and Arora (2023). [9]** The aim of this work is to establish the use of the Homotopy Analysis Method (HAM) for solving the Korteweg-de Vries (KdV) equations in the present work and give an approximately analyzing solution in the series power form. Thus, the convergence region of the solutions is enlarged with the help of the convergence control parameter h. The research paper is published in Communications in Mathematics. Hence, various graphical comparisons are made between the HAM-derived solutions and exact solutions to illustrate the viability of the method, and the absolute errors proved their correctness. All the above solutions were computed using MATHEMATICA, and it proves how the HAM can solve nonlinear KdV equations effectively.

55. **Pal et.al (2023). [56]** In this paper modifies the traditional method by making small changes which make it solve heat diffusion equations better, showing this through various examples. They stress how accurate and quickly converging answers can be found using their approach, especially in complex thermal problems. Engineers and scientists dealing with these types of equations will find this work very useful according to them. The New Homotopy Perturbation Method (HPM) should prove invaluable for engineers and scientists working with heat diffusivity problems, as concluded by the authors.

56. **Madhavi et.al (2024). [47]** In this paper, the authors presented a generalized HAM to approximate the solutions of q-fractional nonlinear differential equations. This paper builds on the conventional HAM and takes into account q-fractional differential equations and provides a solid foundation to obtain series solutions that converge faster. The authors also view the influence of different auxiliary parameters, functions or linear operators on the order of convergence. Moreover, they prove it with several examples to illustrate the fact that this method is accurate and efficient when dealing with those highly nonlinear and non-quadratic problems, therefore, its applicability with other nonlinear problems in complex mathematical modelling.

57. **Qayyum and Hussain (2024)**. **[57]** In this paper, the authors developed a method named as Homotopy Perturbation Laplace Method (HPLM) that can be used to solve boundary value problems. This consequently entails the application of the Laplace transform, combining it with the homotopy perturbation method for solving differential equations better than before. In their work, they also present a broad theoretical background, which, in addition, they illustrate with numerical examples explaining that applying this method can decrease computational costs and increase the precision of solutions compared with other numerical methods. The authors note that it is established here that the extension of this research work indicates the versatility of HPLM in both scientific and engineering disciplines.

58. **Qayyum et.al (2024). [59]** In this research, the authors explained A Modified Optimal Homotopy Asymptotic Method (MOHAM) for solving the KdV family of equations is presented in the International Journal of Emerging Multidisciplinary: Numeracy. The researchers recommend a procedure that is a mix of OHAM and DJ polynomials, and they execute on the (1+1) and (2+1) dimensional soliton KdV equations. They also claimed that this change enhances the accuracy of the result and reduces the number of computations as opposed to conventional OHAMs; as per their outcome, MOHAM is more efficient in handling complex processes, and as such, it may turn out to be efficacious tool for solving non-linear PDEs.

59. Turq et.al (2024). [78] The authors introduce the steps of the Laplace Homotopy Analysis Method (LHAM) in non-linear fractional models for evolution equations and heat of type to discuss that the LHAM is helpful in providing the approximate techniques for solving the involved fractional partial differential equations and it is very useful in obtaining the exact and rapidly convergent solutions. This explains how this method can be applied to multiple scientific and engineering applications as well. In this paper, different examples are given that show that LHAM has the potential ability to solve different scientific and engineering problems. As such, this approach incorporates the homotopy analysis method with the Laplace transform, whereby the convergence rate can be enhanced simultaneously with the solution accuracy, and thus, such an approach will be quite helpful to researchers working on nonlinear fractional models.

60. Wang et.al (2024). [80] An efficient numerical method is presented in this paper for the analysis of the 2-D and 3-D fractional nonlinear Schrödinger equation (FNLSE) with the help of the FCT method. They bring in the fractional Laplace operator by employing the spectral decomposition and discretizing the time by the Crank-Nicolson method. Mass and energy conservation in the context of the fully discrete scheme is also proved. Computational verifications prove the correctness and effectiveness of the technique, suggesting that for soliton modelling, the time of computation significantly decreases if FDCT is used.

VI. CONCLUSION

The Homotopy Analysis Method is a transformative tool for solving linear and nonlinear problems, offering flexibility and control unavailable in traditional methods. Its applications span numerous scientific and engineering domains, demonstrating its robustness and versatility. While challenges remain, ongoing advancements in theory, computation, and hybridization promise to further enhance its capabilities. HAM represents a valuable addition to the analytical toolbox for tackling complex linear and nonlinear systems.

REFERENCES

[1]. Abbas, Z. M. A. (2014). Homotopy analysis method for solving non-linear various problems of partial differential equations. Mathematical Theory and Modeling, 4(14), 113-126.

[2]. Abbasbandy, S. (2007). Homotopy analysis method for heat radiation equations. International Communications in Heat and Mass Transfer, 34(3), 380–387. https://doi.org/10.1016/j.icheatmasstransfer.2006.12.001.

[3]. Asli Alkan. (2016). Improving Homotopy Analysis Method with An Optimal Parameter for Time- Fractional Burgers Equation. KMU Journal of Engineering and Natural Sciences, 4(2), 117–134. https://doi.org/https://doi.org/10.55213/kmujens.1206517 Zaman-Kesirli.

[4]. Baleanu, D., & Jassim, H. K. (2020). Exact solution of two-dimensional fractional partial differential equations. Fractal and Fractional, 4(2), 1–9. https://doi.org/10.3390/fractalfract4020021.

[5]. Baskonus, H. M., Bulut, H., & Pandir, Y. (2014). On the Solution of Nonlinear Time Fractional Generalized Burgers Equation by Homotopy Analysis Method and Modified Trial Equation Method. International Journal of Modeling and Optimization, 4(4), 305–309. https://doi.org/10.7763/IJMO.2014.V4.390.

[6]. Benchohra, M., Cabada, A., & Henderson, J. (2013). Fractional differential equations and their applications. Nonlinear Studies, 20(4), 469–470. https://doi.org/10.20537/vm090101.

[7]. Buhe, E., Rafiullah, M., Jabeen, D., & Anjum, N. (2023). Application of homotopy perturbation method to solve a nonlinear mathematical model of depletion of forest resources. Frontiers in Physics, 11(October), 1–9. https://doi.org/10.3389/fphy.2023.1246884.

[8]. Chang, J. Y., Tsai, C. C., & Young, D. L. (2019). Homotopy method of fundamental solutions for solving nonlinear heat conduction problems. Engineering Analysis with Boundary Elements, 108(March), 179–191. https://doi.org/10.1016/j.enganabound.2019.08.004.

[9]. Chauhan, A., & Arora, R. (2023). Application of homotopy analysis method (HAM) to the non-linear KdV equations. Communications in Mathematics, 31(1), 205–220. https://doi.org/10.46298/cm.10336.

[10]. Chen, X., & Cui, W. (2012, August). The Homotopy Analysis Method to Solve Time Fractional Partial Differential Equations. In 2012 International Conference on Computer Science and Service System, 2012(11), 90-93.

[11]. Dinesha, D. L., & Gurrala, G. (2019). Application of multi-stage homotopy analysis method for power system dynamic simulations. IEEE Transactions on Power Systems, 34(3), 2251–2260. https://doi.org/10.1109/TPWRS.2018.2880605.

[12]. El-Tawil, M. A., & Huseen, S. N. (2013). On convergence of q-homotopy analysis method. International Journal of Contemporary Mathematical Sciences, 8(10), 481–497. https://doi.org/10.12988/ijcms.2013.13048.

[13]. Georgieva, A., & Hristova, S. (2020). Homotopy analysis method to solve two-dimensional nonlinear volterra-fredholm fuzzy integral equations. Fractal and Fractional, 4(1), 1–14. https://doi.org/10.3390/fractalfract4010009.

[14]. Ghazanfari, B., & Veisi, F. (2011). Homotopy analysis method for the fractional nonlinear equations. Journal of King Saud University - Science, 23(4), 389–393. https://doi.org/10.1016/j.jksus.2010.07.019.

[15]. Gorman, A. D. (2002). On the time-dependent parabolic wave equation. International Journal of Mathematics and Mathematical Sciences, 31(5), 291–299. https://doi.org/10.1155/S016117120210915X.

[16]. Hasan, P. M., & Sulaiman, N. A. (2020). Convergence analysis for the homotopy perturbation method for a linear system of mixed volterra-fredholm integral equations. Baghdad Science Journal, 17(3), 1010–1018. https://doi.org/10.21123/BSJ.2020.17.3(SUPPL.).1010.

[17]. He, J. H. (2000). Coupling method of a homotopy technique and a perturbation technique for non-linear problems. International Journal of Non-Linear Mechanics, 35(1), 37–43. https://doi.org/10.1016/S0020-7462(98)00085-7.

[18]. He, J. H. (2004). Comparison of homotopy perturbation method and homotopy analysis method. Applied Mathematics and Computation, 156(2), 527–539. https://doi.org/10.1016/j.amc.2003.08.008.

[19]. Hilton, P. (1953). An Introduction to Homotopy Theory (Cambridge Tracts in Mathematics). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511666278.

[20]. Hussein, N. A., & Tawfiq, L. N. M. (2023). Efficient Approach for Solving (2+1) D- Differential Equations. Baghdad Science Journal, 20(1), 166–174. https://doi.org/10.21123/bsj.2022.6541.

[21]. Ibraheem, K. I., & Mahmood, H. S. (2022). Algorithm for solving fractional partial differential equations using homotopy analysis method with Padé approximation. International Journal of Electrical and ComputerEngineering, 12(3), 3335–3342. https://doi.org/10.11591/ijece.v12i3.pp3335-3342.

[22]. Imran, N., & Mehmood Khan, R. (2022). Homotopy Analysis Method for Solving System of Non-Linear Partial Differential Equations. International Journal of Emerging Multidisciplinaries: Mathematics, 1(2), 35–48. https://doi.org/10.54938/ijemdm.2022.01.2.30.

[23]. Jasrotia, S., & Singh, P. (2022). Accelerated Homotopy Perturbation Elzaki Transformation Method for Solving Nonlinear Partial Differential Equations. Journal of Physics: Conference Series, 2267(1), 1–14. https://doi.org/10.1088/1742-6596/2267/1/012106.

[24]. Jena, R. M., & Chakraverty, S. (2019). Q-homotopy analysis Aboodh transform method based solution of proportional delay time-fractional partial differential equations. Journal of Interdisciplinary Mathematics, 22(6), 931-950.

[25]. Jena, R. M., Chakraverty, S., & Jena, S. K. (2020). Analysis of the dynamics of phytoplankton nutrient and whooping cough models with nonsingular kernel arising in the biological system. Chaos, Solitons & Fractals, 141, 110373.

[26]. Kapoor, M. (2020). Exact solution of coupled 1d non-linear burgers' equation by using homotopy perturbation method (HPM): A review. Journal of Physics Communications, 4(9), 1–6. https://doi.org/10.1088/2399-6528/abb218.

[27]. Khan, A. A., & Akter, M. T. (2020). Solving Highly Nonlinear Partial Differential Equations Using Homotopy Perturbation Method. American Journal of Applied Mathematics, 9(1), 334-343.

[28]. Khan, H., Shah, R., Kumam, P., Baleanu, D., & Arif, M. (2020). Laplace decomposition for solving nonlinear system of fractional order partial differential equations. Advances in Difference Equations, 2020(1), 1–18. https://doi.org/10.1186/s13662-020-02839-y.

[29]. Khan, K., & Akbar, M. A. (2014). Exact solutions of the (2+1)-dimensional cubic Klein-Gordon equation and the (3+1)-dimensional Zakharov-Kuznetsov equation using the modified simple equation method. Journal of the Association of Arab Universities for Basic and Applied Sciences, 15(1), 74–81. https://doi.org/10.1016/j.jaubas.2013.05.001.

[30]. Khan, N. A., Ara, A., Ali, S. A., & Mahmood, A. (2009). Analytical study of Navier-Stokes equation with fractional orders using He's homotopy perturbation and variational iteration methods. International Journal of Nonlinear Sciences and Numerical Simulation, 10(9), 1127-1134.

[31]. Khan, N. A., Ara, A., Ali, S. A., & Mahmood, A. (2009). Analytical study of Navier-Stokes equation with fractional orders using He's homotopy perturbation and variational iteration methods. International Journal of Nonlinear Sciences and Numerical Simulation, 10(9), 1127-1134.

[32]. Kumar, D., Singh, J., & Kumar, S. (2015). Numerical computation of fractional multi-dimensional diffusion equations by using a modified homotopy perturbation method. Journal of the Association of Arab Universities for Basic and Applied Sciences, 17, 20–26. https://doi.org/10.1016/j.jaubas.2014.02.002.

[33]. Kumar, M. (n.d.). Exact solutions of (1+2)-dimensional non-linear time-space fractional PDEs. Arab Journal of Mathematical Sciences, 2022(7), 1–13. https://doi.org/10.1108/ajms-11-2021-0282.

[34]. Kumar, R., Singh, A. K., & Yadav, S. S. (2021). New Homotopy Perturbation Method For Analytical Solution Of Telegraph Equation. Turkish Journal of Computer and Mathematics Education, 12(12), 2144–2155.

[35]. Lesnic, D. (2006). The decomposition method for linear, one-dimensional, time-dependent partial differential equations. International Journal of Mathematics and Mathematical Sciences, 2006(December 2005), 1–29. https://doi.org/10.1155/IJMMS/2006/42389.

[36]. Li, J. L. (2009). Adomian's decomposition method and homotopy perturbation method in solving nonlinear equations. Journal of Computational and Applied Mathematics, 228(1), 168–173. https://doi.org/10.1016/j.cam.2008.09.007.

[37]. Li, J. L. (2009). Adomian's decomposition method and homotopy perturbation method in solving nonlinear equations. Journal of Computational and Applied Mathematics, 228(1), 168-173.

[38]. Liao, S. (2004). On the homotopy analysis method for nonlinear problems. Applied Mathematics and Computation, 147(2), 499–513. https://doi.org/10.1016/S0096-3003(02)00790-7.

[39]. Liao, S. (2005). Comparison between the homotopy analysis method and homotopy perturbation method. Applied Mathematics and Computation, 169(2), 1186–1194. https://doi.org/10.1016/j.amc.2004.10.058.

[40]. Liao, S. (2009). Notes on the homotopy analysis method: Some definitions and theorems. Communications in Nonlinear Science and Numerical Simulation, 14(4), 983–997. https://doi.org/10.1016/j.cnsns.2008.04.013.

[41]. Liao, S. (2011). Homotopy analysis method in nonlinear differential equations. Homotopy Analysis Method in Nonlinear Differential Equations, 9783642251(3), 1–565. https://doi.org/10.1007/978-3-642-25132-0

[42]. Liao, S.J. (2003) Beyond Perturbation: Introduction to Homotopy Analysis Method. Chapman and Hall, CRC Press,1-336 http://dx.doi.org/10.1201/9780203491164.

[43]. Liao, Shijun. (2011). Homotopy Analysis Method in Nonlinear Differential Equations. 9783642251: 1–565.

[44]. Liu, C. shi. (2010). The essence of the homotopy analysis method. Applied Mathematics and Computation, 216(4), 1299–1303. https://doi.org/10.1016/j.amc.2010.02.022.

[45]. Logan, J. D. (1994). An Introduction to Nonlinear Partial Differential Equations. European Journal of Engineering Education, 19(4), 515. https://doi.org/10.1080/03043799408928308.

[46]. M. Matinfar, M. Saeidy, B. G. (2012). A New Homotopy Analysis Method for Finding the Exact Solution of Systems of Partial Differential Equations. Selçuk Journal of Applied Mathematics, 13(1), 41–56.

[47]. Madhavi, B., Kumar, G. S., Nagalakshmi, S., & Rao, T. S. (2024). Generalization of Homotopy Analysis Method for q-Fractional Non-linear Differential Equations. International Journal of Analysis and Applications, 22(4), 1–12. https://doi.org/10.28924/2291-8639-22-2024-22.

[48]. Maitama, S., & Abdullahi, I. (2016). A new analytical method for solving linear and nonlinear fractional partial differential equations. Progress in Fractional Differentiation and Applications, 2(4), 247–256. https://doi.org/10.18576/pfda/020402.

[49]. Mesloub, S., & Gadain, H. E. (2024). Homotopy Analysis Transform Method for a Singular Nonlinear Second-Order Hyperbolic Pseudo-Differential Equation. Axioms, 13(6), 398. https://doi.org/10.3390/axioms13060398.

[50]. Mesloub, S., & Obaidat, S. (2019). Homotopy analysis method for a fractional order equation with dirichlet and non-local integral conditions. Mathematics, 7(12), 1–18. https://doi.org/10.3390/MATH7121167.

[51]. Moazzzam, A., Ayza Anjum, Nimra Saleem, & Emad A. Kuffi. (2023). Study of Telegraph Equation via He-Fractional Laplace Homotopy Perturbation Technique. Ibn AL-Haitham Journal For Pure and Applied Sciences, 36(3), 349–364. https://doi.org/10.30526/36.3.3239.

[52]. Mousa, M. M., & Alsharari, F. (2021). Convergence and error estimation of a new formulation of homotopy perturbation method for classes of nonlinear integral/integro-differential equations. Mathematics, 9(18), 1–16. https://doi.org/10.3390/math9182244.

[53]. Nasabzadeh, H., & Toutounian, F. (2013). Convergent Homotopy Analysis Method for Solving Linear Systems. Advances in Numerical Analysis, 2013(4), 1–6. https://doi.org/10.1155/2013/732032.

[54]. Odibat, Zaid M. (2010). A Study on the Convergence of Homotopy Analysis Method. Applied Mathematics and Computation 217(2), 782–89. http://dx.doi.org/10.1016/j.amc.2010.06.017.

[55]. Özpınar, F. (2018). Applying Discrete Homotopy Analysis Method for Solving Fractional Partial Differential Equations. Entropy, 20(5), 332.https://doi.org/10.3390/e20050332.

[56]. Pal, K., Gupta, V. G., Singh, H., & Pawar, V. (2023). Enlightenment of Heat Diffusion Using New Homotopy Perturbation Method. Journal of Applied Science and Engineering (Taiwan), 27(3), 2211–2214. https://doi.org/10.6180/jase.202403_27(3).0007.

[57]. Qayyum, M., & Hussain, K. (2024). Homotopy Perturbation Laplace Method for Boundary Value Problems. International Journal of Emerging Multidisciplinaries: Mathematics, 3(1), 1–6. https://doi.org/10.59790/2790-3257.1046.

[58]. Qayyum, M., Ahmad, E., Tauseef Saeed, S., Ahmad, H., & Askar, S. (2023). Homotopy perturbation method-based soliton solutions of the time-fractional (2+1)-dimensional Wu–Zhang system describing long dispersive gravity water waves in the ocean. Frontiers in Physics, 11(6), 2–13. https://doi.org/10.3389/fphy.2023.1178154.

[59]. Qayyum, M., Faisal, M., & Imran, N. (2024). Modified Optimal Homotopy Asymptotic Method for KdV Family of Equations. International Journal of Emerging Multidisciplinaries: Mathematics, 3(1), 1–6. https://doi.org/10.59790/2790-3257.1048

[60]. Qu, H., She, Z., & Liu, X. (2020). Homotopy Analysis Method for Three Types of Fractional Partial Differential Equations. Hindawi Complexity, 2020(7), 1–13. https://doi.org/10.1155/2020/7232907.

[61]. Raftari, B., Khosravi, H., & Yildirim, A. (2013). Homotopy analysis method for the one-dimensional hyperbolic telegraph equation with initial conditions. International Journal of Numerical Methods for Heat and Fluid Flow, 23(2), 355–372. https://doi.org/10.1108/09615531311293515.

[62]. Rathore, S., & Saraswat, G. K. (2018). Homotopy Perturbation Approach to the Solution of Non-linear Burger's Equation. International Journal of Mathematics And Its Applications, 6(4), 203–211. http://ijmaa.in.

[63]. Rehman, H. U. R., Saleem, M. S., & Ahmad, A. (2018). Combination of homotopy perturbation method (hpm) and double sumudu transform to solve fractional kdv equations. Open J. Math. Sci., 2(1), 29–38.

[64]. S, D. B., Tg, G., Sangeetha, M., & Sm, A. (2024). Solving Linear Heat Equation And Wave Equation Using Homotopy Perturbation Laplace-Carson Transform Method. Journal of Computational Mathematics, 8(1), 94–104. https://doi.org/https://doi.org/10.26524/cm190.

[65]. Sajid, M., & Hayat, T. (2008). Comparison of HAM and HPM methods in nonlinear heat conduction and convection equations. Nonlinear Analysis: Real World Applications, 9(5), 2296–2301. https://doi.org/10.1016/j.nonrwa.2007.08.007.

[66]. Sakar, M. G., & Erdogan, F. (2013). The homotopy analysis method for solving the time-fractional Fornberg – Whitham equation and comparison with Adomian' s decomposition method. Applied Mathematical Modelling, 37(20–21), 8876–8885. https://doi.org/10.1016/j.apm.2013.03.074.

[67]. Sakar, M. G., Uludag, F., & Erdogan, F. (2016). Numerical solution of time-fractional nonlinear PDEs with proportional delays by homotopy perturbation method. Applied Mathematical Modelling, 40(13-14), 6639-6649.

[68]. Salih, O. M., & Al-Jawary, M. A. (2021). Reliable Iterative Methods for Solving 1d, 2d And 3d Fisher's Equation. In IIUM Engineering Journal (Vol. 22, Issue 1, pp. 138–166). https://doi.org/10.31436/IIUMEJ.V22I1.1413.

[69]. Samajdar, S., Khandakar, M. H., Purkait, A., Das, S., & Sen, B. (2022). The Technique Homotopy Perturbation Method Operated on Laplace Equation. In Asian Journal of Science and Applied Technology (Vol. 11, Issue 2, pp. 13–16). https://doi.org/10.51983/ajsat-2022.11.2.3295.

[70]. Shah, K., Seadawy, A. R., & Arfan, M. (2020). Evaluation of one dimensional fuzzy fractional partial differential equations. Alexandria Engineering Journal, 59(5), 3347-3353.

[71]. Shateyi, S., Motsa, S. S., Sibanda, P., & Marewo, G. T. (2010). A note on improved homotopy analysis method for solving the Jeffery-Hamel flow. Mathematical Problems in Engineering, 2010. https://doi.org/10.1155/2010/359297.

[72]. Singh, B. K., & Kumar, P. (2017). Fractional Variational Iteration Method for Solving Fractional Partial Differential Equations with Proportional Delay. International Journal of Differential Equations, 2017(13), 1–12.

[73]. Singh, G., & Singh, I. (2022). A Comprehensive Review on Several Techniques for Solving 2D and 3D Partial Differential Equations. International Journal of Research Publication and Reviews, 3(9), 469–476. https://doi.org/10.55248/gengpi.2022.3.9.1.

[74]. Singh, P., & Sharma, D. (2020). Comparative study of homotopy perturbation transformation with homotopy perturbation Elzaki transform method for solving nonlinear fractional PDE. Nonlinear Engineering, 9(1), 60–71. https://doi.org/10.1515/nleng-2018-0136.

[75]. Singh, R., Maurya, D. K., & Rajoria, Y. K. (2020). A mathematical model to solve the nonlinear Burger's equation by Homotopy perturbation method. Mathematics in Engineering, Science and Aerospace, 11(1), 115–125.

[76]. Srivastava, H. M., Golmankhaneh, A. K., Baleanu, D., & Yang, X. J. (2014). Local fractional Sumudu transform with application to IVPs on cantor sets. Abstract and Applied Analysis, 2014(26), 1–8. https://doi.org/10.1155/2014/620529.

[77]. Turkyilmazoglu, M. (2010). Convergence of the homotopy analysis method. Mathematical Physics, 2010(1), 1–12. http://arxiv.org/abs/1006.4460.

[78]. Turq, S. M., Nuruddeen, R. I., & Nawaz, R. (2024). Recent advances in employing the Laplace homotopy analysis method to nonlinear fractional models for evolution equations and heat-typed problems. International Journal of Thermofluids, 22(3), 1–10. https://doi.org/10.1016/j.ijft.2024.100681.

[79]. Wang, F., & Hou, E. (2020). A Direct Meshless Method for Solving Two-Dimensional Second-Order Hyperbolic Telegraph Equations. Journal of Mathematics, 2020(10), 1–9. https://doi.org/10.1155/2020/8832197.

[80]. Wang, P., Peng, S., Cao, Y., & Zhang, R. (2024). The Conservative and Efficient Numerical Method of 2-D and 3-D Fractional Nonlinear Schrödinger Equation Using Fast Cosine Transform. Mathematics, 12(7), 1–14. https://doi.org/10.3390/math12071110

[81]. Yang, X. (2022). The Application of Homotopy Perturbation Method: An Overview. Asian Research Journal of Mathematics, 18(2), 27–35. https://doi.org/10.9734/arjom/2022/v18i230357.