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Posteriori Error Estimation for Eigenvalue Problems Using the Mixed Finite Element Method

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***ABSTRACT**: The general second-order elliptic eigenvalue problem is of great significance and is closely related to various fields such as fluid mechanics, quantum mechanics, and structural engineering analysis. Classical finite element methods have been successfully applied to solve such problems, but in some cases, particularly when dealing with complex boundary conditions and non-homogeneous media, the efficiency and accuracy of traditional methods may not meet the requirements. To improve both the solutionaccuracy, the mixed finite element method has been proposed and has achieved significant results in solving secondorder eigenvalue problems. The mixed finite element method introduces auxiliary variables (which generally also have practical physical significance),allowing for a reduction in the order of high-order differential equations, thereby relaxing the smoothness requirements of the finite element space. This paper uses the mixed finite element method to study general secondorder eigenvalue problems, and by introducing asymptotically exact a-posterior error indicators through low-order interpolation, it provides a complete posterior error estimate for the method. The performance of this indicator is verified in an adaptive mesh refiner.*
***KEYWORDS**: Second-Order eigenvalue problems,Mixedfinite element method,A-posterior error, adaptive.*

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I. INTRODUCTION

The second-order eigenvalue problem is widely applied in various fields such as vibration analysis, material mechanics, acoustics, and quantum mechanics. In the classical finite element method, eigenvalue problems are typically solved by discretizing the differential operators being approximated. However, these methods often rely on the direct solution of higher-order differential equations, making the treatment of boundary conditions, material inhomogeneity, and complex geometries relatively challenging. Especially for high-frequency problems with irregular boundaries, traditional methods may struggle to meet the demands of both computational efficiency and accuracy.

As an advanced branch of the finite element method, the mixed finite element method was initially established by Babuška and Brezzi in the early 1970s, who developed the general theory of the method[1, 2]. In the early 1980s, Falk and Osborn proposed an improved version of the method[3].[4,5,6,7]provides extensive research on mixed problems, presenting numerous mixed finite element formulations, and further investigates the theoretical development and practical applications of the mixed finite element method.

There are several works for secondorder elliptic eigenvalue problems by the mixed formulation and their numerical methodssuch as Babuška and Osborn[8,9], Mercier, Osborn, Rappaz, and Raviart[11], etc. Based on thegeneral theory of compact operators[10],Osborn[12],Mercier,Osborn, Rappaz, and Raviart[11] give abstract analysis for the eigenpair approximationsby mixed/hybrid finite element methods. [13]discusses the $L^2(\Omega)$ norm and L^∞ norm estimates of eigenvalues and eigenfunctions for a more general class of eigenvalue problems.[14]propose a method to improve the convergence rate of the lowest order Raviart–Thomas mixed finite element approximations for the second order elliptic eigenvalue problem.[15]based on a class of super-convergence results for eigenfunction approximations, a residualtype a posteriori error estimator for the mixed finite element method in solving general second-order elliptic eigenvalue problems is derived and analyzed.[16]proposes a non-standard mixed finite element method for the Dirichlet boundary value problem of second-order elliptic equations.

This paperusing the ideas of the references above. This paperapplies the two-dimensional mixed finite element method to solve the secondorder elliptic eigenvalue problem. To effectively reduce the computational cost while maintaining good solution accuracy,low-order interpolation is proposed and analyzed, and an asymptotically exact a-posterior error indicator is established. Numerical results demonstrate that, while ensuring computational efficiency, the proposed method significantly improves the solution accuracy and is suitable for solving secondorder elliptic eigenvalue problems on complex geometries and irregular meshes.

In the entire paper, C denotes a general constant that is independent of the mesh sizebut sometimes depend on the eigenvalues of the problem(1).

II. BASIC THEORETICAL PREPARATION

Let $L^q(\Omega)$ bea standard Lebesgue space,where $1 \le q \le \infty, \Omega \subset R^2$,Thecorresponding norm is expressed by $\|\cdot\|_{L^q(\Omega)}$ .In this paper,the norm of $L^q(\Omega)$ is represented by $\|\cdot\|_{\Omega}$ .We also use $H^s(\Omega)$ to express the standard Hilbert Sobolev space of real functions defined at $\Omega \subset R^2$ with index $s \ge 0$ and the corresponding norm and semi-norm are $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ .Let Ω be thebounded open polygon regionof R^2 ,and let $\partial\Omega$ represent its boundary. In this paper, we are concernedwith the following second order elliptic eigenvalue problem:

$$\begin{cases} -\nabla \cdot (K(x,y)\,\nabla p) = \lambda p, & \text{in } \Omega, \\ p = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $K = (a_{ij})_{2\times 2}$ is a symmetric positive definite matrix with $a_{ij} \in W^{1,\infty}(\Omega)$ for $1 \le i, j \le 2$, $K^{-1} = (a_{ij})_{2\times 2}$ is also a symmetric positive definite matrix, $\Omega \subset R^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$, ∇ and $\nabla\cdot$ denote the gradient and divergence operators.

III. MIXED FINITE ELEMENT METHOD

We define a new vector-valued function $\boldsymbol{\mu} = K\nabla p$.

Then(1)can be transformed into the following equivalent formulation

$$\begin{cases} K^{-1}\boldsymbol{\mu} - \nabla p = 0, & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\mu} = \lambda p, & \text{on } \partial\Omega, \\ p = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Next, define the spaces

$$W = L^2(\Omega)\,,\; G = L^2(\Omega)\,,\; \mathbf{H} = [L^2(\Omega)]^2\,,$$

$$\mathbf{V} = H(div,\Omega) = \left\{ \boldsymbol{\mu} \in [L^2(\Omega)]^2 : \nabla\cdot\boldsymbol{\mu} \in L^2(\Omega) \right\},$$

equipped with the norm

$$\|\boldsymbol{\mu}\|^2_{H(div,\Omega)} = (\|\boldsymbol{\mu}\|_0^2 + \|\nabla\cdot\boldsymbol{\mu}\|_0^2)\,.$$

Then,the weak form for the problem(1) can be defined as follows:

Find $(\lambda, \boldsymbol{\mu}, p) \in R \times \mathbf{V} \times W$, $(\boldsymbol{\mu}, p) \ne (0,0)$,suchthat

$$\begin{cases} a(\boldsymbol{\mu}, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, p) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}, \\ b(\boldsymbol{\mu}, v) = \lambda(p, v), & \forall v \in W, \end{cases} \tag{3}$$

where $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are bilinear forms defined by

$$a(\boldsymbol{\mu}, \boldsymbol{\varphi}) = \int_\Omega \boldsymbol{\mu} \cdot K^{-1} \boldsymbol{\varphi}\, dx\,,\; b(\boldsymbol{\varphi}, p) = -\int_\Omega div\boldsymbol{\varphi} \cdot p\,dx\,,\; (p,v) = \int_\Omega pv\,dx\,.$$

Clearly,the bilinear forms $a(\cdot,\cdot)$ issymmetricand the bilinear forms defined above have the following characteristics:

$$|a(\boldsymbol{\mu}, \boldsymbol{\mu})| \le C_0 \|\boldsymbol{\mu}\|_{\mathbf{H}} \|\boldsymbol{\mu}\|_{\mathbf{H}}\,, \tag{4}$$

$$|a(\boldsymbol{\mu}, \boldsymbol{\varphi})| \le C_1 \|\boldsymbol{\mu}\|_{\mathbf{H}} \|\boldsymbol{\varphi}\|_{\mathbf{H}}\,, \tag{5}$$

$$|\mathrm{b}(\boldsymbol{\varphi}, p)| \le C_2 \|\boldsymbol{\varphi}\|_{\mathbf{V}} \|p\|_W\,, \tag{6}$$

where $C_i\,(i = 0,1,2)$ represents a constant independent of h .

For the eigenvalue λ ,there exists the following Rayleigh quotient expression

$$\lambda = \frac{-a(\boldsymbol{\mu}, \boldsymbol{\mu}) + 2b(\boldsymbol{\mu}, p)}{(p,p)}\,.$$

Form[9],we know eigenvalue problem(3)has an eigenvalue sequence $\left\{\lambda_j\right\}$:

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \lim_{k\to\infty} \lambda_k = \infty\,,$$

and the associated eigenfunctions

$$(\boldsymbol{\mu}_1, p_1), (\boldsymbol{\mu}_2, p_2), \cdots (\boldsymbol{\mu}_k, p_k) \cdots,$$

where $(p_i, p_j) = \delta_{ij}$.

Theorem1. Let (λ, p) be an eigenpair of equation (1), $\boldsymbol{\mu} = K\nabla p$,then $(\lambda, \boldsymbol{\mu}, p)$ satisfies equation (3); if $(\lambda, \boldsymbol{\mu}, p)$ satisfies equation (3), then (λ, p) is an eigenpair of equation(1), and $\boldsymbol{\mu} = K\nabla p$.

Proof. From the derivation above, the first part of the theorem has been established. Now we will prove the second part of the theorem.

Let $(\lambda, \boldsymbol{\mu}, p)$ satisfy equation (3), and consider the auxiliary problem

$$\begin{cases} -\nabla \cdot (K(x,y)\nabla \tilde{p}) = \lambda p, & \text{in } \Omega, \\ \tilde{p} = 0, & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Let $\tilde{\boldsymbol{\mu}} = K\nabla \tilde{p}$,then the mixed variational form of(7)is：

Find $(\lambda, \tilde{\boldsymbol{\mu}}, \tilde{p}) \in R \times \mathbf{V} \times W$,such that

$$\begin{cases} a(\tilde{\boldsymbol{\mu}}, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, \tilde{p}) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}, \\ b(\tilde{\boldsymbol{\mu}}, v) = \lambda(p, v), & \forall v \in W. \end{cases} \tag{8}$$

From the subtraction of (3) and(8), we get:find $(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, p - \tilde{p}) \in \mathbf{V} \times W$,such that

$$\begin{cases} a(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, p - \tilde{p}) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}, \\ b(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, v) = 0, & \forall v \in W. \end{cases} \tag{9}$$

Take form(9),let $\boldsymbol{\varphi} = \boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}$, $v = p - \tilde{p}$,then

$$\begin{cases} a(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, \boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}) - b(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, p - \tilde{p}) = 0 \\ b(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, p - \tilde{p}) = 0 \end{cases}$$

Add the above two equations, and we get $a(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, \boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}) = 0$,this illustrates $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}}$.

Substitute $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}}$ into the first equation of (9), and we get $b(\boldsymbol{\varphi}, p - \tilde{p}) = 0$,i.e $\int_\Omega (p - \tilde{p}) \cdot div\boldsymbol{\varphi} dx = 0, \forall \boldsymbol{\varphi} \in \mathbf{V}$.

Take ω satisfied $\Delta\omega = p - \tilde{p}$, and let $\boldsymbol{\varphi} = \nabla\omega$,then by $div\boldsymbol{\varphi} = p - \tilde{p}$, pushed $p = \tilde{p}$.

This proves (λ, p) is aneigenpair ofequation(1), and $\boldsymbol{\mu} = K\nabla p$.

We complete the proof.

Now,let's define the mixed finite element approximations of the problem(3).Let $\mathfrak{I}_h$ be a partition of Ω into finite elements(triangles),which is regular and has a mesh size h .Associated with the partition $\mathfrak{I}_h$,we define the finite dimensional spaces W_h and $\mathbf{V}_h$ (see[4]),where for any $\kappa \in \mathfrak{I}_h$, $P_n(\kappa)(n \geq 0)$ denotes the spaces of polynomial of degree not greater than n on κ .

Define

$$\mathbf{V}_h^L = \left\{ q_h \in \mathbf{V} \cap C^0(\bar{\Omega})^2 : q_h |_\kappa \in P_1(\kappa)^2, \forall \kappa \in \mathfrak{I}_h \right\},$$

for each $\kappa \in \mathfrak{I}_h$, and the barycentric coordinates $\lambda_1 \lambda_2 \lambda_3$ on κ ,define

$$B = (span\left\{ \lambda_1 \lambda_2 \lambda_3 \lambda_j : \kappa \in \mathfrak{I}_h, j = 1,2,3 \right\})^2,$$

and

$$\mathbf{V}_h = \mathbf{V}_h^L \oplus B.$$

Apparently, we have $\mathbf{V}_h^L \subset \mathbf{V}_h \subset \mathbf{V}$.

Afterward,define

$$W_h = \left\{ v_h \in H_0^1(\Omega) \cap C^0(\bar{\Omega}) : v_h |_\kappa \in P_2(\kappa), \forall \kappa \in \mathfrak{I}_h \right\}.$$

Apparently, we have $W_h \subset W$.

With the discrete spaces defined above,the mixed finite element approximation of (3) is given by:

Find $(\lambda_h, \boldsymbol{\mu}_h, p_h) \in R \times \mathbf{V}_h \times W_h$, $(\boldsymbol{\mu}_h, p_h) \neq (0,0)$,such that

$$\begin{cases} a(\boldsymbol{\mu}_h, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, p_h) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}_h, \\ b(\boldsymbol{\mu}_h, v) = \lambda_h(p_h, v), & \forall v \in W_h. \end{cases} \tag{10}$$

For the eigenvalue λ_h ,there exists the following Rayleigh quotient expression

$$\lambda_h = \frac{-a(\boldsymbol{\mu}_h, \boldsymbol{\mu}_h) + 2b(\boldsymbol{\mu}_h, p_h)}{(p_h, p_h)}.$$

Form[9] theeigenvalue problem(10)has eigenvalues

$$0 \leq \lambda_{1,h} \leq \cdots \lambda_{k,h} \leq \cdots \leq \lambda_{N,h},$$

and the corresponding eigenfunctions

$$(\boldsymbol{\mu}_{1,h}, p_{1,h}),(\boldsymbol{\mu}_{2,h}, p_{2,h}),\cdots(\boldsymbol{\mu}_{k,h}, p_{k,h})\cdots(\boldsymbol{\mu}_{N,h}, p_{N,h}),$$

where $(p_{i,h}, p_{j,h}) = \delta_{ij}, 1 \le i, j \le N, N = \dim W_h$.

For any $f \in L^2(\Omega)$,consider the boundary value problem corresponding to(3)and its mixed finite element approximation: find $(\boldsymbol{\sigma}, u) \in \mathbf{V} \times W, (\boldsymbol{\sigma}, u) \ne (0,0)$,such that

$$\begin{cases} a(\boldsymbol{\sigma}, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, u) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}, \\ b(\boldsymbol{\sigma}, v) = (f, v), & \forall v \in W. \end{cases} \tag{11}$$

find $(\boldsymbol{\sigma}_h, u_h) \in \mathbf{V}_h \times W_h, (\boldsymbol{\sigma}_h, u_h) \ne (0,0)$,such that

$$\begin{cases} a(\boldsymbol{\sigma}_h, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, u_h) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}_h, \\ b(\boldsymbol{\sigma}_h, v) = (f, v), & \forall v \in W_h. \end{cases} \tag{12}$$

IV. OPERATORFORMANDITSPROPERTIES

For any $f \in L^2(\Omega)$, assume that (11) has a unique solution $(\boldsymbol{\sigma}, u)$, and since $\mathbf{V}_h \subset \mathbf{V}, W_h \subset W$, it is known that (12)has a unique solution $(\boldsymbol{\sigma}_h, u_h)$.Thus, a linear bounded operator can be defined

$$T: G \to W \subset G, Tf = u.\ T_h: G \to W_h \subset G, T_h f = u_h.$$

$$\mathbf{S}: G \to \mathbf{V} \subset \mathbf{H}, \mathbf{S}f = \boldsymbol{\sigma}.\ \mathbf{S}_h: G \to \mathbf{V}_h \subset \mathbf{H}, \mathbf{S}_h f = \boldsymbol{\sigma}_h.$$

Thus, the eigenvalue problems (3) and (10) have equivalent operator forms, respectively.

$$\begin{cases} \lambda T p = p \\ \mathbf{S}(\lambda p) = \boldsymbol{\mu} \end{cases} \tag{13}$$

$$\begin{cases} \lambda_h T_h p_h = p_h \\ \mathbf{S}_h(\lambda_h p_h) = \boldsymbol{\mu}_h \end{cases} \tag{14}$$

Therefore, solving for the eigenpair of (3) for $(\lambda, \boldsymbol{\mu}, p)$ can be reduced to solving for the eigenpair of the operator T for $((\tau = \lambda^{-1}), p)$ and $\boldsymbol{\mu} = \mathbf{S}(\lambda p)$; similarly,solving for the eigenpair of (10) for $(\lambda_h, \boldsymbol{\mu}_h, p_h)$ can be reduced to solving for the eigenpair of the operator T_h for $((\tau_h = \lambda_h^{-1}), p_h)$ and $\boldsymbol{\mu}_h = \mathbf{S}_h(\lambda_h p_h)$.

For the linear bounded operators T and $\mathbf{S}$ defined in above,for any $f \in L^2(\Omega)$, the following relations hold:

$$\begin{cases} a(\mathbf{S}f, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, Tf) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}, \\ b(\mathbf{S}f, v) = (f, v), & \forall v \in W. \end{cases} \tag{15}$$

For this elliptic problem, the following regularity estimate holds

$$\| Tf \|_{1+r_0} \le C \| f \|_0 .$$

Where $\frac{1}{2} < r_0 \le 1$, depends on the shape of the domain.

For the discrete version of the linear bounded operators T_h and $\mathbf{S}_h$ defined in above, for any $f \in L^2(\Omega)$, thefollowing relationshold:

$$\begin{cases} a(\mathbf{S}_h f, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, T_h f) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}_h, \\ b(\mathbf{S}_h f, v) = (f, v), & \forall v \in W. \end{cases} \tag{16}$$

Lemma1.(Lemma1 in[13]) T and T_h are self-adjoint operators.

Proof. For any $g \in L^2(\Omega)$,let $\boldsymbol{\sigma} = \mathbf{S}g, u = Tg$,similarly, we have

$$\begin{cases} a(\mathbf{S}g, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, Tg) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}, \\ b(\mathbf{S}g, v) = (g, v), & \forall v \in W. \end{cases} \tag{17}$$

Bytaking $\boldsymbol{\varphi} = \mathbf{S}g, v = Tg$ in(15), we get

$$\begin{cases} a(\mathbf{S}f, \mathbf{S}g) - b(\mathbf{S}g, Tf) = 0 \\ b(\mathbf{S}f, Tg) = (f, Tg) \end{cases} \tag{18}$$

Bytaking $\boldsymbol{\varphi} = \mathbf{S}f, v = Tf$ in(17), we get

$$\begin{cases} a(\mathbf{S}g, \mathbf{S}f) - b(\mathbf{S}f, Tg) = 0 \\ b(\mathbf{S}g, Tf) = (g, Tf) \end{cases} \tag{19}$$

From the symmetry of $a(\cdot,\cdot)$,(18)and (19),we can obtain

$$(f,Tg)_G = b(\mathbf{S}f,Tg) = a(\mathbf{S}g,\mathbf{S}f) = a(\mathbf{S}f,\mathbf{S}g) = b(\mathbf{S}g,Tf) = (g,Tf)_G.$$

Thus, T is self-adjoint; similarly, it can be proven that T_h is self-adjoint.

V. A PRIORI ERROR ESTIMATEFOR EIGENFUNCTIONS

Lemma2. For any $\boldsymbol{\varphi}_h \in z_h$,there exists a constant α independent of h , such that the following inequality holds:

$$a(\boldsymbol{\varphi}_h,\boldsymbol{\varphi}_h) \gtrsim \alpha \|\boldsymbol{\varphi}_h\|_{\mathbf{H}}^2, \forall \boldsymbol{\varphi}_h \in Z_h , \tag{20}$$

where $Z_h = \{\boldsymbol{\varphi}_h \in \mathbf{V}_h : b(\boldsymbol{\varphi}_h, v_h) = 0, \forall v_h \in W_h\}$

Proof. The property (20) is obvious.

Lemma3.(Lemma 2.27 in[4])For any $\boldsymbol{\varphi} \in \mathbf{V}$ and $v \in H_0^1(\Omega)$,both

$$(\mathrm{div}\boldsymbol{\varphi}, v) = -(\boldsymbol{\varphi}, \nabla v) , \tag{21}$$

where $(\cdot,\cdot)$ represents the inner product of $L^2(\Omega)^2$.

Proof. The property (21) can be proven using the divergence theorem.

Corollary 1. For any $u \subset H^3(\Omega)$, we have

$$| u - \rho_h u |_1 \lesssim h^t | u |_{1+t}, 1 \le t \le 2 , \tag{22}$$

where $\rho_h : H^3(\Omega) \to W_h$ is the L^2 -projection operator.

Proof.see[4].

Let $Q_h : \mathbf{V} \to \mathbf{V}_h^L$ be the L^2 -projection, such that for any $\boldsymbol{\varphi} \in \mathbf{V}$, it holds that

$$(\boldsymbol{\varphi} - Q_h\boldsymbol{\varphi}, \boldsymbol{\varphi}_h) = 0, \forall \boldsymbol{\varphi}_h \in \mathbf{V}_h^L , \tag{23}$$

then

$$\|\boldsymbol{\varphi} - Q_h\boldsymbol{\varphi}\|_0 \lesssim h^k |\boldsymbol{\varphi}|_k , \tag{24}$$

where $\boldsymbol{\varphi} \in [H^k(\Omega)]^2, 0 \le k \le 2$.

Define theoperator $r_h : \mathbf{V} \to \mathbf{V}_h$,such that for any $\boldsymbol{\varphi} \in \mathbf{V}$,it holds that $r_h\boldsymbol{\varphi}|_\kappa = Q_h\boldsymbol{\varphi}|_\kappa + \sum_{j=1}^{3}\alpha_j\lambda_1\lambda_2\lambda_3\lambda_j, \forall \kappa \in \mathfrak{I}_h$,

where Q_h is defined by(23), and $\alpha_j \in R^2 (j = 1,2,3)$ is an undetermined constant vector.

Assume that there exists an operator $r_h : \mathbf{V} \to \mathbf{V}_h$,such that for any $\boldsymbol{\varphi} \in \mathbf{V}$,it holds that

$$b(\boldsymbol{\varphi} - r_h\boldsymbol{\varphi}, v_h) = 0, \forall v_h \in W_h , \tag{25}$$

for any $v_h \in W_h$,since $\nabla v_h|_\kappa$ is a first-degree polynomial vector, without loss of generality, let $\nabla v_h|_\kappa = \sum_{i=1}^{3}\beta_i\lambda_i$, where $\beta_i (i = 1,2,3)$ is a constant vector. Then, by Lemma 3, we have

$$\begin{aligned} b(\boldsymbol{\varphi} - r_h\boldsymbol{\varphi}, v_h) &= -(\mathrm{div}(\boldsymbol{\varphi} - r_h\boldsymbol{\varphi}), v_h) \\ &= (\boldsymbol{\varphi} - r_h\boldsymbol{\varphi}, \nabla v_h) \\ &= \sum_{\kappa\in\mathfrak{I}_h}\int_\kappa (\boldsymbol{\varphi} - r_h\boldsymbol{\varphi})\nabla v_h \mathrm{d}x \\ &= \sum_{\kappa\in\mathfrak{I}_h}\sum_{i=1}^{3}\beta_i\int_\kappa \lambda_i(\boldsymbol{\varphi} - r_h\boldsymbol{\varphi})\mathrm{d}x \end{aligned}$$

To make equation(25)hold, it is sufficient to $\int_\kappa \lambda_i(\boldsymbol{\varphi} - r_h\boldsymbol{\varphi})\mathrm{d}x = 0, i = 1,2,3, \forall \kappa \in \mathfrak{I}_h$.

From the definition of r_h , it is enough to choose α_j ,such that

$$\sum_{j=1}^{3}\alpha_j\int_\kappa \lambda_1\lambda_2\lambda_3\lambda_j\lambda_i\mathrm{d}x = \int_\kappa \lambda_i(\boldsymbol{\varphi} - Q_h\boldsymbol{\varphi})\mathrm{d}x, i = 1,2,3 , \tag{26}$$

upon calculation, the determinant of the coefficient matrix of the system (26)is non-zero, which implies that the system(26) has a unique solution $\alpha_j (j = 1,2,3)$, ensuring that r_h satisfies equation(25).

Lemma4. Existence of the operator $r_h : \mathbf{V} \to \mathbf{V}_h$,such that for any $\boldsymbol{\varphi} \in \mathbf{V}$,it holds that

$$b(\boldsymbol{\varphi} - r_h\boldsymbol{\varphi}, v_h) = 0, \forall v_h \in W_h ,$$

when $\boldsymbol{\varphi} \in [H^k(\Omega)]^2, 0 \le k \le 2$,we have

$$\|\boldsymbol{\varphi} - r_h \boldsymbol{\varphi}\|_{0,\Omega} \lesssim h^k |\boldsymbol{\varphi}|_k . \quad (27)$$

Proof. When solving for $\alpha_j (j = 1,2,3)$ using the Grammer rule, from Section 1.4 of[4], specifically equations (1.4.9), (1.4.32), and Lemma 1.20–Lemma 1.22, we have

$$|\alpha_j| \lesssim h^{-1} \|\boldsymbol{\varphi} - Q_h \boldsymbol{\varphi}\|_{0,\kappa},\ j = 1,2,3. \quad (28)$$

From the definition of r_h ,(28), $H\ddot{o}lder$inequality, we have

$$\begin{aligned}\|\boldsymbol{\varphi} - r_h \boldsymbol{\varphi}\|_{0,\kappa} &\lesssim \|\boldsymbol{\varphi} - Q_h \boldsymbol{\varphi}\|_{0,\kappa} + \sum_{j=1}^{3} |\alpha_j| \|\lambda_1\lambda_2\lambda_3\lambda_j\|_{0,\kappa} \\ &\lesssim \|\boldsymbol{\varphi} - Q_h \boldsymbol{\varphi}\|_{0,\kappa} + h^{-1} \|\boldsymbol{\varphi} - Q_h \boldsymbol{\varphi}\|_{0,\kappa} \cdot [\mathrm{mes}(\kappa)]^{1/2} \\ &\lesssim \|\boldsymbol{\varphi} - Q_h \boldsymbol{\varphi}\|_{0,\kappa}\end{aligned}$$

From (24) we can obtain

$$\|\boldsymbol{\varphi} - r_h \boldsymbol{\varphi}\|_{0,\Omega} = \left(\sum_{\kappa \in \mathfrak{I}_h} \|\boldsymbol{\varphi} - r_h \boldsymbol{\varphi}\|_{0,\kappa}^2 \right)^{1/2} \lesssim \|\boldsymbol{\varphi} - Q_h \boldsymbol{\varphi}\|_{0,\Omega} \lesssim h^k |\boldsymbol{\varphi}|_k ,$$

where $\boldsymbol{\varphi} \in [H^k(\Omega)]^2, 0 \le k \le 2$.

Theorem2. Assume that there exists an operator $r_h : \mathbf{V} \to \mathbf{V}_h$,such that for any $\boldsymbol{\varphi} \in \mathbf{V}$,it holds that

$$b(\boldsymbol{\varphi} - r_h \boldsymbol{\varphi}, v_h) = 0, \forall v_h \in W_h ,$$

moreover, $(\boldsymbol{\sigma}, u) \in \mathbf{V} \times W$ is the solution to problem (11), and $(\boldsymbol{\sigma}_h, u_h) \in \mathbf{V}_h \times W_h$ is the solution to problem (12). Then the following error estimate holds:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \lesssim \|\boldsymbol{\sigma} - r_h \boldsymbol{\sigma}\|_0 + |u - v_h|_1, \forall v_h \in W_h \cap H_0^1(\Omega), \quad (29)$$

$$\begin{aligned}\|u - u_h\|_0 \lesssim \sup_{0 \ne d \in L^2(\Omega)} \frac{1}{\|d\|_0} &[b(\boldsymbol{\lambda}_d - r_h \boldsymbol{\lambda}_d, u - v_h) + a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, r_h \boldsymbol{\lambda}_d - \boldsymbol{\lambda}_d) \\ &+ b(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, y_d - \eta_h)], \forall v_h, \eta_h \in W_h.\end{aligned} \quad (30)$$

Where for any $d \in L^2(\Omega)$, the function pairsare defined in $(\boldsymbol{\lambda}_d, y_d) \in \mathbf{V} \times [H_0^1(\Omega) \cap H^{1+r}(\Omega)]$ and satisfies

$$\begin{cases} a(\boldsymbol{\lambda}_d, \boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, y_d) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}, \\ b(\boldsymbol{\lambda}_d, v) = (d, v), & \forall v \in W. \end{cases} \quad (31)$$

and we know a priori estimate:

$$\|\boldsymbol{\lambda}_d\|_{r_0} + \|y_d\|_{1+r_0} \lesssim \|d\|_0 . \quad (32)$$

Where $\frac{1}{2} < r \le 2$, depends on the shape of the domain.

Proof. From equations (11) and (12), by subtracting the corresponding terms, we obtain the error equation

$$\begin{cases} a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\varphi}) - b(u - u_h, \boldsymbol{\varphi}) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}_h, \\ b(v, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = 0, & \forall v \in W_h. \end{cases} \quad (33)$$

From the error equation(33), $b(\boldsymbol{\varphi} - r_h \boldsymbol{\varphi}, v_h) = 0$,it follows that for any $v_h \in W_h$, we have

$$\begin{aligned} b(r_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v_h) &= b(r_h \boldsymbol{\sigma} - \boldsymbol{\sigma} + \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v_h) \\ &= b(r_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, v_h) + b(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v_h) \\ &= 0 \end{aligned}$$

Thus

$$r_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \in Z_h .$$

By combining inequality (20) and the error equation (33), we have

$$
\begin{aligned}
\alpha \| r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h \|_0^2 \le\ & a(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\\
&= a(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\\
&= a(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+b(u-u_h, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\\
&= a(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+b(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, u-u_h)+b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-u_h)\\
&= a(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+b(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, u-v_h+v_h-u_h)+b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-v_h+v_h-u_h)\\
&= a(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+b(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, u-v_h)+b(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, v_h-u_h)+b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-v_h)+b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, v_h-u_h)\\
&= I_1
\end{aligned}
$$

Also, since the operator r_h satisfies that for any $\boldsymbol{\varphi}\in\mathbf{V}, v_h\in W_h$, $b(\boldsymbol{\varphi}-r_h\boldsymbol{\varphi}, v_h)=0$ holds, we obtain

$$b(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, v_h-u_h)=0.$$

Next, from the error equation (33), we obtain

$$b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, v_h-u_h)=0.$$

Thus, from the above two expressions, Lemma 3, and the $H\ddot{o}lder$ inequality, we have

$$
\begin{aligned}
I_1 &= a(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+b(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, u-v_h)+b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-v_h)\\
&= a(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+b(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-v_h)\\
&= a(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+(r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \nabla(u-v_h))\\
&\le \| r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}\|_0 \| r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 + \| r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 \|\nabla(u-v_h)\|_0
\end{aligned}
$$

Therefore, we have

$$\| r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 \le \| r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}\|_0 + \|\nabla(u-v_h)\|_0, \forall v_h\in W_h\cap H_0^1(\Omega).$$

Using the triangle inequality and the above conclusions, we obtain

$$
\begin{aligned}
\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 &= \|\boldsymbol{\sigma}-r_h\boldsymbol{\sigma}+r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\\
&\le \|\boldsymbol{\sigma}-r_h\boldsymbol{\sigma}\|_0 + \| r_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\\
&\le \|\boldsymbol{\sigma}-r_h\boldsymbol{\sigma}\|_0 + \|\nabla(u-v_h)\|_0\\
&= \|\boldsymbol{\sigma}-r_h\boldsymbol{\sigma}\|_0 + |u-v_h|_1, \forall v_h\in W_h\cap H_0^1(\Omega)
\end{aligned}
$$

Thus, we obtain(29).

By combining inequality(31) and the error equation (33), for any $v_h\in W_h$,we have

$$
\begin{aligned}
(d, u-u_h) &= b(\boldsymbol{\lambda}_d, u-u_h)\\
&= b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d + r_h\boldsymbol{\lambda}_d, u-u_h)\\
&= b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-u_h)+b(r_h\boldsymbol{\lambda}_d, u-u_h)\\
&= b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-v_h+v_h-u_h)+a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d)\\
&= b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-v_h)+a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d)+b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, v_h-u_h)\\
&= I_2
\end{aligned}
$$

Furthermore, since the operator r_h satisfies that for any $\boldsymbol{\varphi}\in\mathbf{V}, v_h\in W_h$, $b(\boldsymbol{\varphi}-r_h\boldsymbol{\varphi}, v_h)=0$ holds, we obtain

$$b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, v_h-u_h)=0.$$

For any $\eta_h\in W_h$, by combining the above expressions and(31), we get

$$
\begin{aligned}
I_2 &= b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-v_h)+a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d)\\
&= b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-v_h)+a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d-\boldsymbol{\lambda}_d)+a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \boldsymbol{\lambda}_d)\\
&= b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-v_h)+a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d-\boldsymbol{\lambda}_d)+b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, y_d)\\
&= b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-v_h)+a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d-\boldsymbol{\lambda}_d)+b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, y_d-\eta_h)+b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \eta_h)
\end{aligned}
$$

Then, from the error equation (33), we obtain $b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \eta_h)=0$.So we have

$$
\begin{aligned}
(d, u-u_h) &= b(\boldsymbol{\lambda}_d, u-u_h)\\
&= b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-v_h)+a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d-\boldsymbol{\lambda}_d)+b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, y_d-\eta_h)
\end{aligned}
$$

Substitutingthe above results into the following norm expression,

$$\| u-u_h\|_G = \sup_{0\ne d\in L^2(\Omega)} \frac{(d, u-u_h)}{\| d\|_G} = \sup_{0\ne d\in L^2(\Omega)} \frac{b(\boldsymbol{\lambda}_d, u-u_h)}{\| d\|_G}.$$

we can obtain

$$\|u-u_h\|_0 \lesssim \sup_{0\neq d\in L^2(\Omega)} \frac{1}{\|d\|_0}[b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-v_h) + a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d - \boldsymbol{\lambda}_d) + b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, y_d - \eta_h)], \forall v_h, \eta_h \in W_h,$$

we obtain equation (30).

Theorem3.Suppos $(\boldsymbol{\sigma},u)\in[H^r(\Omega)]^2\times(H^{r+1}(\Omega)\cap H_0^1(\Omega))$ is the solution of problem (11),and $(\boldsymbol{\sigma}_h,u_h)\in\mathbf{V}_h\times W_h$ is the solution of problem(12),then the following error estimate holds:

$$h^{r_0}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 + \|u-u_h\|_0 \lesssim h^{r_0+r}\|u\|_{1+r}. \tag{34}$$

Proof. From(29),(27) and (22),we can give the estimate for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$ as

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 &\lesssim \|\boldsymbol{\sigma}-r_h\boldsymbol{\sigma}\|_0 + |u-\rho_h u|_1\\ &\lesssim h^r|\boldsymbol{\sigma}|_r + h^r|u|_{1+r}\\ &= h^r|K\nabla u|_r + h^r|u|_{1+r}\\ &\lesssim h^r\|u\|_{1+r}.\end{aligned} \tag{35}$$

Where $\rho_h : H^3(\Omega)\to W_h$ is the L^2-projection operator.

Next, we estimate the three terms on the right-hand side of the inequality in (30).

Here, $(\boldsymbol{\lambda}_d, y_d)$ is the solution to the auxiliary problem (31) introduced in Theorem 2.

From (21),(22),(27),(32)and $H\ddot{o}lder$ inequality, we have

$$\begin{aligned}b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-\rho_h u) &= -(\mathrm{div}(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d), u-\rho_h u)\\ &= (\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, \nabla(u-\rho_h u))\\ &\lesssim \|\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d\|_{r_0}|u-\rho_h u|_1\\ &\lesssim h^{r_0}\|\boldsymbol{\lambda}_d\|_{r_0}\cdot h^r|u|_{1+r}\\ &\lesssim h^{r_0+r}\|u\|_{1+r}\|\boldsymbol{\lambda}_d\|_{r_0}\\ &\lesssim h^{r_0+r}\|u\|_{1+r}\|d\|_0.\end{aligned} \tag{36}$$

From (27),(32),(35)and $H\ddot{o}lder$inequality, we have

$$\begin{aligned}a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d - \boldsymbol{\lambda}_d) &\lesssim \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\|r_h\boldsymbol{\lambda}_d - \boldsymbol{\lambda}_d\|_0\\ &\lesssim h^r\|u\|_{1+r}\cdot h^{r_0}\|\boldsymbol{\lambda}_d\|_{r_0}\\ &\lesssim h^{r_0+r}\|u\|_{1+r}\|d\|_0.\end{aligned} \tag{37}$$

From (21),(32),(35)and $H\ddot{o}lder$inequality, we have

$$\begin{aligned}b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, y_d - \rho_h y_d) &= -(\mathrm{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), y_d - \rho_h y_d)\\ &= (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \nabla(y_d - \rho_h y_d))\\ &\lesssim \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0|y_d - \rho_h y_d|_1\\ &\lesssim h^r\|u\|_{1+r}\cdot h^{r_0}|y_d|_{1+r_0}\\ &\lesssim h^r\|u\|_{1+r}\cdot h^{r_0}\|y_d\|_{1+r_0}\\ &\lesssim h^{r_0+r}\|u\|_{1+r}\|d\|_0.\end{aligned} \tag{38}$$

Thus, substituting (36),(37)and (38) into (30) from Theorem 2, we can obtain

$$\begin{aligned}\|u-u_h\|_0 &\lesssim \sup_{0\neq d\in L^2(\Omega)} \frac{1}{\|d\|_0}[b(\boldsymbol{\lambda}_d - r_h\boldsymbol{\lambda}_d, u-\rho_h u)\\ &\quad + a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, r_h\boldsymbol{\lambda}_d - \boldsymbol{\lambda}_d) + b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, y_d - \rho_h y_d)]\\ &\lesssim h^{r_0+r}\|u\|_{1+r}.\end{aligned} \tag{39}$$

Finally combining(35) and (39),we can get the desired result (34).

Theorem4. For the previously defined T and T_h, we have $\|T-T_h\|_G\to 0$, as $h\to 0$.

Proof. Let $Tf=u, T_h f = u_h$, then we have

$$\|T-T_h\|_G=\sup_{0\neq f\in L^2(\Omega)}\frac{\|Tf-T_hf\|_0}{\|f\|_0}=\sup_{0\neq f\in L^2(\Omega)}\frac{\|u-u_h\|_0}{\|f\|_0}$$
$$\lesssim \sup_{0\neq f\in L^2(\Omega)}\frac{h^{r_0+r_0}\|u\|_{1+r_0}}{\|f\|_0}\lesssim \sup_{0\neq f\in L^2(\Omega)}\frac{h^{r_0+r_0}\|f\|_0}{\|f\|_0}\lesssim h^{2r_0}\to 0,(h\to 0).$$

We complete the proof.

Corollary 2. Assume that $u\in H^3(\Omega)$,the following error estimate holds:

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div,\Omega)}\lesssim h\|u\|_3 \tag{40}$$

Proof.

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div,\Omega)}&=\|\boldsymbol{\sigma}-I_h^1\boldsymbol{\sigma}+I_h^1\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div,\Omega)}\\ &\lesssim h\|\boldsymbol{\sigma}\|_2+h^{-1}\|I_h^1\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\\ &=h\|\boldsymbol{\sigma}\|_2+h^{-1}\|I_h^1\boldsymbol{\sigma}-\boldsymbol{\sigma}+\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\\ &\lesssim h\|\boldsymbol{\sigma}\|_2+h^{-1}h^2\|\boldsymbol{\sigma}\|_2+h^{-1}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\\ &\lesssim h\|\boldsymbol{\sigma}\|_2\\ &=h\|K\nabla u\|_2\\ &\lesssim h\|u\|_3 .\end{aligned}$$

Where I_h^1 is the linear finite element interpolation

VI. A PRIORI ERROR ESTIMATEFOR EIGENVALUES

Let $(\lambda,\boldsymbol{\mu},p)$ be an eigenpair of (3), and $(\lambda_h,\boldsymbol{\mu}_h,p_h)$ be an eigenpair of (10); $(\lambda_h,\boldsymbol{\mu}_h,p_h)$ approximates $(\lambda,\boldsymbol{\mu},p)$. Let M_λ be the space spanned by the eigenfunctions $\{u_j\}$ corresponding to the eigenvalue λ of (3).

Lemma 5.(Lemma2 in[13])Suppose the multiplicity of the eigenvalue λ is *m*, then the following estimate holds:

$$|\lambda-\lambda_{l,h}|\lesssim C\left\{\left\|(\mathbf{S}-\mathbf{S}_h)|_{M_\lambda}\right\|^2_{G\to\mathbf{H}}+\left\|(\mathbf{S}-\mathbf{S}_h)|_{M_\lambda}\right\|_{G\to\mathbf{V}}\cdot\left\|(T-T_h)|_{M_\lambda}\right\|_{G\to W}+\left\|(T-T_h)|_{M_\lambda}\right\|^2_{G\to G}\right\} \tag{41}$$

Proof. Since the multiplicity of the eigenvalues of a self-adjoint operator is equal to the dimension of the eigenspace, let $u_1,u_2,\cdots u_m$ be an orthonormal basis of M_λ .

By Theorem 3 of [9] and the steepness $\alpha=1$ of the self-adjoint operator, we have the followingestimate.

$$\left|\lambda^{-1}-\lambda_{l,h}^{-1}\right|\lesssim C\left\{\sum_{i,j=1}^{m}\left|((T-T_h)u_i,u_j)\right|+\left\|(T-T_h)|_{M_\lambda}\right\|^2_{G\to G}\right\}. \tag{42}$$

For any $f,g\in L^2(\Omega)$,let us consider $|(T-T_h)g,f|$.

By the two equations of(15), we obtain the following

$$(f,v)=-a(\mathbf{S}f,\boldsymbol{\varphi})+b(\boldsymbol{\varphi},Tf)+b(\mathbf{S}f,v),\forall(\boldsymbol{\varphi},v)\in\mathbf{V}\times W .$$

For $g\in L^2(\Omega)$,let $\boldsymbol{\varphi}=(\mathbf{S}-\mathbf{S}_h)g$, $v=(T-T_h)g$,then

$$(f,(T-T_h)g)=-a(\mathbf{S}f,(\mathbf{S}-\mathbf{S}_h)g)+b((\mathbf{S}-\mathbf{S}_h)g,Tf)+b(\mathbf{S}f,(T-T_h)g) , \tag{43}$$

replacing $g\in L^2(\Omega)$ for f from(15), we have

$$\begin{cases}a(\mathbf{S}g,\boldsymbol{\varphi})-b(\boldsymbol{\varphi},Tg)=0, & \forall\boldsymbol{\varphi}\in\mathbf{V},\\ b(\mathbf{S}g,v)=(g,v), & \forall v\in W.\end{cases} \tag{44}$$

From(16), we have

$$\begin{cases}a(\mathbf{S}_hg,\boldsymbol{\varphi})-b(\boldsymbol{\varphi},T_hg)=0, & \forall\boldsymbol{\varphi}\in\mathbf{V}_h,\\ b(\mathbf{S}_hg,v)=(g,v), & \forall v\in W.\end{cases} \tag{45}$$

By subtracting (44) and (45), we have

$$\begin{cases}a((\mathbf{S}-\mathbf{S}_h)g,\boldsymbol{\varphi})-b(\boldsymbol{\varphi},(T-T_h)g)=0\\ -b((\mathbf{S}-\mathbf{S}_h)g,v)=0\end{cases}$$

Adding the two above equations yields the following.

$$a((\mathbf{S}-\mathbf{S}_h)g,\boldsymbol{\varphi})-b(\boldsymbol{\varphi},(T-T_h)g)-b((\mathbf{S}-\mathbf{S}_h)g,v)=0 . \tag{46}$$

Since $a(\cdot,\cdot)$ is symmetric, adding (43) and (46) gives the result

$$(f,(T-T_h)g)=-a((\mathbf{S}-\mathbf{S}_h)g,\boldsymbol{\varphi}-\mathbf{S}f)+b((\mathbf{S}-\mathbf{S}_h)g,Tf-v)+b(\mathbf{S}f-\boldsymbol{\varphi},(T-T_h)g) .$$

From equations (4) to (6), for any $\boldsymbol{\varphi} \in \mathbf{V}_h$, $v \in W_h$, we have

$$\begin{aligned}|(f,(T-T_h)g)| \lesssim\ & C_1 \|(\mathbf{S}-\mathbf{S}_h)g\|_{\mathbf{H}} \|\boldsymbol{\varphi}-\mathbf{S}f\|_{\mathbf{H}} + C_2 \|(\mathbf{S}-\mathbf{S}_h)g\|_{\mathbf{V}} \|v-Tf\|_W \\ & + C_2 \|\boldsymbol{\varphi}-\mathbf{S}f\|_{\mathbf{V}} \|(T-T_h)g\|_W .\end{aligned} \quad (47)$$

Taking $\boldsymbol{\varphi} = \mathbf{S}_h f, v = T_h f$ in (47), we get

$$\begin{aligned}|(f,(T-T_h)g)| \lesssim\ & C_1 \|(\mathbf{S}-\mathbf{S}_h)g\|_{\mathbf{H}} \|(\mathbf{S}-\mathbf{S}_h)f\|_{\mathbf{H}} \\ & + C_2 \|(\mathbf{S}-\mathbf{S}_h)g\|_{\mathbf{V}} \|(T-T_h)f\|_W \\ & + C_2 \|(\mathbf{S}-\mathbf{S}_h)f\|_{\mathbf{V}} \|(T-T_h)g\|_W \end{aligned} \quad (48)$$

In(48), replacing u_i for g and u_j for f, we have

$$|((T-T_h)u_i,u_j)| \lesssim C_1 \|(\mathbf{S}-\mathbf{S}_h)|_{M_\lambda}\|_{G\to\mathbf{H}}^2 + 2C_2 \|(\mathbf{S}-\mathbf{S}_h)|_{M_\lambda}\|_{G\to\mathbf{V}} \|(T-T_h)|_{M_\lambda}\|_{G\to W} . \quad (49)$$

Substituting (49)into (42), we arrive at (41).

The mixed discretisedsource problemis well-posed and has a unique solutionwhen h is small enough. Based on (34) and(40), we can obtain the following a priori error estimate.

For any $f \in L^2(\Omega)$, the following hold:

$$\|Tf - T_h f\|_0 \lesssim h^{r_0+r} \|Tf\|_{1+r}, \frac{1}{2} < r \le 2 . \quad (50)$$

$$\|\mathbf{S}f - \mathbf{S}_h f\|_0 \lesssim h^{r} \|Tf\|_{1+r}, \frac{1}{2} < r \le 2 . \quad (51)$$

$$\|\mathbf{S}f - \mathbf{S}_h f\|_{\mathbf{V}} \lesssim h^{2} \|Tf\|_{3}, \frac{1}{2} < r \le 2 . \quad (52)$$

If $f \in M_\lambda$, then $Tf = \lambda_l^{-1} f$,and we can obtain the following estimate:

$$\|(T-T_h)|_{M_\lambda}\|_0 \lesssim h^{r_0+r}, \text{if } M_\lambda \subset H^{1+r}(\Omega).$$

$$\|(\mathbf{S}-\mathbf{S}_h)|_{M_\lambda}\|_0 \lesssim h^{r}, \text{if } M_\lambda \subset H^{1+r}(\Omega).$$

$$\|(\mathbf{S}-\mathbf{S}_h)|_{M_\lambda}\|_{\mathbf{V}} \lesssim h^{2}, \text{if } M_\lambda \subset H^{3}(\Omega).$$

Lemma 6. Let $(\lambda_h, \boldsymbol{\mu}_h, p_h)$ be a mixed finite element eigenpair of (10), then there exists an eigenpair $(\lambda, \boldsymbol{\mu}, p)$ of (3), such that the following a priori error estimate holds:

$$h^{r_0} \|\boldsymbol{\mu}-\boldsymbol{\mu}_h\|_0 + \|p-p_h\|_0 \lesssim h^{r_0+r} \quad (53)$$

$$|\lambda - \lambda_h| \le h^{2(r_0+r)} \quad (54)$$

$$\|\boldsymbol{\mu}_h - \boldsymbol{\mu}\|_{\mathbf{V}} \lesssim h^2 \quad (55)$$

VII.CONVERGENCERESULTANDNEWADAPTIVEALGORITHMSFOREIGENVALUEPROBLEM

In this subsection, for the 2nd order elliptic eigenvalue problems,we shall give the corresponding convergence result for finite element eigenpair,anddesign low order interpolationbaseda-posteriori indicator and new adaptive algorithms. We consider the following eigenvalue problem:

Find $(\lambda, \boldsymbol{\mu}, p) \in R \times \mathbf{V} \times W$, $(\boldsymbol{\mu}, p) \ne (0,0)$, suchthat

$$\begin{cases} a(\boldsymbol{\mu},\boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, p) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}, \\ b(\boldsymbol{\mu}, v) = \lambda(p, v), & \forall v \in W, \end{cases} \quad (3)$$

The mixed finite element approximation of (3) is given by:

Find $(\lambda_h, \boldsymbol{\mu}_h, p_h) \in R \times \mathbf{V}_h \times W_h$, $(\boldsymbol{\mu}_h, p_h) \ne (0,0)$,such that

$$\begin{cases} a(\boldsymbol{\mu}_h,\boldsymbol{\varphi}) - b(\boldsymbol{\varphi}, p_h) = 0, & \forall \boldsymbol{\varphi} \in \mathbf{V}_h, \\ b(\boldsymbol{\mu}_h, v) = \lambda_h(p_h, v), & \forall v \in W_h. \end{cases} \quad (10)$$

Lemma7. Assume that $\boldsymbol{\mu} \in [H^2(\Omega)]^2$, $p \in H^3(\Omega)$, for the zero-order interpolation $I_h^0 \boldsymbol{\mu}_h$ of the linear element eigenfunction and the first order interpolation $I_h^1 p_h$ of the quadratic element eigenfunction, the following inequality holds:

$$\| I_h^0 \boldsymbol{\mu}_h - \boldsymbol{\mu}_h \|_0 \lesssim h \|\boldsymbol{\mu}\|_{1,\Omega} \quad (56)$$

$$\| I_h^0 \boldsymbol{\mu}_h - \boldsymbol{\mu} \|_0 \lesssim h \|\boldsymbol{\mu}\|_{1,\Omega} \quad (57)$$

$$\| I_h^1 p_h - p_h \|_0 \lesssim h^2 \| p \|_{2,\Omega} \tag{58}$$

$$\| I_h^1 p_h - p \|_0 \lesssim h^2 \| p \|_{2,\Omega} \tag{59}$$

Proof.

$$\begin{aligned}
\| I_h^0 \boldsymbol{\mu}_h - \boldsymbol{\mu}_h \|_0 &\lesssim h \| \boldsymbol{\mu}_h \|_{1,\Omega} \\
&= h \| \boldsymbol{\mu}_h - \Theta\boldsymbol{\mu} + \Theta\boldsymbol{\mu} \|_{1,\Omega} \\
&\lesssim h \| \boldsymbol{\mu}_h - \Theta\boldsymbol{\mu} \|_{1,\Omega} + h \| \Theta\boldsymbol{\mu} \|_{1,\Omega} \\
&\lesssim \| \boldsymbol{\mu}_h - \Theta\boldsymbol{\mu} \|_0 + h \| \boldsymbol{\mu} \|_{1,\Omega} \\
&= \| \boldsymbol{\mu}_h - \boldsymbol{\mu} + \boldsymbol{\mu} - \Theta\boldsymbol{\mu} \|_0 + h \| \boldsymbol{\mu} \|_{1,\Omega} \\
&\lesssim \| \boldsymbol{\mu}_h - \boldsymbol{\mu} \|_0 + \| \boldsymbol{\mu} - \Theta\boldsymbol{\mu} \|_0 + h \| \boldsymbol{\mu} \|_{1,\Omega} \\
&\lesssim h \| \boldsymbol{\mu} \|_{1,\Omega}
\end{aligned}$$

where Θ is the Clement interpolation corresponding to the linear element, then

$$\begin{aligned}
\| I_h^0 \boldsymbol{\mu}_h - \boldsymbol{\mu} \|_0 &= \| I_h^0 \boldsymbol{\mu}_h - \boldsymbol{\mu}_h + \boldsymbol{\mu}_h - \boldsymbol{\mu} \|_0 \\
&\lesssim \| I_h^0 \boldsymbol{\mu}_h - \boldsymbol{\mu}_h \|_0 + \| \boldsymbol{\mu}_h - \boldsymbol{\mu} \|_0 \\
&\lesssim h \| \boldsymbol{\mu} \|_{1,\Omega}
\end{aligned}$$

Now,let us consider $\| I_h^1 p_h - p_h \|_0$ and $\| I_h^1 p_h - p \|_0$.

$$\begin{aligned}
\| I_h^1 p_h - p_h \|_0 &\lesssim h^2 \| p_h \|_{2,\Omega} \\
&= h^2 \| p_h - \tilde{\Theta} p + \tilde{\Theta} p \|_{2,\Omega} \\
&\lesssim h^2 \| p_h - \tilde{\Theta} p \|_{2,\Omega} + h^2 \| \tilde{\Theta} p \|_{2,\Omega} \\
&\lesssim h \| p_h - \tilde{\Theta} p \|_1 + h^2 \| p \|_{2,\Omega} \\
&= h \| p_h - p + p - \tilde{\Theta} p \|_1 + h^2 \| p \|_{2,\Omega} \\
&\lesssim h(\| p_h - p \|_1 + \| p - \tilde{\Theta} p \|_1) + h^2 \| p \|_{2,\Omega} \\
&\lesssim h^2 \| p \|_{2,\Omega}
\end{aligned}$$

where $\tilde{\Theta}$ is the Clement interpolation corresponding to the quadratic element, then

$$\begin{aligned}
\| I_h^1 p_h - p \|_0 &= \| I_h^1 p_h - p_h + p_h - p \|_0 \\
&\lesssim \| I_h^1 p_h - p_h \|_0 + \| p_h - p \|_0 \\
&\lesssim h^2 \| p \|_{2,\Omega}
\end{aligned}$$

Lemma 8. Let $(\lambda, \boldsymbol{\mu}, p)$ be an eigenpair of(3),then for any $0 \neq \psi \in L^2(\Omega)$ and $\boldsymbol{\omega} \in \mathbf{V}$,the Rayleigh quotient

$$\hat{\lambda} = \frac{-a(\boldsymbol{\omega}, \boldsymbol{\omega}) + 2\mathrm{b}(\boldsymbol{\omega}, \psi)}{(\psi, \psi)},$$

satisfies

$$\hat{\lambda} - \lambda = \frac{-a(\boldsymbol{\omega} - \boldsymbol{\mu}, \boldsymbol{\omega} - \boldsymbol{\mu}) + 2b(\boldsymbol{\omega} - \boldsymbol{\mu}, \psi - p)}{(\psi, \psi)} - \frac{\lambda(\psi - p, \psi - p)}{(\psi, \psi)}. \tag{60}$$

Proof.

$$\begin{aligned}
&-a(\boldsymbol{\omega} - \boldsymbol{\mu}, \boldsymbol{\omega} - \boldsymbol{\mu}) + 2b(\boldsymbol{\omega} - \boldsymbol{\mu}, \psi - p) - \lambda(\psi - p, \psi - p) \\
&= -a(\boldsymbol{\omega}, \boldsymbol{\omega}) + 2a(\boldsymbol{\omega}, \boldsymbol{\mu}) - a(\boldsymbol{\mu}, \boldsymbol{\mu}) + 2b(\boldsymbol{\omega}, \psi) - 2b(\boldsymbol{\omega}, p) - 2b(\boldsymbol{\mu}, \psi) + 2b(\boldsymbol{\mu}, p) \\
&\quad -\lambda(\psi, \psi) + 2\lambda(\psi, p) - \lambda(p, p) \\
&= -a(\boldsymbol{\omega}, \boldsymbol{\omega}) + 2b(\boldsymbol{\omega}, \psi) - 2[b(\boldsymbol{\mu}, \psi) - \lambda(\psi, p)] + 2[a(\boldsymbol{\omega}, \boldsymbol{\mu}) - b(\boldsymbol{\omega}, p)] \\
&\quad +2b(\boldsymbol{\mu}, p) - \lambda(p, p) - a(\boldsymbol{\mu}, \boldsymbol{\mu}) - \lambda(\psi, \psi) \\
&= -a(\boldsymbol{\omega}, \boldsymbol{\omega}) + 2b(\boldsymbol{\omega}, \psi) - \lambda(\psi, \psi)
\end{aligned}$$

divide both sides by (ψ, ψ) .This completes the proof.

Notice: Let $(\lambda_h, \boldsymbol{\mu}_h, p_h)$ be an eigenpair of (10). In (60), take $\boldsymbol{\omega} = \boldsymbol{\mu}_h$, $\psi = p_h$,then we have

$$\lambda_h - \lambda = \frac{-a(\boldsymbol{\mu}_h - \boldsymbol{\mu}, \boldsymbol{\mu}_h - \boldsymbol{\mu}) + 2b(\boldsymbol{\mu}_h - \boldsymbol{\mu}, p_h - p)}{(p_h, p_h)} - \frac{\lambda(p_h - p, p_h - p)}{(p_h, p_h)}.$$

Theorem 5. Assume that $\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}, p)\| \gtrsim h$,then we have

$$\left| \frac{\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}_h, p_h)\|_0}{\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}, p)\|_0} - 1 \right| \lesssim h.$$

Proof.

$$\begin{aligned}
&\left| \frac{\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}_h, p_h)\|_0}{\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}, p)\|_0} - 1 \right| \\
&= \left| \frac{\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}_h, p_h)\|_0 - \|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}, p)\|_0}{\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}, p)\|_0} \right| \\
&\lesssim \frac{\|(\boldsymbol{\mu}, p) - (\boldsymbol{\mu}_h, p_h)\|_0}{\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}, p)\|_0} \\
&\lesssim h
\end{aligned}$$

Theorem5 indicates that $\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}_h, p_h)\|_0$ isasymptotically exact a-posteriori indicator of $\|(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - (\boldsymbol{\mu}, p)\|_0$.

Theorem 6.Assume that $\boldsymbol{\mu} \in H^2(\Omega)$, $p \in H^3(\Omega)$,let $(\lambda, \boldsymbol{\mu}, p)$ be an eigenpair of(3) and $(\lambda_h, \boldsymbol{\mu}_h, p_h)$ be the associated discrete eigenpair. Assume that

$$\left| \lambda - \hat{\lambda}_h(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) \right| \gtrsim h^2,$$

then

$$\left| 1 - \frac{\eta_h(\boldsymbol{\mu}_h, p_h, \Omega)}{\hat{\lambda}_h(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - \lambda} \right| \lesssim h,$$

where

$$\eta_h(\boldsymbol{\mu}_h, p_h, \Omega) = \frac{-a(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h) + 2b(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, I_h^1 p_h - p_h)}{(I_h^1 p_h, I_h^1 p_h)}.$$

Proof. By Lemma 8, we have

$$\hat{\lambda}_h(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - \lambda = \frac{-a(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}, I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}) + 2b(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}, I_h^1 p_h - p) - \lambda(I_h^1 p_h - p, I_h^1 p_h - p)}{(I_h^1 p_h, I_h^1 p_h)}.$$

Thus, the left-hand side is

$$\begin{aligned}
&\left| 1 - \frac{\eta_h(\boldsymbol{\mu}_h, p_h, \Omega)}{\hat{\lambda}_h(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - \lambda} \right| \\
&= \left| \frac{\hat{\lambda}_h(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - \lambda - \eta_h(\boldsymbol{\mu}_h, p_h, \Omega)}{\hat{\lambda}_h(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - \lambda} \right| \\
&= \left| \frac{-a(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}, I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}) + 2b(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}, I_h^1 p_h - p) - \lambda(I_h^1 p_h - p, I_h^1 p_h - p)}{[\hat{\lambda}_h(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - \lambda](I_h^1 p_h, I_h^1 p_h)} + \frac{a(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h) - 2b(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, I_h^1 p_h - p_h)}{[\hat{\lambda}_h(I_h^0\boldsymbol{\mu}_h, I_h^1 p_h) - \lambda](I_h^1 p_h, I_h^1 p_h)} \right|
\end{aligned}$$

Let

$$\begin{aligned}
C_h &= -a(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}, I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}) + a(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h) \\
&\quad + 2b(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}, I_h^1 p_h - p) - 2b(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, I_h^1 p_h - p_h) \\
&= -a(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h + \boldsymbol{\mu}_h - \boldsymbol{\mu}, I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h + \boldsymbol{\mu}_h - \boldsymbol{\mu}) + a(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h) \\
&\quad + 2b(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h + \boldsymbol{\mu}_h - \boldsymbol{\mu}, I_h^1 p_h - p_h + p_h - p) - 2b(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, I_h^1 p_h - p_h) \\
&= -2a(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, \boldsymbol{\mu}_h - \boldsymbol{\mu}) - a(\boldsymbol{\mu}_h - \boldsymbol{\mu}, \boldsymbol{\mu}_h - \boldsymbol{\mu}) + 2b(I_h^0\boldsymbol{\mu}_h - \boldsymbol{\mu}_h, p_h - p) + 2b(\boldsymbol{\mu}_h - \boldsymbol{\mu}, I_h^1 p_h - p)
\end{aligned}$$

According toLemma 7,Corollary 2and $H\ddot{o}lder$inequality, we have

$$|C_h| \lesssim h^3.$$

According to the assumption of this theorem and

$$-a(I_h^0\boldsymbol{\mu}_h-\boldsymbol{\mu},I_h^0\boldsymbol{\mu}_h-\boldsymbol{\mu})+a(I_h^0\boldsymbol{\mu}_h-\boldsymbol{\mu}_h,I_h^0\boldsymbol{\mu}_h-\boldsymbol{\mu}_h)+2b(I_h^0\boldsymbol{\mu}_h-\boldsymbol{\mu},I_h^1p_h-p)-2b(I_h^0\boldsymbol{\mu}_h-\boldsymbol{\mu}_h,I_h^1p_h-p_h)-\lambda(I_h^1p_h-p,I_h^1p_h-p)$$
$$=C_h-\lambda(I_h^1p_h-p,I_h^1p_h-p)$$

Therefore, the right-hand side term in the above equation is bounded by $O(h)$ .A combination of the above estimates give the assertion.

Theorem6indicates that $\eta_h(\boldsymbol{\mu}_h,p_h,\Omega)$ isasymptotically exact a-posteriori indicator of $\hat{\lambda}_h(I_h^0\boldsymbol{\mu}_h,I_h^1p_h)-\lambda$.

Next we establish the adaptive version of a-posteriori error result.This paper uses the new indicator $\eta_h(\boldsymbol{\mu}_h,p_h,\Omega)$ to modify the algorithm.Next we shall give a new indicator $\eta_h(\boldsymbol{\mu}_h,p_h,\Omega)$ based adaptive algorithm for the eigenvalue problem(3) as follow:

Step 1: Let $l=0$ .Pick any initial mesh Π_{h_0} with mesh size h_0 .

Step 2: Solve(10)on Π_{h_l} for discrete solution $(\lambda^{h_l},\boldsymbol{\mu}^{h_l},p^{h_l})$.

Step 3: Compute the local indicators $\eta_{h_l}(\boldsymbol{\mu}^{h_l},p^{h_l},\kappa)$.

Step 4: Construct $\Pi_{h_l}\subset\Pi_{h_l}$ by Marking Strategy E1 and parameter θ .

Step 5: Refine Π_{h_l} to get a new mesh $\Pi_{h_{l+1}}$ by Procedure REFINE.

Step 6: Solve (10)on $\Pi_{h_{l+1}}$ for discrete solution $(\lambda^{h_l},\boldsymbol{\mu}^{h_l},p^{h_l})$.

Step 7: Let $l=l+1$ and go to Step 3.

Marking Strategy E1

Given parameter $0<\theta<1$:

Step 1: Construct a minimal subset $\Pi_{h_l}\subset\Pi_{h_l}$ by selecting some elements in Π_{h_l} such that

$$\sum_{\kappa\in\hat{\Pi}_{h_l}}\eta_{h_l}(\boldsymbol{\mu}^{h_l},p^{h_l},\kappa)\geq\theta\eta_{h_l}(\boldsymbol{\mu}^{h_l},p^{h_l},\Omega).$$

Step 2: Mark all the elements in Π_{h_l} .

VIII. NUMERICAL RESULTS

In this section,we report some numerical experiments to demonstrate the effectiveness of our approach.Considering the problem (1), our program is compiled under the iFEM package.Throughout this section, we give the numerical results for the first eigenvalue and its corresponding eigenfunction.Of course, we should point out that our method can also be used for other simple eigenvalues.

Example 1:When K is the identity matrix,consider the following equation

$$\begin{cases}-\Delta p=\lambda p, & \text{in } \Omega,\\ p=0, & \text{on } \partial\Omega.\end{cases}$$

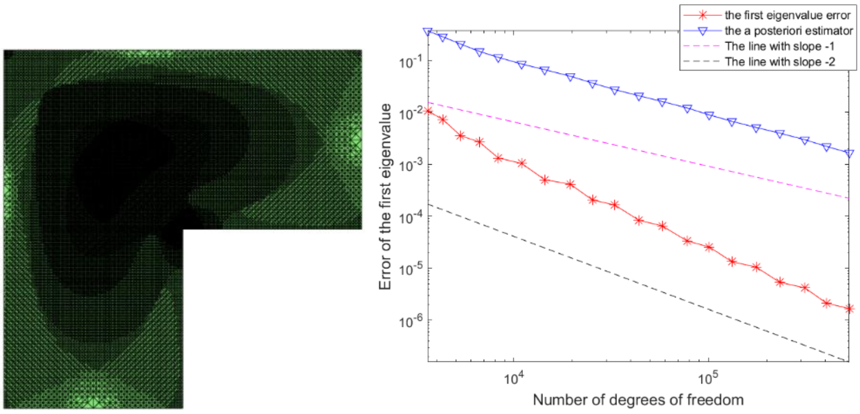
where $\Omega=[-1,1]\times[-1,1]\backslash[-1,0]\times[0,1]$.

Since the exact first eigenvalue is unknown, we choose a sufficiently accurate approximation $\lambda=9.6397238440219$ as the exact value for our numerical tests. Here, we present the numerical results of the adaptive mixed finite element algorithm for the first eigenpair approximation of the parameter $\theta=0.5$.

Table 1: Results of numerical solutions of quadratic eigenvalues for region Ω_L ,with an initial grid of 1/8

Domain	h	*dof*	λ_1	*Error*
Ω_L	1/4	185073	9.639720015638154	0.00490596566668596
	1/8	314483	9.639719641767611	0.00298391055419493
	1/16	389417	9.639718270448762	0.00234939293266335
	1/32	504793	9.639712771932910	0.00180819752180261
	1/64	554043	9.639695775258204	0.00175056001581682
	1/128	929675	9.639653032429592	0.00165141811437118

Figure 1: On the test domain Ω_L , the initial grid is $1/8$ quadratic adaptive mesh and error curve



Example 2:consider the following equation

$$\begin{cases} -\nabla\cdot(K(x,y)\,\nabla p)=\lambda p, & \text{in } \Omega, \\ p=0, & \text{on } \partial\Omega. \end{cases}$$

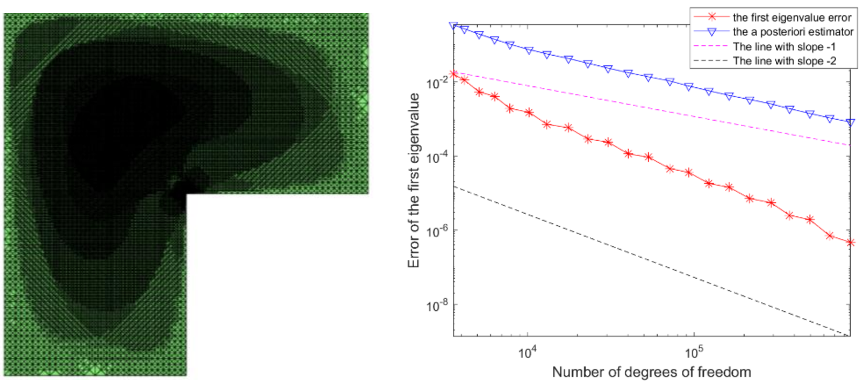
where $K=(2,1;1,2)$, $\Omega=[-1,1]\times[-1,1]\setminus[-1,0]\times[0,1]$.

Since the exact first eigenvalue is unknown, we choose a sufficiently accurate approximation $\lambda=14.459585291043194$ as the exact value for our numerical tests. Here, we present the numerical results of the adaptive mixed finite element algorithm for the first eigenpair approximation of the parameter $\theta=0.5$.

Table 2: Results of numerical solutions of quadratic eigenvalues for region Ω_L ,with an initial grid of 1/8

Domain	*h*	*dof*	λ_1	*Error*
Ω_L	1/4	174959	14.459579059619493	0.00398978295837711
	1/8	289205	14.459579822702214	0.00252226765181019
	1/16	362877	14.459577656883361	0.00194570028372228
	1/32	487025	14.459569278849736	0.00143109094655359
	1/64	566961	14.459543714784211	0.00134665473265421
	1/128	965333	14.459479551173692	0.00112614034170943

Figure 2: On the test domain Ω_L, the initial grid is $1/8$ quadratic adaptive mesh and error curve



Example 3:consider the following equation

$$\begin{cases} -\nabla\cdot(K(x,y)\,\nabla p)=\lambda p, & \text{in } \Omega, \\ p=0, & \text{on } \partial\Omega. \end{cases}$$

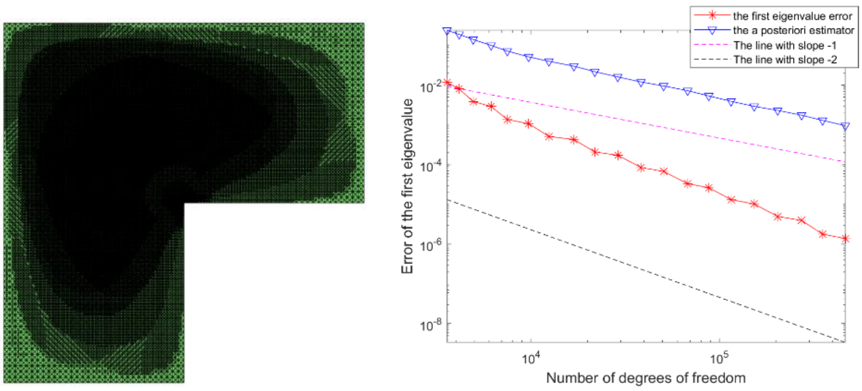
where $K=1+x^2y^2$, $\Omega=[-1,1]\times[-1,1]\setminus[-1,0]\times[0,1]$.

Since the exact first eigenvalue is unknown, we choose a sufficiently accurate approximation $\lambda=10.233889180562130$ as the exact value for our numerical tests. Here, we present the numerical results of the adaptive mixed finite element algorithm for the first eigenpair approximation of the parameter $\theta=0.5$.

Table 3: Results of numerical solutions of quadratic eigenvalues for region Ω_L ,with an initial grid of 1/8

Domain	*h*	*dof*	λ_1	*Error*
Ω_L	1/4	150631	10.233878953138044	0.00299194549108736
	1/8	273475	10.233885228135211	0.00176079131288725
	1/16	346147	10.233883604835187	0.00133160286204250
	1/32	472895	10.233877479692703	0.000953363610147218
	1/64	553731	10.233858840744100	0.000910692995065630
	1/128	960401	10.233812028648238	0.000802153441326506

Figure 3: On the test domain Ω_L, the initial grid is $1/8$ quadratic adaptive mesh and error curve



The numerical eigenvalue results obtained through adaptive calculations are presented in Tables1 to 3, and the figures illustrate the adaptive mesh and error curves. From Figures 1 to 3, we can observe that the error curve of the numerical eigenvalues is approximately parallel to the error index curve to some extent. This indicates that all the posterior error indices for the numerical eigenvalues are reliable and effective. The results demonstrate that the adaptive algorithm achieves the optimal convergence rate. Additionally, from the error curves, it is evident that for the same degrees of freedom, the approximation obtained by the adaptive algorithm is more accurate than that obtained through uniform grid calculations.

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