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On the construction of new stochastic fractional analysis

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Abstract*. Stochastic processes governed by a new fractional Brownian motion are thoroughly examined in this paper. Building on the foundational work of El-Borai and El-Nadi, we offer advanced analytical techniques to study these processes, which are distinguished by their memory effects and long-range interdependence. Our approach uses fractional stochastic calculus, fractional stochastic analysis, and fixed-point theorems to provide a solid mathematical basis to deal with these complicated systems.*
The study makes a substantial contribution to the subject by explicitly solving fractional stochastic differential equations, analyzing their statistical properties with characteristic and moment-generating functions, and proving the existence and uniqueness of solutions within this new framework.

Keywords: *Fractional Brownian motion, stochastic processes, fractional calculus, stochastic differential equations, existence and uniqueness theorem*

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I. Introduction

Fractional calculus and stochastic analysis have grown in significance in applied and theoretical mathematics during the past few decades. It is now clear that many physical systems, financial models, and natural phenomena display memory effects or non-local interactions as our understanding of complex systems advances. Conventional tools for stochastic analysis are unable to capture these features. New Fractional Brownian motion (NfBm), which provides a potent model for systems with long-range dependencies, enters the picture here. Because of these dependencies, which imply that future values of a process are dependent on past values to differing degrees rather than being independent of them, NfBm is a particularly helpful tool in a variety of domains, including finance, engineering, and physics

New Fractional Brownian motion is an advanced form of classical Brownian motion, where the system's behavior is influenced not just by the current state but also by its historical states. fractional Brownian motion bases a system's future evolution on both its current state and its previous states. Understanding systems with intricate memory dynamics, like turbulence, biological systems, and financial markets, is made easier by the study of this kind of motion.

Fractional stochastic calculus, which enables the modeling and analysis of stochastic systems with long-range dependencies, is the result of the integration of fractional calculus into stochastic analysis. The mathematical tools needed to model non-local interactions and systems where the memory of previous states is essential are provided by fractional calculus. This is especially important for systems that are described by stochastic differential equations (SDEs), which control how dynamic systems behave when subjected to both random fluctuations and deterministic forces. Understanding how processes change over time, particularly when their future behavior is connected to their complete history rather than just their current state, depends on these equations.

A fundamental component of stochastic analysis, stochastic differential equations are used to model systems that change because of randomness. Fractional derivatives, which add a memory effect that goes beyond the recent past, are what give fractional systems their complexity. Understanding phenomena in a variety of domains, including economics and the physical sciences, depends on solving these equations. We extend the classical calculus to model processes that display memory effects and historical dependence, which are observed in real-world systems, by introducing fractional derivatives.

The development of fractional stochastic analysis has been greatly aided by the work of El-Bori and El-Nadi.to address stochastic processes with fractional components, they have created sophisticated approaches that offer profound insights into the mathematical foundations of these systems. Their work provided useful

tools for scientists and engineers working with complicated systems that display both randomness and memory effects, laying the groundwork for our knowledge of the existence and uniqueness of solutions to fractional stochastic differential equations.
This study is divided into several key sections, each focusing on fundamental aspects of new fractional stochastic analysis:

1. Introduction to New Fractional Brownian Motion and New Fractional Stochastic calculus: This section provides the necessary theoretical background to understand the mathematical framework of the new fractional Brownian motion and its significance in stochastic analysis. It introduces key concepts such as the Hurst exponent and the properties of the new fractional Brownian motion
2. Fractional Stochastic Differential Equations: The second section focuses on the theory of fractional stochastic differential equations. It presents methods for solving these equations, along with an exploration of their statistical properties
3. characteristic and moment-generating functions: In this sectionwe computed the characteristic and moment-generating functions for the new fractional Brownian motion.

Thecharacteristicfunctioniscrucial.Itprovidesinsightintothedistributionanddependencestructure of the new fractional Brownian motion, which is significant because it exhibits long-range dependence properties.The moment-generatingfunctionisalsoimportantbecauseitallowsustocomputemomentsoftheprocess,helpingin analyzing its behavior over time, which is critical in finance and signal processing applications

4. Existence and Uniqueness Theorems for New Fractional Stochastic Processes:

This section delves into the theoretical foundations of fractional stochasticprocesses, discussing the existence and uniqueness of solutions. It coversmathematical results and theorems that guarantee the well-defined nature ofsolutions within the framework of fractional stochastic analysis

5. Applications of Fractional Brownian Motion: This section highlights the applications of fractional Brownian and how to solve different equations
6. Future Directions and Conclusion

Let $(\Omega, \mathrm{f}, \mathrm{P})$ be a probability space. According to the previous results in [1]if a random variable $X: \Omega \to (-\infty, \infty)$ has a probability density function f defined by

$$f(X) = \int_0^\infty \frac{1}{\sqrt{2\pi t^\alpha \theta}} \zeta_\alpha(\theta) e^{\frac{-(x-m)^2}{2t^\alpha \theta}} d\theta \qquad (1)$$

We say as in [1] that X has a fractional Gaussian (or fractional normal) distribution with mean m and variance $\frac{t^\alpha}{\Gamma(\alpha+1)}$, where $\zeta_\alpha(\theta)$ is the stable distribution density function and $\Gamma(.)$ is the gamma function see [2]. In this case, we write X is $N_\alpha(m, \frac{t^\alpha}{\Gamma(\alpha+1)})$

Again, according to the previous results in [1], we call a real values stochastic process $W_\alpha(.)$ a fractional Brownian motion if the following conditions are satisfied:

i. $W_\alpha(0) = 0$

ii. $W_\alpha(t) - W_\alpha(s)$ is $N_\alpha(0, \frac{t^\alpha - t^s}{\Gamma(\alpha+1)})$ for all $0 < s < t$

iii. For all times $0 < t_1 < \cdots < t_n$ the random variables $W_\alpha(t_1), W_\alpha(t_2) - W_\alpha(t_1), \ldots \ldots \ldots \ldots, W_\alpha(t_n) - W_\alpha(t_{n-1})$ are independent,(with independent increments)

Notice that

$E(W_\alpha(t)) = 0$, $E(W_\alpha^2(t)) = \frac{t^\alpha}{\Gamma(\alpha+1)}$,$E(W_\alpha(t)W_\alpha(s)) = \frac{s^\alpha}{\Gamma(\alpha+1)}$, $s \le t$

where E(X) is the expectation of X

Let $\mathcal{L}^2(0,T)$ be the space of all real-valued, progressively measurable stochastic presses G(.) such that $E\left(\int_0^T G^2 dt\right) < \infty$

The fractional stochastic integral $\int_0^T G \, dW_\alpha$ is defined in [1]

It's proved that

i. $\int_0^T W_\alpha \, dW_\alpha = \frac{W_\alpha^2(T)}{2} - \frac{T^\alpha}{2\Gamma(\alpha+1)}$

ii. $d(tW_\alpha) = tdW_\alpha + W_\alpha dt$

iii. $\int_0^T (aG + bH) dW_\alpha = a\int_0^T G dW_\alpha + b\int_0^T H \, dW_\alpha$

iv. $E\left(\int_0^T G \, dW_\alpha\right) = 0$

v. $E\left(\int_0^T G \, dW_\alpha \int_0^T H \, dW_\alpha\right) = \frac{1}{\Gamma(\alpha)} E(\int_0^T t^{\alpha-1} GH dt)$

For all $G, H \in \mathcal{L}^2(0,T)$ and all real numbers a, b

II. New stochastic analysis

Theorem 1. New Fractional stochastic calculus with Hermite polynomials
Let $h_1(x, t\ ), \dots \dots \dots \dots \dots \dots, h_n(x, t)$ be the Hermite polynomials then,

$$\int_0^t h_n(W_\alpha(t), \frac{s^\alpha}{\Gamma(\alpha+1)}) dW_\alpha(s) = h_{n+1}(W_\alpha(t), \frac{t^\alpha}{\Gamma(\alpha+1)}) \quad (2)$$

Which implies

$$dh_{n+1}\left(W_\alpha, \frac{t^\alpha}{\Gamma(\alpha+1)}\right) = h_n\left(W_\alpha, \frac{t^\alpha}{\Gamma(\alpha+1)}\right) dW_\alpha(t) \quad (3)$$

Thus, in the fractional stochastic calculus the expression$h_n(W_\alpha(t), \frac{t^\alpha}{\Gamma(\alpha+1)})$takes the role of$\frac{t^n}{n!}$ In theordinary calculus

.Proof

we begin by considering the function $e^{-(x-\lambda)^2 \backslash 2t}$ and compute its n-th derivative with respect to

$$\frac{d^n}{d\lambda^n}[e^{-(x-\lambda)^2 \backslash 2t}]|_{\lambda=0} = (-1)^n e^{\frac{-x^2}{2t}} \frac{d^n}{dx^n}(e^{\frac{-x^2}{2t}}) \quad (4)$$

Using the definition of Hermite polynomials, this simplifies to:

$$\frac{d^n}{d\lambda^n}[e^{-(x-\lambda)^2 \backslash 2t}]|_{\lambda=0} = n!\, h_n(x, t)(5)$$

The exponential function $e^{\lambda x - \frac{\lambda^2 t}{2}}$ can be explained in terms of Hermite polynomials as

$$e^{\lambda x - \frac{\lambda^2 t}{2}} = \sum_{n=0}^{\infty} \lambda^n h_n(x, t)(6)$$

Using the expansion, we can represent $Y(t) = e^{\lambda W_\alpha(t) - \frac{\lambda^2 t^\alpha}{2\Gamma(\alpha+1)}}$ as

$$Y(t) = \sum_{n=0}^{\infty} \lambda^n h_n\left(W_\alpha(t), \frac{t^\alpha}{\Gamma(\alpha+1)}\right)(7)$$

To fractional stochastic differential equation $dY = \lambda\, dW_\alpha(t)$ with the initial condition $Y(0) = 1$ can be solved by substituting the series expression of $Y(t)$ these yields

$$\sum_{n=0}^{\infty} \lambda^n h_n\left(W_\alpha(t), \frac{t^\alpha}{\Gamma(\alpha+1)}\right) = 1 + \lambda \int_0^t \sum_{n=0}^{\infty} \lambda^n h_n\left(W_\alpha(s), \frac{s^\alpha}{\Gamma(\alpha+1)}\right) dW_\alpha(s) \quad (8)$$

By matching coefficients of λ^non both sides, we obtain the recursive relation for Hermite polynomials

$$\sum_{n=0}^{\infty} \lambda^n h_n(W_\alpha(t), \frac{t^\alpha}{\Gamma(\alpha+1)}) = 1 + \sum_0^t \lambda^n \int_0^t h_{n-1}(W_\alpha(s), \frac{s^\alpha}{\Gamma(\alpha+1)}) dW_\alpha(s) \quad (9)$$

This completes the proof. see [3-6]

Theorem 2. Let$W_\alpha(t)$ and $\widetilde{W_\alpha}(t)$ be independent one-dimensional fractional Brownian motions. Then, the stochastic differential of their products is given by:

$$dW_\alpha \widetilde{W}_\alpha = W_\alpha d\widetilde{W}_\alpha + \widetilde{W}_\alpha dW_\alpha \quad (10)$$

Since $w_\alpha(t)$ and $\widetilde{w_\alpha}(t)$ are independent, no correction term involving “dt” appears in the expression
To proceed, define a new process X(t) as:

$$X(t) = \frac{\widetilde{W}_\alpha + W_\alpha}{\sqrt{2}} \quad (11)$$

This process X(t) is a one-dimensional fractional Brownian motion because it is a linear combination of two independent fractional Brownian motions. The distribution of X(t) is $N_\alpha \sim (0, \frac{t^\alpha}{\Gamma(\alpha+1)})$ which confirms that X(t) has the standard properties of a fractional Brownian motion
Using fractional Itô Lemma, the differentials of $X^2, {W_\alpha}^2$ and ${\widetilde{W_\alpha}}^2$are given by

$$\begin{aligned} d(X^2) &= 2XdX + \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \\ dW_\alpha^2 &= 2W_\alpha dW_\alpha + \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \quad (12) \\ d\widetilde{W}_\alpha^2 &= 2\widetilde{W}_\alpha d\widetilde{W}_\alpha + \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \end{aligned}$$

Substituting these into the expression of $d(W\widetilde{W})$,we get:

$dW_\alpha\widetilde{W}_\alpha = dX^2 - \frac{1}{2}dW_\alpha^2 - \frac{1}{2}d\widetilde{W}_\alpha^2$(13) $dW_\alpha\widetilde{W}_\alpha = 2XdX + \frac{t^{\alpha-1}}{\Gamma(\alpha)}dt - W_\alpha dW_\alpha - \frac{1}{2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}dt - \widetilde{W}_\alpha d\widetilde{W}_\alpha - \frac{1}{2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}dt$ (14)

$dW_\alpha\widetilde{W}_\alpha = 2XdX - W_\alpha dW_\alpha - \widetilde{W}_\alpha d\widetilde{W}_\alpha$ (15)

$$dW_\alpha\widetilde{W}_\alpha = \frac{2}{\sqrt{2}}[W_\alpha + \widetilde{W}_\alpha]d[\frac{W_\alpha+\widetilde{W}_\alpha}{2}] - W_\alpha dW_\alpha - \widetilde{W}_\alpha d\widetilde{W}_\alpha \quad (16)$$

$$dW_\alpha\widetilde{W}_\alpha = W_\alpha dW_\alpha + W_\alpha d\widetilde{W}_\alpha + \widetilde{W}_\alpha dW_\alpha + \widetilde{W}_\alpha d\widetilde{W}_\alpha - \widetilde{W}_\alpha d\widetilde{W}_\alpha - W_\alpha dW_\alpha \quad (17)$$

It easy to see that then

$$dW_\alpha\widetilde{W}_\alpha = W_\alpha d\widetilde{W}_\alpha + \widetilde{W}_\alpha dW_\alpha \quad (18)$$

See [7-11]

2.1 New fractional stochastic calculus and Langevin's Equation

Integratingfractionalforcesintotheone-dimensionalscenariosuggestsapotentialadjustmenttothemathematical framework of motion in a Brownian system.

$$X = -\beta X+\gamma\epsilon \quad (19)$$

At an instance whereϵ(.) represents 'white noise', β>0 symbolizes a friction coefficient and$\gamma$is a coefficient of diffusion

Rewriting this in terms of fractional stochastic differential equations, we have

$$dX = -\beta Xdt + \gamma dW_\alpha(t) \quad (20)$$

Which the initial condition

$$X(0) = X_0 \quad (21)$$

The solution of the problem (20), (21) is given by

$$X(t) = e^{-\beta t}X_0 + \gamma\int_0^t e^{-\beta(t-s)}dW_\alpha(s) \quad (22)$$

Where the term $e^{-\beta t}X_0$ represents the deterministic decay of the initial velocity due to friction and the term $\gamma\int_0^t e^{-\beta(t-s)}dW_\alpha(s)$accounts for random fluctuations driven by noise

The expected value of X(t) is

$$E[X(t)] = e^{-\beta t E[X_0]} \quad (23)$$

This shows that the mean velocity decays exponentially to zero as time in-creases

To compute the variance let us compute $E[X^2(t)]$

$$E[X^2(t)] = E[e^{-2\beta t}X_0^2 + 2e^{-\beta t}X_0\gamma\int_0^t e^{-\beta(t-s)}dW_\alpha(t) + \gamma^2(\int_0^t e^{-\beta(t-s)}dW_\alpha(s))^2] \quad (24)$$

$$E[X^2(t)] = e^{-2\beta t}E[X_0^2] + \frac{\gamma^2}{\Gamma(\alpha)}\int_0^t e^{-2\beta(t-s)}s^{\alpha-1}ds \quad (25)$$

The variance of X(t) is given by

$$var\,[X(t)] = e^{-2\beta t}var(X_0) + \frac{\gamma^2}{\Gamma(\alpha)}\int_0^t e^{-2\beta(t-s)}s^{\alpha-1}ds \quad (26)$$

Here the first term $e^{-2\beta t}var(X_0)$ represents the decaying contribution form the initial value. The second term $\frac{\gamma^2}{\Gamma(\alpha)}\int_0^t e^{-2\beta(t-s)}s^{\alpha-1}ds$ accounts for the steady-state fluctuation due to noise see [12-13]

III. Characteristic and Moment generating function

From the definition of the α-fractional Brownian motion

We can write

$$p(\rho < W_\alpha(t) < \omega) = \int_\rho^\omega\int_0^\infty \frac{1}{\sqrt{2t^\alpha\pi\theta}}\zeta_\alpha(\theta)e^{\frac{-x^2}{2t^\alpha\theta}}d\theta dx \quad (27)$$

Then the characteristic function is provided by

$$E[e^{i\lambda W\alpha(t)}] = \sum_{k=0}^{\infty} \frac{(-t^{\alpha}\frac{\lambda^2}{2})^k}{\Gamma(\alpha k+1)} \qquad (28)$$

Proof

$$E[e^{i\lambda W\alpha(t)}] = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{1}{\sqrt{2t^{\alpha}\pi\theta}} e^{i\lambda\theta}\zeta_{\alpha}(\theta)e^{\frac{-x^2}{2t^{\alpha}\theta}}d\theta dx$$

$$E[e^{i\lambda W\alpha(t)}] = \int_{0}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2t^{\alpha}\pi\theta}} e^{i\lambda\theta}\zeta_{\alpha}(\theta)e^{\frac{-x^2}{2t^{\alpha}\theta}}dxd\theta$$

$$E[e^{i\lambda W\alpha(t)}] = \int_{0}^{\infty} e^{\frac{-\lambda^2\theta t^{\alpha}}{2}}\zeta_{\alpha}(\theta)d\theta$$

Thus

$$[e^{i\lambda W\alpha(t)}] = \sum_{k=0}^{\infty} \frac{(-t^{\alpha}\frac{\lambda^2}{2})^k}{\Gamma(\alpha k+1)} \qquad (29)$$

And the moment generating function is given by

$$E[e^{\lambda W\alpha(t)}] = \sum_{k=0}^{\infty} \frac{(t^{\alpha}\frac{\lambda^2}{2})^k}{\Gamma(\alpha k+1)} \qquad (30)$$

See [14]

IV. Existence and Uniqueness theorem

We begin with a useful lemma

Lemma3.Fractional Gronwall's Lemma

Let ϕ and f be nonnegative, continuous functions defined for ($0 \le t \le T$) and let $C_0 > 0$ denote a constant. If

$$\phi(t) \le C_0 + C_1\int_0^t s^{\alpha-1}\phi(s)ds \text{ for all } (0 \le t \le T) \qquad (31)$$

Where C_0 and C_1 are positive, then

$$\phi(t) \le C_0 e^{C_1\frac{t^{\alpha}}{\alpha}} \qquad (32)$$

Proof

Define

$$\psi(t) = C_0 + C_1\int_0^t s^{\alpha-1}\phi(s)ds \ge \phi(t) \qquad (33)$$

Then

$$\psi'(t) = C_1 t^{\alpha-1}\phi(t) \le C_1 t^{\alpha-1}\psi \qquad (34)$$

And so

$$\frac{d}{dt}\left[e^{-\frac{C_1t^{\alpha}}{\alpha}}\psi\right] = e^{-\frac{C_1t^{\alpha}}{\alpha}}\psi' - t^{\alpha-1}C_1e^{-\frac{C_1t^{\alpha}}{\alpha}}\psi \qquad (35)$$

$$\frac{d}{dt}\left[e^{-\frac{C_1t^{\alpha}}{\alpha}}\psi\right] = e^{-\frac{C_1t^{\alpha}}{\alpha}}[\psi' - C_1t^{\alpha-1}\psi] \qquad (36)$$

And thus

$$\phi(t) \le C_0 e^{\frac{C_1t^{\alpha}}{\alpha}} \qquad (37)$$

4.1 Existence and uniqueness theorem

SupposethatV_2: Rx [0, T]andV_1: Rx[0,T]areuninterruptedandmeettheupcomingrequirements1. $|V_2(x,t) - V_2(\tilde{x},t)| \le L|x-\tilde{x}|$, $|V_1(x,t) - V_1(\tilde{x},t)| \le L|x-\tilde{x}|$

2. $|V_2(x,t)| \le L(1+|x|)$,$|V_1(x,t)| \le L(1+|x|)$

3.let X_0be any $\mathbb{R}^n$-valued random variable such that

$$E[|X_0|^2] < \infty$$

4.And X_0 is independent of $W_\alpha^*(0)$ where $W_\alpha(0)$ is a given fractional Brownian motion following that here is a unique solution X $\in L^2(0,T)$of the fractional stochasticdifferential equation

$$X(t) = X_0 + \int_o^t (t-s)^{\beta-1} V_2(X,s)ds + \int_0^t V_1(X,s)dW_\alpha(s) \quad (38)$$

And

$$X(0) = X_0 \quad (39)$$

Remake (i) "unique" denotes that in case $X, \tilde{X} \in L^2(0,T)$with continuous sample paths roughly surely, and either resolve the fractional stochastic differential equation then

$$P(X(t) = \tilde{X}(t) \text{for all} 0 \leq t \leq T) = 1$$

(ii)hypotheses(1)statesthatV_2andV_1areuniformlylipschitzcontinualwithininthevariablex Notice also that hypothesis (2) follows from (1)

Proof 1. uniqueness
Suppose that X(t) and $\tilde{X}(t)$ are the solutions as above, them

$$X(t) - \tilde{X}(t) = \int_0^t (t-s)^{\beta-1}[V_2(X,s) - V_2(\tilde{X},s)]ds + \int_0^t [V_1(X,s) - V_1(\tilde{X},s)]dW_\alpha(s) \quad (40)$$

We can estimate

$$E[|X(t) - \tilde{X}(t)|^2] \leq 2E(|\int_0^t (t-s)^{\beta-1}[V_2(X,s) - V_2(\tilde{X},s)]ds|^2) + 2E(|\int_0^t [V_1(X,s) - V_1(\tilde{X},s)]dW_\alpha(s)|^2) \quad (41)$$

Here, we apply *Cauchy-Schwarz inequality * to estimate the first term as follows

$$|\int_0^t (t-s)^{\beta-1} V_2(s)ds|^2 \leq \int_0^t (t-s)^{2\beta-2}ds \int_0^t {V_2}^2(s)ds$$
$$\leq \frac{t^{2\beta-1}}{2\beta-1} \int_0^t {V_2}^2(s)ds \quad (42)$$

By assuming the Lipschitz condition on 1, we have

$$E(|\int_0^t (t-s)^{\beta-1} V_2(X,s) - V_2(\tilde{X},s)ds|^2) \leq \int_0^t L^2 T\, E(X(s) - \tilde{X}(s))^2 ds \quad (43)$$

So

$$E(|\int_0^t (t-s)^{\beta-1}[V_2(X(s),s) - V_2(\tilde{X}(s),s)]ds|^2) \leq L^2 T \int_0^t (t-s)^{\alpha-1}(t-s)^{1-\alpha} E(X-\tilde{X})^2 ds$$
$$\leq L^2 T t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} E(X(s) - \tilde{X}(s))^2 ds$$

(44)

For the fractional stochastic term

$$E(|\int_0^t V_1(X(s),s) - V_1(\tilde{X}(s),s)dW_\alpha(s)|^2) = E\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)}|V_1(X,s) - V_1(\tilde{X},s)|^2 ds \quad (45)$$

Combining the above results, we have

$$E(|X(t) - \tilde{X}(t))|^2 \leq C\int_0^t s^{\alpha-1} E|X(s) - \tilde{X}(s)ds|^2 \quad (46)$$

Where C is a constant depending on L and T
By Fractional Gromwell’s lemma

$$E(|X(t) - \tilde{X}(t)|^2) = 0 \quad (47)$$

Thus, X(t)=$\tilde{X}(t)$almost surely, which proves the uniqueness of the solution

Proof 2. Existence
To prove existence, we use an iterative scheme defined as:

$$\begin{cases} X_0(t) = X_0 \\ X_{n+1}(t) = X_0 + \int_0^t (t-s)^{\beta-1} V_2(X_n(s),s)ds + \int_0^t V_1(X_n(s),s)dW_\alpha(s) \end{cases} \quad (48)$$

Define the mean squared error:

$$d_n(t) = E(|X_{n+1} - X_n|^2) \quad (49)$$

For n=0, we estimate

$$d_0(t) = E(|X_1(t) - X_0(t)|^2) \quad (50)$$

$$= E[|\int_0^t (t-s)^{\beta-1} V_2(X_0(s), s)ds - \int_0^t V_1(X_0(s), s)dW_\alpha(s)|^2] \quad (51)$$

$$\leq 2E[|\int_0^t (t-s)^{\beta-1} V_2(X_0(s), s)ds|^2] + 2E[|\int_0^t V_1(X_0(s), s)dW_\alpha(s)|^2] \quad (52)$$

$$\leq 2E[\int_0^t (t-s)^{2\beta-2} ds \int_0^t V_2^2(s, X_0(s))ds] + 2E\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} V_1^2(s, X_0(s))ds \quad (53)$$

$$\leq 2\frac{t^{\beta-1}}{2\beta-1}\int_0^t L^2[1+|X_0|]^2 ds + 2E[\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} L^2[1+|X_0|]^2 ds] \quad (54)$$

$$\leq \frac{T^{2\beta-1}}{2\beta-1}\int_0^t s^{\alpha-1}s^{1-\alpha}L^2[1+|X_0|]^2 ds + E\int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} L^2[1+|X_0|^2]ds \quad (55)$$

Thus

$$d_0(\mathrm{t}) \leq M\frac{t^\alpha}{\Gamma(\alpha)} \quad (56)$$

For some constant M
Next. It easy to see that

$$E[|X_2(t) - X_1(t)|^2] \leq$$

$$2\frac{t^{2\beta-1}}{2\beta-1}E[\int_0^t |V_2(s, X_1(s)) - V_2(s, X_0(s))|^2]ds + 2E\int_0^t [V_1(s, X_1(s)) - V_1(s, X_0(s))]^2 \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \quad (57)$$

AccordingtotheLipschitzcondition(1), Wecanfigureoutaconstant*M*>0asforinstance

$$E[|X_2(t) - X_1(t)|^2] \leq ME\int_0^t |X_1(s) - X_0(s)|^2 \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \quad (58)$$

Thus

$$E[|X_2(t) - X_1(t)|^2] \leq \frac{M^2 t^{2\alpha}}{2(\Gamma(\alpha+1))^2} \quad (59)$$

By induction, we get the required result.
According to the Martingale inequality

$$[\max_{0\leq t\leq T} |X(s)|^p] \leq (\frac{p}{1-p})^p E[|X(t)|^p] \quad (60)$$

We can write

$$E(\max_{0\leq t\leq T} |X_{n+1}(t) - X_n(t)|^2) \leq 2\frac{T^{2\beta-1}}{2\beta-1}L^2\int_0^T E[|X_n(s) - X_{n-1}(s)|^2]ds + 8L^2\int_0^T E|X_n(s) - X_{n-1}(s)|^2 \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds$$
$$\leq \frac{CM^n T^{n\alpha}}{n!(\Gamma(\alpha+1))^n}$$

(61)
By the claim above, where C is a constant
Since

$$P(\max_{0\leq t\leq T} |X_{n+1}(t) - X_n(t)| \leq 2^{2n}E[\max_{0\leq t\leq T} |X_{n+1}(t) - X_n(t)|^2]$$
$$\leq \frac{2^{2n}CM^n T^{n\alpha}}{n!(\Gamma(\alpha+1))^n} \quad (62)$$

And

$$\sum_{n=1}^{\infty} \frac{2^{2n} M^n T^{n\alpha}}{n!(\Gamma(\alpha+1))^n} < \infty \quad (63)$$

Itfollowsthat,TheBorel-Contellilemmacouldbeemployed
Thus

$$P[\max_{0\le t\le T} |X_{n+1}(t) - X_n(t)| > \frac{1}{2^n} i.o] = 0 \quad (64)$$

(i.o “infinitely often”)
Taking thisintoconsideration, for nearlyany$\omega\in\Omega$

$$X_n(t) = X_0(t) + \sum_{j=0}^{n} (X_{j+1}(t) - X_j(t)) \quad (65)$$

Evenlyconvergesuniformlyon$[0,T]$toastochasticprocess$X(t)$.

Carryingoutofpassingtolimitsinthedefinitionof$X_n(.)$isperformedtoprove

$$X(t) = X_0 + \int_0^t (t-s)^{\beta-1} V_2(X,s)ds + \int_0^t V_1(X,s)dW_\alpha(s) \quad (66)$$

For $0 \le t \le T$
Nowwearestillobligedtodemonstrate$X(.)\in L^2(0,t)$
A constant$C>0$couldbedetermined, suchthat

$$E|X_{n+1}(t)|^2 \le C[E|X_0|^2 + CE[|\int_0^t (t-s)^{\beta-1} V_2(X_n(s),s)ds|^2] + CE[|\int_0^t V_1(X_n(s),s)dW_\alpha(s)|^2]$$

(67)

$$\le C[E|X_0|^2 + C\int_0^t (t-s)^{2\beta-2}ds \int_0^t E[V_2^2(s,X_n(s))]ds + C\int_0^t E|V_1(s,X_n(s))|^2 \frac{s^{\alpha-1}}{\Gamma(\alpha)}ds \quad (68)$$

$$\le C[E|X_0|^2 + \frac{Ct^{2\beta-1}}{2\beta-1} t^{1-\alpha} \int_0^t s^{\alpha-1} L^2[1+X_n(s)]^2 ds + CL^2 \int_0^t [1+X_n(s)]^2 \frac{s^{\alpha-1}}{\Gamma(\alpha)}ds \quad (69)$$

Subsequently

$$E|X_{n+1}(t)|^2 \le C[1+E|X_0|^2] + C\int_0^t E[|X_n(s)|^2 \frac{s^{\alpha-1}}{\Gamma(\alpha)}]ds \quad (70)$$

Forsomeconstant$C>0$.
So, by induction, we get

$$E|X_{n+1}(t)|^2 \le C[1+E|X_0|^2]e^{Ct^\alpha} \quad (71)$$

Let $n\longrightarrow\infty$

$$E[|X_n(t)|^n] \le C[1+E|X_0|^2]e^{Ct^\alpha} \quad (72)$$

And inthesamemanner

$$X \in L^2[0,T]$$

Thisconcludestheproofofthetheory. See [15-20]

V. Some applications on the new fractional itô formula

5.1 Solution of fractional stochastic differential equations
Consider the following fractional stochastic differential equations

$$\begin{cases} dX = qXdW_\alpha(t) \\ X(0) = 1 \end{cases} \quad (73)$$

where q is not representing a random variable but continuous function the unique solution to equation (76) is given by:

$$X(t) = \exp\left[\frac{-1}{2\Gamma(\alpha)} \int_0^t s^{\alpha-1} q^2(s)ds + \int_0^t q(s)dW_\alpha(s)\right] \quad (74)$$

To verify the solution, define the fractional auxiliary function

$Y(t) = \frac{-1}{2\Gamma(\alpha)}\int_0^t s^{\alpha-1}q^2(s)ds + \int_0^t q(s)dW_\alpha(s)$ (75)

It can be shown that Y(t) satisfies the fractional differential equation

$dY(t) = \frac{-t^{\alpha-1}}{2\Gamma(\alpha)}q^2(t)dt + q(t)dW_\alpha(t)$ (76)

Using fractional Itô's Lemma for the exponential function $u(x) = e^x$ we get

$du = e^Y dY + \frac{t^{\alpha-1}}{2\Gamma(\alpha)}q^2(t)e^Y dt$ (77)

Then

$$du = e^Y[\frac{-t^{\alpha-1}}{2\Gamma(\alpha)}q^2(t)dt + q(t)dW_\alpha(t)] + \frac{t^{\alpha-1}}{2\Gamma(\alpha)}q^2(t)e^Y dt \quad (78)$$

Simplifying, this becomes

$$du = e^Y dt$$

$$du = e^Y q dW_\alpha(t)$$

$$du = qXdW_\alpha(t) \quad (79)$$

Thus, the proposed solution satisfies the fractional stochastics differential equation.see [21-24]

5.2 Solving a fractional stochastic differential equation with logarithmic transformation

We are given the fractional stochastic differential equation

$$\begin{cases} dX = f(t)X(t)dt + q(t)X(t)dW_\alpha(t) \\ X(0) = 1 \end{cases} \quad (80)$$

We shall that the unique solution of this function is given by

$$X(t) = exp[\int_0^t [f(s) - \frac{s^{\alpha-1}}{2\Gamma(\alpha)}q^2(s)]ds + \int_0^t q(s)dW_\alpha(s)] \quad (81)$$

Proof

Set

$$Y(t) = \int_0^t[f(s) - \frac{s^\alpha}{2\Gamma(\alpha)}q^2]ds + \int_0^t q(s)dW_\alpha(s) \quad (82)$$

Using Fractional Itô's lemma, we compute

$$dY(t) = [f(t) - \frac{t^{\alpha-1}}{2\Gamma(\alpha)}q^2(t)]dt + q(t)dW_\alpha(t) \quad (83)$$

Set $u = e^Y$we get

$$du = \frac{\partial u}{\partial Y}dY + \frac{1}{2}q^2\frac{t^{\alpha-1}}{\Gamma(\alpha)}\frac{\partial^2 u}{\partial^2 Y}dt \quad (84)$$

$$du = e^Y dY + \frac{1}{2\Gamma(\alpha)}q^2 e^Y t^{\alpha-1}dt$$

The final solution is

$$du = f(u)dt + q(u)dW_\alpha(t) \quad (85)$$

Which the required solution see [25-27]

5.3 Appling Fractional Itô's formula to power functions

Let $X(t) = W_\alpha(t)$ which $W_\alpha(t)$represents α-fractional Brownian motion and choose $u(x) = x^m$, where m is a constant. Using fractional Itô's formula which defined as

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX + \frac{1}{2}G^2\frac{t^{\alpha-1}}{\Gamma(\alpha)}\frac{\partial^2 u}{\partial x^2}dt \quad (86)$$

We get

$$du = mX^{m-1}dx + \frac{1}{2}m(m-1)X^{m-2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}dt \quad (87)$$

Thus

$$dW_\alpha^m(t) = mW_\alpha^{m-1}dW_\alpha + \frac{1}{2}m(m-1)\frac{t^{\alpha-1}}{\Gamma(\alpha)}W_\alpha^{m-2}dt \quad (88)$$

See [28-29]

5.4 Applying fractional Itô formula with the exponential function

Let $X(t) = W_\alpha(t)$ and choose $u = exp[\lambda w_\alpha(t) - \frac{\lambda^2}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}]$ where λ is a constant

Using the Fractional Itô formula, we compute

$$du = -\frac{\lambda^2 t^{\alpha-1}}{2\Gamma(\alpha)}exp[\lambda W_\alpha(t) - \frac{\lambda^2}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}]dt + \lambda exp[\lambda W_\alpha(t) - \frac{\lambda^2}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}]dW_\alpha(t) + \frac{1}{2}\frac{\lambda^2 t^{\alpha-1}}{\Gamma(\alpha)}exp[\lambda W_\alpha(t) - \frac{\lambda^2}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}]dt$$

(89)

Finally, we have

$$du = \lambda exp[\lambda W_\alpha(t) - \frac{\lambda^2}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}]dW_\alpha(t) \quad (90)$$

Then

$$\begin{cases} du = \lambda u dW_\alpha(t) \\ u(0) = 1 \end{cases} \quad (91)$$

Which means that in the new fractional stochastic calculus the expression $u = exp[\lambda W_\alpha(t) - \frac{\lambda^2}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}]$ plays the role of $e^{\lambda t}$ in the ordinary calculus see [30-32]

VI. Conclusion

In this investigation, we have expanded the theoretical structure of stochastic analysis by exploring newaspects ofthe new fractional Brownian motion. We thoroughly examined the characteristic function and the moment generating function, which are essential for describing the distribution and moments of this process. Additionally, we established the conditionsneeded for the existence and uniqueness of solutions for this fractional Brownian motion

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