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Hemipolyhedra and their Duals

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There exist 9 nonconvex uniform polyhedra whose name includes "hemi". They are called hemipolyhedra. The duals of these hemipolyhedra share a unique feature, i.e. they are represented by a compound of prisms interpenetrating the center of symmetry. The number of these prisms corresponds to the number of facial planes passing through the center of symmetry of the original hemipolyhedron and the cross section of the prisms is the same as the shape of these planes that pass through the center of symmetry. A symmetry of points and lines arises in a projective plane: just as a pair of points determine a line, so a pair of lines determine a point. The existence of parallel lines leads to establishing a point at infinity which represents the intersection of these parallels. Thus, the lateral faces of the prism are considered to converge at infinity, becoming the vertices which correspond to the original hemipolyhedron's facial planes passing through the center of symmetry. All these are already known and not new. However, the way to identify the vertices corresponding to the other facial planes of the original hemipolyhedron in question has not yet been clearly established. The author gave a consideration to this point and found a simple proper way to enable it. The purpose of this paper is to explain it in an easy-to-understand manner.

Key words. *Hemipolyhedron, dual, prism, convex hull, inner core*

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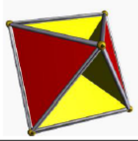
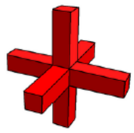
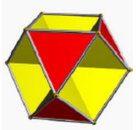
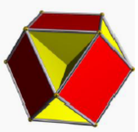
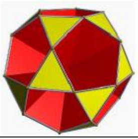
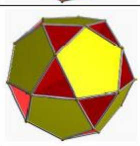
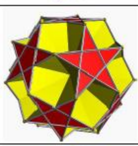
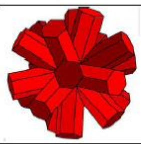
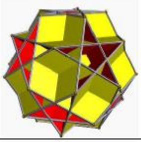
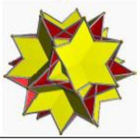
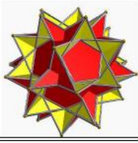
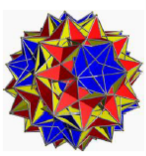
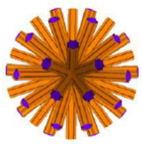
I. Introduction

Since the hemipolyhedra have faces passing through the center, their dual solids have corresponding vertices at infinity; properly speaking, on the real projective plane at infinity. In [1], they are represented by intersecting prisms, each extending in both directions to the same vertex at infinity, in order to maintain symmetry. When a model of such duals is made, these prisms are cut off at a certain point for convenience sake. It is suggested in [1] that these solids be members of a new class of stellation figures, called *stellation to infinity*. However, it is also suggested in [1] that, strictly speaking, they be not polyhedra because their construction does not conform to the usual definitions.

There are nine nonconvex uniform polyhedra whose name includes "hemi". Each of them has their respective dual; therefore, there exist 9 such solids. However, there are only 5 distinct outward forms. This is because the duals of the following pairs share the identical outward form: (1) octahemioctahedron and cubohemioctahedron, (2) small icosihemidodecahedron and small dodecahemidodecahedron, (3) small dodecahemicosahedron and great dodecahemicosahedron, (4) great icosihemidodecahedron and great dodecahemidodecahedron. These pairs share the same convex hull as described in the Table. Its rightmost column shows the outward forms of the duals. The graphics of 9 nine nonconvex uniform polyhedra whose name includes "hemi" and their respective duals were taken from [3] and those of the dirhombicosidodecahedron and its dual were taken from [4].

II. Vertices of Duals corresponding to the Faces of the Original Hemipolyhedra other than Those passing through the Center

The hemipolyhedra, in addition to the facial planes passing through the center, have also those which do not pass through the center. Accordingly, their duals must have vertices corresponding to these facial planes.

Name of hemipolyhedra	Shape	Convex hull	Facial planes passing through the center of symmetry & their quantity	Facial planes other than those passing through the center of symmetry	Quantity of Vertices	Polyhedra composed of such planes	Duals of such polyhedra	Facial planes	Quantity of such planes	Dual Shape	Inner core of such dual
Tetrahemi-hexahedron		Regular octahedron	Cube 3	Regular Triangle	4	Regular tetrahedron	Regular tetrahedron	Regular Triangle	4		Cube
Octahemi-octahedron		Cubocta-hedron	Regular hexagon 4	Regular Triangle	8	Regular octahedron	Cube	Square	8		Rhombic dodeca-hedron
Cubohemi-octahedron		Cubocta-hedron	Regular hexagon 4	Square	6	Cube	Regular octahedron	Regular Triangle	6		Rhombic dodeca-hedron
Small icosihemi-dodecahedron		Icosidodeca-hedron	Regular decagon 6	Regular Triangle	20	Regular icosahedron	Regular dodecahedron	Regular pentagon	20		Rhombic triaconta-hedron
Small dodecahemi-dodecahedron		Icosidodeca-hedron	Regular decagon 6	Regular pentagon	12	Regular dodecahedron	Regular icosahedron	Regular Triangle	12		Rhombic triaconta-hedron
Small dodecahemi-cosahedron		Icosidodeca-hedron	Regular hexagon 10	Regular pentagram	12	Small stellated dodecahedron	Great dodecahedron	Regular pentagon	12		Rhombic triaconta-hedron
Great dodecahemi-cosahedron		Icosidodeca-hedron	Regular hexagon 10	Regular pentagon	12	Regular dodecahedron	Regular icosahedron	Regular Triangle	12		Rhombic triaconta-hedron
Great icosihemi-dodecahedron		Icosidodeca-hedron	Star polygon {10/3} 6	Regular Triangle	20	Regular icosahedron	Regular dodecahedron	Regular pentagon	20		Rhombic triaconta-hedron
Great dodecahemi-dodecahedron		Icosidodeca-hedron	Star polygon {10/3} 6	Regular pentagram	12	Small stellated dodecahedron	Great dodecahedron	Regular pentagon	12		Rhombic triaconta-hedron
Great dirhombicosi-dodecahedron		Variation of rhombicosi-dodecahedron	Two-ply square 30	Two-ply regular pentagram Two-ply regular triangle	12 20	Small stellated dodeca-hedron Regular icosa-hedron	Great dodecahedron Regular dodecahedron	Regular pentagon Regular pentagon	12 20		Deltoidal hexeconta-hedron

However, the way to identify them has not been clearly established yet. They are outwardly difficult to be discovered. For the purpose of finding them out, the author, first of all, carefully studied polyhedra composed of facial planes other than those passing through the center and then researched into vertices of their duals.

In case of the tetrahemihexahedron which is the simplest hemipolyhedron, facial planes other than those passing through the center are four regular triangles and the polyhedron composed of them is a regular tetrahedron. The dual of the regular tetrahedron also is a regular tetrahedron and the number of its vertices is 4. These 4 vertices correspond to the aforementioned 4 regular triangles. In case of the octahemioctahedron, facial planes other than those passing through the center are 8 regular triangles and the polyhedron composed of them is a regular octahedron. The dual of the regular octahedron is a cube and the number of its vertices is 8. These 8 vertices correspond to the above-mentioned 8 regular triangles.

Such outcomes of the study are put together in the Table. It covers 9 kinds of hemipolyhedron that include “hemi” in their respective name and the great dirhombicosidodecahedron that does not include “hemi” in its name.

III. Dirhombicosidodecahedron and its Dual

The great dirhombicosidodecahedron also is a hemipolyhedron. As mentioned above, its name does not include ”hemi” for the reason that it is one of the nonconvex snub polyhedra. As described in [2], it is the only uniform polyhedron with as many as 8 faces meeting at each vertex (i.e. 2 regular pentagrams, 2 regular triangles, and 4 squares). A model of this polyhedron is shown in Figure 1 below. This model was made by the author. It consists of 30 two-ply squares, 12 two-ply regular pentagrams, and 20 two-ply regular triangles. The total number of these faces is 124. A square has 4 vertices and there exist 60 squares (30 two-ply squares); therefore, the total number of vertices is 240. However, 4 squares share their vertices; accordingly, the total number of vertices of this solid is 60.

The dual of the great dirhombicosidodecahedron is a compound of prisms interpenetrating the center of symmetry. A model of this solid is shown in Figure 2 below. This model was also made by the author. The prisms are cut off at a certain point for convenience sake.

Figure 1 Great dirhombicosidodecahedron

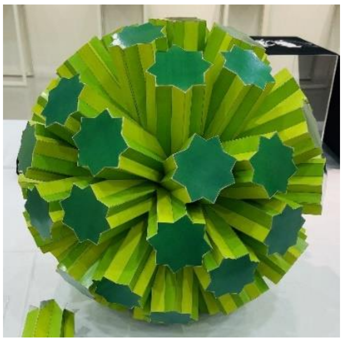


Figure 2 Dual of the great dirhombicosidodecahedron

The number of these prisms is 30 and the cross section of these prisms is the two-ply square that is considered to correspond to the two-ply square passing through the center of symmetry of the great dirhombicosidodecahedron. These prisms are symmetrical with respect to the center. It appears, as mentioned in [1], that 60 prisms radiate out from the center of the solid. Their cross sections are not the regular octagram but the eight-angled star polygon. As each prism has 8 lateral faces, the total number of facial planes of the dual is 240 that correspond to the total number of vertices of the great dirhombicosidodecahedron. All lateral faces of each prism are considered to converge to the same vertex at infinity, in order to maintain symmetry. The convex hull of this polyhedron is variation of rhombicosidodecahedron and its dual is the deltoidal hexecontahedron.

As mentioned earlier, the prisms of the model were cut off at a certain point when it was made, just for convenience sake. Figure 2 above indicates that the dual consists of 12 groups of such prisms, with each group composed of 5 prisms. At the center of each group, there exists an inward-focused vertex which is the one of ten-sided pyramid. This inward-focused vertex is shown in Figure 3 below.

Figure 3 One of 5-prism groups and an inward-focused vertex located at its center

There are 12 such vertices and they correspond to 12 two-ply pentagrams of the great dirhombicosidodecahedron. The number of lateral faces of the ten-sided pyramids is 120 which are included in the total number of facial planes of the dual that is 240. Each of the 12 two-ply pentagrams has 10 vertices.

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These 10 vertices correspond to 10 lateral faces of the pyramid.

The inner core of the dual is the deltoidal hexecontahedron composed of 60 faces which are kites. When 5 kites assemble with the smallest acute angles getting together, a vertex is created in the center of the 5-kite group. See Figure 4 below. It was taken from [5].

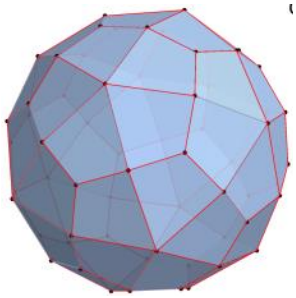

Figure 4 Deltoidal hexecontahedron

The deltoidal hexecontahedron has 12 such vertices. They coincide with the 12 vertices of the great dodecahedron or the regular icosahedron which is inscribed in the deltoidal hexecontahedron. If the length of five prisms shown in Figure 3 decreases without limit, 5 cross sectional planes will get close to each other, illimitably approaching the 5 kites of the deltoidal hexecontahedron. Then, the aforementioned inward-focused vertex will coincide with the above-mentioned vertex created in the center of the 5-kite group.

The polyhedron consisting of 12 pentagrams is the small stellated dodecahedron. Its dual is the great dodecahedron with 12 vertices and they coincide with the aforementioned 12 inward-focused vertices. The great dodecahedron is inscribed in the deltoidal hexecontahedron, i.e. inner core of the dual, and they share vertices.

The dual also has a different group of prisms. This group consists of 3 prisms which forms a regular triangle and, at the center of it, there also exists an inward-focused vertex which is the one of six-sided pyramid. This inward-focused vertex is shown in Figure 5 below.

Figure 5 One of 3-prism groups and an inward-focused vertex located at its center

There are 20 such vertices and they correspond to 20 two-ply regular triangles of the great dirhombicosi-dodecahedron. The total number of lateral faces of the 6-sided pyramids is 120. These 120 faces are also included in the total number of facial planes of the dual that is 240. Each of 20 two-ply regular triangles has 6 vertices. These 6 vertices correspond to 6 lateral faces of the pyramid.

When 3 kites of the deltoidal hexecontahedron assemble with the largest angles getting together, a vertex is created in the center of the 3-kite group. See Figure 4 above. The deltoidal hexecontahedron has 20 such vertices. They coincide with the 20 vertices of the regular dodecahedron which is inscribed in the deltoidal hexecontahedron. If the length of three prisms shown in Figure 5 above decreases without limit, 3 cross sectional planes will get close to each other, illimitably approaching the 3 kites of the deltoidal hexecontahedron. Then, the inward-focused vertex shown in Figure 5 will coincide with the above-mentioned vertex created in the center of the 3-kite group.

The polyhedron consisting of 20 regular triangles is the regular icosahedron. Its dual is the regular dodecahedron with 20 vertices and they coincide with the aforementioned 20 inward-focused vertices. The regular dodecahedron is inscribed in the deltoidal hexecontahedron and they share vertices.

IV. Other Hemipolyhedra and their Duals

As mentioned above, it is not so difficult to identify vertices of the dual corresponding to facial planes other than those passing through the center of symmetry in case of the great dirhombicosidodecahedron. In case of the other hemipolyhedra, however, it is quite difficult to do so. This is because the dual of polyhedron composed of such facial planes is contained in its inner core. Close observation of the dual, however, makes it possible to identify the vertices in question because such dual is inscribed in its inner core and because these two kinds of polyhedron share vertices. It is to be noted in this regard that the convex hull of hemipolyhedra and the inner core of its dual are in a dual relationship with each other. It is indicated in the Table. For example, the convex hull of the tetrahemihexahedron is the regular octahedron and the inner core of its dual is the cube. They are in a dual relationship with each other.

The dual of the tetrahemihexahedron clearly shows the aforementioned fact. See Figure 7. Segments created by 3 intersecting quadratic prisms correspond to edges of its inner core which is the cube. The number of such segments is 12. With these segments crossing each other, 8 inward-focused vertices are created. The tetrahemihexahedron consists of 3 squares and 4 regular triangles. It is shown in Figure 6. The former pass through its center of symmetry. The polyhedron composed of 4 regular triangles is the regular tetrahedron and its dual is the same. It is inscribed in the cube and they share vertices, which are identifiable.

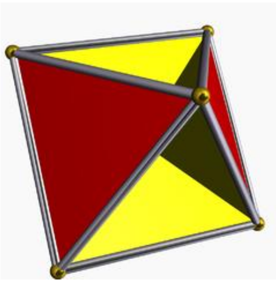

Figure 6 Tetrahemihexahedron

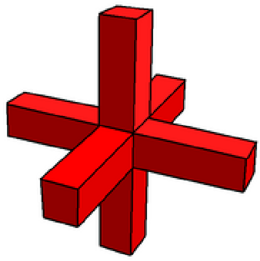

Figure 7 Dual of the tetrahemihexahedron

In case of the dual of the octahemioctahedron, this is not so clear. See Figure 9 below. Twenty-four segments are created by 4 intersecting six-sided prisms and many of them correspond to edges of its inner core which is the rhombic dodecahedron. With these segments crossing each other, 16 vertices are created. The octahemioctahedron consists of 4 regular hexagons and 8 regular triangles. It is shown in Figure 8 below.

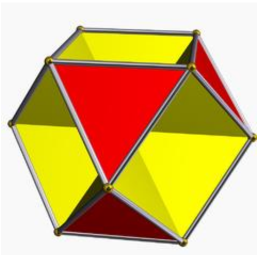

Figure 8 Octahemioctahedron

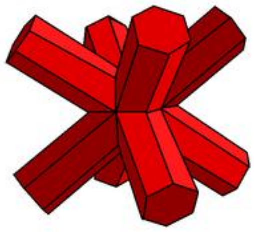

Figure 9 Dual of the octahemioctahedron

The former pass through its center of symmetry. The polyhedron composed of 8 regular triangles is the regular octahedron and its dual is the cube. The cube is inscribed in the rhombic dodecahedron and they share vertices, which are identifiable. The Figures 6 through 9 were all taken from [3].

Similar observation and analysis can be made in case of the other hemipolyhedra and their duals. The following are considered to be important and to be noted in this respect.

- The regular octahedron is inscribed in the rhombic dodecahedron and they share vertices.
- The regular dodecahedron, the regular icosahedron and the great dodecahedron are respectively inscribed in the rhombic triacontahedron and each of the former share vertices with the latter.
- The great dodecahedron and the regular dodecahedron are respectively inscribed in the deltoidal hexecontahedron and each of the former share vertices with the latter.

Acknowledgement

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