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An Extension of Generalized $U_{|h}$ −Birecurrent Finsler Space

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Abstract: *This paper has focuses on a specific class of Finsler spaces known as generalized birecurrent Finsler space. By introducing a new geometric structure, we investigate the properties of these spaces and establish several theorems. Our results generalize previous work on birecurrent Finsler spaces and provide a deeper understanding of their geometry. In this paper, we introduced an extension of the generalized U −birecurrent Finsler spaces. i.e., we define a Finsler space F_n which the curvature tensor U^i_{jkh} satisfies the extension for generalized birecurrence property in sense of Cartan. Further, we get the relations among different curvature tensors in the main space.*

Keyword: *Generalization generalized birecurrent Finsler space, h −covariant derivative, Curvature tensor U^i_{jkh}.*

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I. Introduction and Preliminaries

The Finsler geometry considers as generalization of Riemannian geometry, our research draws inspiration from several prominent studies in the field. The generalized birecurrent spaces for different curvature tensors in sense of Cartan and Berwald discussed by [2, 3, 6, 7, 9, 10, 12, 13, 14]. The extension for generalized $\mathcal{B}K-$ recurrent Finsler space has been introduced by [8].

The recurrence and birecurrence property for the curvature tensor U^i_{jkh} in sense of Cartan and Berwald studied by [15, 19, 20]. Also, the generalized $U_{|l}$ −recurrent space and generalized $\mathcal{B}_m U$ −recurrent space have been introduced by [17, 18].

Let F_n be an n –dimensional Finsler space equipped with the metric function $F(x, y)$ satisfying the request conditions [4, 5, 11, 21]. The vectors y_i and y^i defined by

(1.1) $y_i = g_{ij}(x,y)y^j$.

The metric tensor g_{ij} and its associative g^{ij} are connected by

(1.2) $g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & if\ \ j = k\ , \\ 0 & if\ \ j \neq k\ . \end{cases}$

In view of (1.1) and (1.2), we have

(1.3) a) $\delta^i_j y_i = y_j$, b) $\delta^i_j\, y^j = y^i$ and c) $\delta^i_j g_{ir} = g_{jr}$.

The h −covariant derivative of the metric tensor g_{ij}, associate metric tensor g^{ij}, the vectors y^i and y_i vanish identically, i. e.

(1.4) a) $g_{ij|k} = 0$, b) $g^{ij}_{|k} = 0$, c) $y^i_{|k} = 0$ and d) $y_{j|k} = 0$.

The tensor K^i_{jkh} called *Cartan's fourth curvature tensor* is positively homogeneous of degree zero in y^i and defined by [16]

$$K^{\ i}_{jkh} = \partial_h \Gamma^{*i}_{kj} + (\dot{\partial}_\ell \Gamma^{*i}_{jh}) G^{\ \ell}_{k} + \Gamma^{*i}_{mh}\Gamma^{*m}_{kj} - h/k^*.$$

The associate tensor K_{ijkh}, K −Ricci tensor K_{jk} , curvature scalar K and deviation tensor K_j^i of the curvature tensor $K^{\ i}_{jkh}$ are given by

(1.5) a) $K^i_{jkh} = -K^{\ i}_{jhk}$, b) $K^i_{jki} = K_{jk}$, c) $K_{jk}g^{ij} = K^i_k$,

d) $K_{jk}g^{jk} = K$, e) $K^i_{jkh}\ y^j = H^i_{kh}$, e) $K^i_k g_{jh} = K^i_{jkh}$,

and f) $H^i_{jkh} = K^i_{jkh} + y^s(\dot{\partial}_j K^i_{skh})$,

where g) $H^i_{jki} = H_{jk}$.

The tensor R^i_{jkh} called *Cartan's third curvature tensor* is positively homogeneous of degree zero in y^i and defined by [1, 16]

$$R^i_{jkh} = \partial_h\Gamma^{*i}_{jk} + (\dot{\partial}_\ell\Gamma^{*i}_{jh})G^\ell_k + G^i_{jm}(\partial_h G^m_k - G^{\ m}_{h\ell}G^\ell_k) + \Gamma^{*i}_{mh}\Gamma^{*m}_{jk} - h/k ,$$

This tensor satisfies the following

(1.6) a) $R^{\ i}_{jkh}\ y^j = H^{\ i}_{kh}$, b) $R_{ijkh} = g_{rj}R^r_{ikh}$, c) $R^i_{jkh} = -R^i_{jhk}$

d) $R^i_{jkh} = K^i_{jkh} + C^i_{jm}\ H^m_{kh}$ and e) $R_{jk} = K_{jk} + C^r_{jm}\ H^m_{kr}$.

where R_{ijkh} is the associative tensor of R^i_{jkh} .

The curvature tensor U^i_{jkh} that homogeneous of degree -1 in y^i and symmetric in its last two indices is defined by

$$U^i_{jkh} = G^i_{jkh} + \frac{1}{n+1}(\delta^i_j\ G^r_{khr} + y^i\ G^r_{jkhr})$$

And satisfies the following

(1.7) a) $U^i_{jkh} = U^i_{jhk}$, b) $U^i_{jkh}y^h = U^i_{jhk}y^h = U^i_{jk}$, c) $U^i_{jkh}y^j = 0$,

d) $U^h_{jkh} = U_{jk}$ and e) $U^r_{jr} = G^r_{jr}$.

The U^h −recurrent space, U^h −birecurrent space and generalized $U_{|l}$ −recurrent space are characterized by [18 - 20]

$$U^i_{jkh|l} = \lambda_l U^i_{jkh} , \quad U^i_{jkh} \neq 0 ,$$

$$U^i_{jkh|l|m} = a_{lm}U^i_{jkh}$$

(1.8) $U^i_{jkh|l} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where λ_l and μ_l are non – zero covariant tensors field of first order. $|l$ is called h – covariant derivative with respect to x^l. Taking the h – covariant derivative for (1.8) with respect to x^m, using (1.4a), we get

$$U^i_{jkh|l|m} = \lambda_{l|m}U^i_{jkh} + \lambda_l U^i_{jkh|m} + \mu_{l|m}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$

Using (1.8) in above equation, we get

$$U^i_{jkh|l|m} = \lambda_{l|m}U^i_{jkh} + \lambda_l\{\lambda_m U^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \mu_{l|m}(\delta^i_k g_{jh} - \delta^i_h g_{jk})\}$$

or

(1.9) $U^i_{jkh|l|m} = w_{lm}U^i_{jkh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $U^i_{jkh} \neq 0$,

where $w_{lm} = \lambda_{l|m} + \lambda_l\lambda_m$ and $v_{lm} = \lambda_l\mu_m + \mu_{l|m}$ are non – zero covariant tensor fields of second order. A Finsler space F_n which the curvature tensor U^i_{jkh} satisfies the condition (1.9) is called *a generalized* $U_{|l}$*– Birecurrent space* and denote it briefly by $GU_{|h} - BRF_n$.

III. The Extension of Generalized $U_{|h}$ – Birecurrent Finsler Space

In this section, we discuss a new extension for a generalized $U_{|h}$ – birecurrent Finsler space. The extension for a generalized $U_{|h}$ – recurrent Finsler space is written as

(3.1) $U^i_{jkh|l} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}(U^i_k g_{jh} - U^i_h g_{jk})$

A Finsler space F_n which the curvature tensor U^i_{jkh} satisfies the condition (3.1) is called *the generalization generalized* $U_{|l}$*–recurrent space* and denote it briefly by $G^{2nd}U_{|h} - RF_n$.

Taking h– covariant derivative for (3.1) with respect to x^m, using (1.4a), we get

$$U^i_{jkh|l|m} = (\lambda_{l|m})U^i_{jkh} + \lambda_l(U^i_{jkh|m}) + \mu_{l|m}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$

$+\frac{1}{4}(U^i_{k|m}g_{jh} - U^i_{h|m}g_{jk})$.

Using (3.1) in above equation, we get

$$U^i_{jkh|l|m} = (\lambda_{l|m})U^i_{jkh} + \lambda_l[\lambda_m U^i_{jkh} + \mu_m(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}(U^i_k g_{jh} - U^i_h g_{jk})$$
$$+\mu_{l|m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}(U^i_{k|m}g_{jh} - U^i_{h|m}g_{jk}),$$

or

(3.2) $$U^i_{jkh|l|m} = w_{lm}U^i_{jkh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}\lambda_l(U^i_k g_{jh} - U^i_h g_{jk})$$
$$+\frac{1}{4}(U^i_{k|m}g_{jh} - U^i_{h|m}g_{jk}).$$

where $w_{lm} = (\lambda_{l|m} + \lambda_l\lambda_m)$ and $v_{lm} = (\lambda_l\mu_m + \mu_{l|m})$.

Definition 3.1. A Finsler space F_n which the curvature tensor U^i_{jkh} satisfies the condition (3.2) is called *a generalization generalized $U_{|h}$ −Birecurrent space* and the tensor will be called a *generalization generalized h –birecurrent tensor*. These space and tensor denote them briefly by $G^{2nd}U_{|h} - BRF_n$ and $G^{2nd}h - BR$, respectively.

Transvecting the condition (3.2) by y^h , using (1.4c), (1.3b), (1.1) and (1.7b), we get

(3.3) $$U^i_{jk|l|m} = w_{lm}U^i_{jk} + v_{lm}(\delta^i_k y_j - y^i g_{jk}) + \frac{1}{4}\lambda_l(U^i_k y_j - U^i_h\, y^h g_{jk})$$
$$+\frac{1}{4}(U^i_{k|m}y_j - U^i_{h|m}\, y^h\, g_{jk}) .$$

Contracting the indices i and k in (3.3) and using (1.7e), (1.3a) and (1.1), we get

(3.4) $$G^r_{jr|l|m} = w_{lm}G^r_{jr} + (n-1)v_{lm}y_j + \frac{1}{4}\lambda_l(U^r_r y_j - U^r_h y^h g_{jr})$$
$$+\frac{1}{4}(U^r_{r|m}y_j - U^r_{h|m}y^h g_{jr}).$$

Contracting the indices i and k in (3.2) and using (1.7d), (1.2) and (1.3c), we get

(3.5) $$U_{jh|l|m} = w_{lm}U_{jh} + v_{lm}(\mathrm{n}-1)g_{jh} + \frac{1}{4}\lambda_m(Ug_{jh} - U_{hj}) + \frac{1}{4}(U_{|m}g_{jh} - U_{hj|m}),$$

where $U^i_i = U$ and $U^i_h g_{ji} = U_{hj}$. Thus, we conclude

Theorem 3.1. *In $G^{2nd}U_{|h} - BRF_n$, the torsion tensor U^i_{jk} , tensor G^r_{jr} and $U -$ Ricci tensor U_{jh} are given by (3.3), (3.4) and (3.5), respectively.*

Let us consider a Finsler space F_n which $h(v) -$ curvature tensor U^i_{jkh} satisfies the following condition

(3.6) $$U^i_{jkh|l} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}(K^i_k g_{jh} - K^i_h g_{jk}) , \qquad U^i_{jkh} \neq 0 .$$

Taking the h −covariant derivative for (3.6) with respect to x^m, using (1.4a), we get

$$U^i_{jkh|l|m} = (\lambda_{l|m})U^i_{jkh} + \lambda_l(U^i_{jkh|m}) + \mu_{l|m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}(K^i_{k|m}g_{jh} - K^i_{h|m}g_{jk}).$$

Using (3.6) in above equation, we get

$$U^i_{jkh|l|m} = (\lambda_{l|m})U^i_{jkh} + \lambda_l[\lambda_m U^i_{jkh} + \mu_m(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}(K^i_k g_{jh} - K^i_h g_{jk})]$$
$$+\mu_{l|m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}(K^i_{k|m}g_{jh} - K^i_{h|m}g_{jk})$$

Using (1.5e) and (1.4a) in above equation, we get

$$U^i_{jkh|l|m} = (\lambda_{l|m} + \lambda_l\lambda_m)U^i_{jkh} + (\lambda_l\mu_m + \mu_{l|m})(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$
$$+\frac{1}{4}\lambda_l(K^i_{jhk} - K^i_{jkh}) + \frac{1}{4}(K^i_{jhk|m} - K^i_{jkh|m}) .$$

Using (1.5a) in above equation, we get

(3.7) $$U^i_{jkh|l|m} = w_{lm}U^i_{jkh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) - \frac{1}{2}\lambda_l K^i_{jkh} - \frac{1}{2}K^i_{jkh|m} ,$$

where $w_{lm} = \lambda_{l|m} + \lambda_l\lambda_m$ and $v_{lm} = \lambda_l\mu_m + \mu_{l|m}$.

From (3.7), we get

(3.8) $$U^i_{jkh|l|m} + \frac{1}{2}K^i_{jkh|m} = w_{lm}U^i_{jkh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) - \frac{1}{2}\lambda_l K^i_{jkh} .$$

Thus, we conclude

Theorem 3.2. *In $G^{2nd}U_{|h} - BRF_n$, the relationship between the curvature tensor U^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} is given by (3.8).*

Using (1.5f) in (3.7), we get

$$U^i_{jkh|l|m} = w_{lm}U^i_{jkh} + v_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - \frac{1}{2}\lambda_l\left[H^i_{jhk} - y^s\left(\dot{\partial}_j K^i_{skh}\right)\right] - \frac{1}{2}K^i_{jkh|m} .$$

Above equation can be written

(3.9) $U^i_{jkh|l|m} + \frac{1}{2}K^i_{jkh|m} = w_{lm}U^i_{jkh} - \frac{1}{2}\lambda_l H^i_{jkh} + w_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{2}\lambda_l y^s(\dot{\partial}_j K^i_{skh})$.

Thus, we conclude

Corollary 3.1. *In $G^{2nd}U_{|h} - BRF_n$, the relationship among the curvature tensor U^i_{jkh}, Cartan's fourth curvature tensor K^i_{jkh} and Berwald's curvature tensor H^i_{jkh} are given by (3.9).*

Contracting i and h in the conditions (3.8) and (3.9), using (1.7d), (1.5b), (1.3c), (1.2) and (1.5g), we get

(3.10) $U_{jk|l|m} + \frac{1}{2}K_{jk|m} = w_{lm}U_{jk} + (1-n)v_{lm}g_{jk} - \frac{1}{2}\lambda_m K_{jk}$

and

(3.11) $U_{jk|l|m} + \frac{1}{2}K_{jk|m} = w_{lm}U_{jk} - \frac{1}{2}\lambda_m H_{jk} + (1-n)v_{lm}g_{jk} - \frac{1}{2}\lambda_m\, y^s\left(\dot{\partial}_j K_{sk}\right)$.

Now, by using (1.6e) in (3.10), we get

(3.12) $U_{jk|l|m} + \frac{1}{2}K_{jk|m} = w_{lm}U_{jk} - \frac{1}{2}\lambda_m R_{jk} + (1-n)v_{lm}\, g_{jk} + \frac{1}{2}\lambda_m C^r_{jm}H^m_{kr}$.

Thus, we conclude

Theorem 3.3. *In $G^{2nd}U_{|h} - BRF_n$, we have the identities (3.10), (3.11) and (3.12).*

Let us consider a Finsler space F_n which $h(v) -$ curvature tensor U^i_{jkh} satisfies the following condition

(3.13) $U^i_{jkh|l} = \lambda_l U^i_{jkh} + \mu_l\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}(R^i_k g_{jh} - R^i_h g_{jk})$, $U^i_{jkh} \neq 0$,

Taking the $h-$ covariant derivative for the condition (3.13) with respect to x^m, using (1.4a) and (3.13), then using (1.6b) and (1.6c) respectively, we get

(3.14) $U^i_{jkh|l|m} = w_{lm}U^i_{jkh} + v_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - \frac{1}{2}\lambda_l R^i_{jkh} - \frac{1}{2}R^i_{jkh|m}$,

where $w_{lm} = \lambda_{l|m} + \lambda_l\lambda_m$ and $v_{lm} = \lambda_l\mu_m + \mu_{l|m}$.

From (3.14), we get

(3.15) $U^i_{jkh|l|m} + \frac{1}{2}R^i_{jkh|m} = w_{lm}U^i_{jkh} + v_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - \frac{1}{2}\lambda_l R^i_{jkh}$.

Thus, we conclude

Theorem 3.4. *In $G^{2nd}U_{|h} - BRF_n$, the relationship between the curvature tensor U^i_{jkh} and Cartan's third curvature tensor R^i_{jkh} is given by (3.15).*

IV. Conclusions

The introduction of generalized U −birecurrent Finsler spaces has significantly enriched the field of Finsler geometry. This paper introduced the extension for generalized U −birecurrence property in sense of Cartan. i.e. we studied the generalization generalized U −birecurrent Finsler space. Also, we obtained certain identities belong to $G^{2nd}U_{|h} - BRF_n$.

V. Recommendations for Future Research

- Explore the physical applications of generalized U −birecurrent Finsler spaces.
- Investigate the relationship between generalized U −birecurrent Finsler spaces and other geometric structures, such as conformal Finsler geometry.
- Develop numerical methods for studying the properties of the generalization generalized U −birecurrent Finsler space.

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