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Existence solution of nonlinear functional quadratic type integral equation of fractional order

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Abstract: *In this paper we discus some results concerning existence and extremal solution of nonlinear functional quadratic type integral equation of fractional order in* $\mathbb{R}_+$*.By using Hybrid fixed point theorem on Banach algebras due to B.C. Dhage.*

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I. Introduction

Nonlinear functional integral equations which contains various type of integral and functional equations that are considered in nonlinear analysis [1, 2]. They developing with the help of several tools of finding and applications are describing of real world problems in functional analysis for examples quadratic integral equation are applicable in theory of physics, electrochemistry, economic, kinetic theory of gases, transportation theory, and traffic theory and other fields [1-3, 4-12]. Nonlinear functional quadratic integral equations have been discussed in literature extensively, for long time see Dhage [17-20,23] and. D. O. Regan [3].Numerous many research papers and monographs devoted to differential and integral equations of fractional order [9,10,13-15]. The fixed point theory is useful for proving the existence of solutions of the problem governed by nonlinear integral equations [5-7.].

In this paper we study the existence of solution and locally attractive solutions of the following nonlinear functional quadratic type integral equations of fractional order.

Statement of the problem

Let $\alpha \in [0,1]$ and $\mathbb{R}$ be denote the real line on $\mathbb{R}_+$, the set of nonnegative real numbers. Consider the following nonlinear functional fractional quadratic integral equation (NFQIE)

$$x(t) = a(t) + f_1\left(t, x(\alpha(t))\right) + \frac{(f_2(t, x(\beta_1(t)), x(\beta_2(t)))}{\Gamma(\alpha)} \times \int (t-s)^{\alpha-1} f_3(t_1, x(\gamma_1(s)), x(\gamma_1(s)))ds \quad (1.1)$$

for all $t \in \mathbb{R}_+$ where $v(t,x) = v: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$g(t,x) = g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and function $\theta, \eta, \mu: \mathbb{R}_+ \to \mathbb{R}_+$

By the solution of NFQIE (1.1) we mean the function $x \in \mathbb{BC}(\mathbb{R}_+, \mathbb{R})$ that satisfies NFQIE (1.1) on $\mathbb{R}_+$. where

$x \in \mathbb{BC}(\mathbb{R}_+, \mathbb{R})$the space of continuous real valued functions is defined on$\mathbb{R}_+$.

In this paper we prove that locally attractive of solution as well as positivity and monotonic results functional NFQIE (1.1) by using fixed point theorem of B.C. Dhage [2]. We need some preliminary definition and auxiliary results.

II. Preliminaries

Let the $\mathbb{X} = \mathbb{BC}(\mathbb{R}_+, \mathbb{R})$ be Banach algebra with norm $\|.\|$and let Φ be subset of $\mathbb{X}$. Let a mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ be an operator equation in $\mathbb{X}$ namely,

$$x(t) = (\mathbb{A}x)(t) \qquad t \in \mathbb{R}_+ \tag{2.1}$$

Below we give different characterization of the solution of the operator (2.1) on $\mathbb{R}_+$ we need following definitions of the sequel.

Definition2.1 [15]. We say that the solution (2.1) are locally attractive if there exists an closed ball $\overline{\mathbb{B}_r(0)}$.in the space $\mathbb{BC}(\mathbb{R}_+, \mathbb{R})$ such that for arbitrary solution $x = x(t)$ and $y = y(t)$ of equation (5.2.1) belonging to $\overline{\mathbb{B}_r(0)} \cap \phi$, we have that $\lim_{t\to\infty}\left(x(t) - y(t)\right) = 0$. (2.2)

Definition2.2 [15]: Let $\mathbb{X}$ be Banach space, A mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ is called Lipschtiz if there is a constant $\alpha > 0$ such that $\|\mathbb{A}x - \mathbb{A}y\| \leq \alpha\|x - y\|$ for all $, y \in \mathbb{X}$, if $\alpha < 1,$ then $\mathbb{A}$ is called a contraction $\mathbb{X}$ with contraction constant α.

Definitation2.3 [16] (Dugundji and Granss): An operator $\mathbb{A}$ on Banach Space $\mathbb{X}$ into itself is called compact if for any bounded subset $\mathbb{S}$ of $\mathbb{X}$. $\mathbb{A}(\mathbb{S})$ is relatively compact subset of $\mathbb{X}$ .If $\mathbb{A}$ is continuous and compact then it is called completely continuous on $\mathbb{X}$.

Let $\mathbb{X}$ be a Banach space with norm $\|.\|$ and let $\mathbb{A}: \mathbb{X} \to \mathbb{X}$, be a nonlinear operator .Then $\mathbb{A}$ is called,

a) Compact if $\mathbb{A}(\mathbb{X})$ is relatively compact subset of $\mathbb{X}$.

b) Totally bounded if $\mathbb{A}(\mathbb{S})$ is totally bounded subset of $\mathbb{X}$ for any bounded subset $\mathbb{S}$ of $\mathbb{X}$.

c) Completely continuous if it is continuous and totally bounded operator on$\mathbb{X}$.

It is clearly that every compact operator is totally bounded but they converse need not true.

We seek the solution of NFQIE (1.1) in the space $\mathbb{BC}(\mathbb{R}_+, \mathbb{R})$of continuous and bounded real –valued functions defined on $\mathbb{R}_+$.Define a standard superemum norm $\|.\|$ and multiplication "." in $\mathbb{BC}(\mathbb{R}_+, \mathbb{R})$by

$$\|x\| = sup\{|x(t)|: t \in \mathbb{R}_+\}, \tag{2.3}$$

$$(xy)(t) = x(t)\,y(t), t \in \mathbb{R}_+ \tag{2.4}$$

Clearly $\mathbb{BC}(\mathbb{R}_+, \mathbb{R})$ becomes a Banch space with respect to the above norm and the multiplication in it. By $L^1(\mathbb{R}_+, \mathbb{R})$ we denote the space of Lebesguge integrable function on $\mathbb{R}_+$ with the norm $\|.\|_{L^1}$ defined by,

$$\|x\|_{L^1} = \int_0^\infty |x(t)|\, dt \tag{2.5}$$

Denote by $L^1(a, b)$be the space of Lebesguge integrable function on the interval (a, b) which is equipped with standard norm.

Definition2.4: [9] The Riemann Liouville fractional integral order $\alpha \in (0,1)$ of the function $f \in L^1[0,T]$ is defined by the formula,

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \quad , t \in [0,T] \qquad (2.6)$$

Where $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds$ $\Gamma(\alpha)$ denote the Euler gamma function.

It may be shown that the fractional operator I^α transformed the space $L^1(\mathbb{R}_+,\mathbb{R})$ into itself and has some other properties (see [9, 10, 13, and 14]).

Theorem 2.1 (Arzela-Ascoli theorem) [2]: Every uniformly bounded and equi-continuous sequence $\{f_n\}$ of function in $\mathbb{C}(\mathbb{R}_+,\mathbb{R})$then it has a convergent subsequence.

Theorem2.2 [2]: A metric space $\mathbb{X}$ is compact iff every sequence in $\mathbb{X}$ has convergent subsequence.

The existence result will be based on hybrid fixed point theorem in Banach Algebra $\mathbb{X}$ due to Dhage [17] is well-known in the literature on fixed point theory. The details appears in Dhage and their references there in [18, 19].

Theorem2.3 (Dhage[20]): Let $\mathbb{S}$ be a non-empty, closed convex and bounded subset of a Banach algebra $\mathbb{X}$ and let $\mathbb{A}, \mathbb{C}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: \mathbb{S} \to \mathbb{X}$ be three operators satisfying,

a) $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian with Lipscthiz constant $\phi_A$and ϕ_C respectively,

b) $\mathbb{B}$ is completely continuous, and

c) $x = \mathbb{A}x\mathbb{B}x + \mathbb{C}x \in \mathbb{S}$ for each $x \in \mathbb{S}$

Then the operator equation $\mathbb{A}x\mathbb{B}x + \mathbb{C}x = x$ has a solution whenever $\alpha M + \beta < 1$ where $\mathbb{M} = \|\mathbb{B}(S)\| = sup\{\|\mathbb{B}(x)\| : x \in S\}$

III. Existence Result

Definition3.1 [2]: A mapping $g(t,x) = h: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. Is said to be caratheodory,

i) The map $t \to g(t,x)$ is measurable for each$x,y \in \mathbb{R}$, and

ii) The map $x \to g(t,x)$ is continuous almost everywhere for each $t \in \mathbb{R}_+$.

Again a caratheodory function g is called L^1- caratheodoryif

iii) for each real number $r > 0$ there exist a function $h_r \in L^1(\mathbb{R}_+,\mathbb{R})$ such that

$|g(t,x)| \le h_r(t)$ a.e. $t \in \mathbb{R}_+$ for all $x \in \mathbb{R}$ with $|x| \le r$

Finally a Caratheodory function $h_1 \in L^1(\mathbb{R}_+,\mathbb{R})$ such that

$|g(t,x,y)| \le h(t)$ $a.e.$ $t \in \mathbb{R}_+$for all$x,y \in \mathbb{R}$

We need the following hypothesis by solving equation (1.1)

$H_1)$ The function $\theta, \mu, \eta: \mathbb{R}_+ \to \mathbb{R}_+$are continuous.

$H_2)$ The function $f(t,x) = f:\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded with bound $F = \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} |f(t,x)|$ there exist and bounded function $l:\mathbb{R}_+ \to \mathbb{R}$ with bound L satisfying $|f(t,x) - f(t,y)| \le n(t)|x-y|$ for all $t \in \mathbb{R}_+$ and $x,y \in \mathbb{R}$ and vanish $t \to \infty$.

$H_3)$ The function $\mathbb{K}(t,x) = \mathbb{K}:\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded with bound $\mathbb{K} = \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} |f(t,x)|$ thee exist and bounded function $n:\mathbb{R}_+ \to \mathbb{R}$ with bound N satisfying $|\mathbb{K}(t,x) - \mathbb{K}(t,y)| \le n(t)|x-y|$ for all $t \in \mathbb{R}_+$ and $x,y \in \mathbb{R}$ and vanish $t \to \infty$.

$H_4)$ The function $|v(t,s)| = h:\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfy caratheodory condition and there exist a function $h \in L^1(\mathbb{R}_+ \times \mathbb{R})$ such that,

$|h(t,x)| \le (t,x) \in \mathbb{R}_+ \times \mathbb{R}$ where $\lim_{n\to\infty} \int_0^t h_1(s)\,ds = 0$.

and Moreover, $V = \sup_{t\in R_+} |v(t,s)|$ then the uniform continuous $V:\mathbb{R}_+ \to \mathbb{R}_+$. Defined formula $v(t) = \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}\,ds$ bounded on $\mathbb{R}_+$ and Vanish that is $\lim_{t\to\infty} v(t) = 0$.

$H_5)$ The function g is $\mathbb{BC}$ $-$caratheodory $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$.

Remark3.1: Note that if the hypothesis (H_4) and (H_5) holds. Then there exist constant $K > 0$ and $K_1 > 0$ such that

$$K_1 = \sup \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}\,ds$$

Now we are redy to prove the existence theorem.

IV. Main result

Theorem4.1: suppose that the hypothesis$(H_1) - (H_4)$holds. Furthermore if $L(K_1 + K_2) + N < 1$, K_1 and K_2 are defined remark (3.1.1).Then NFQIE(3.1.1) has solution in the space $\mathbb{BC}(\mathbb{R}_+, \mathbb{R})$. Moreover,solution of the equation (3.1.1) are ;locally attractive on $\mathbb{R}_+$,

Proof: By the solution of the NFQIE (1.1) we mean the continuous function $x \in \mathbb{R}_+ \to \mathbb{R}$ that satisfies NFQIE (1.1) on$\mathbb{R}_+$.

Let $\mathbb{X} = \mathbb{BC}(\mathbb{R}_+, \mathbb{R})$ be a Banach algebra of all continuous and bounded real valued function on $\mathbb{R}_+$ .Define $\mathbb{S}$ of $\mathbb{X}$ by,

$$\|x\| = \sup_{t\in\mathbb{R}_+} |x(t)| \qquad (4.1)$$

We shall obtain solution of NFQIE (1.1) under some suitable conditions on the functions involved in (1.1)

Consider the closed ball $B_r[0]$ in X centeredat origin O and of radius r where

$r = F + G[K_1 + K_2] > 0$

Let s define three operators, $\mathbb{A}: \mathbb{X} \to \mathbb{X}, \mathbb{B}: \mathbb{S} \to \mathbb{X}$ *and* $\mathbb{C}: \mathbb{X} \to \mathbb{X}$ by,

$$\mathbb{A}x(t) = f\left(t, x(\theta(t))\right), \ t \in \mathbb{R}_+ \qquad (4.4)$$

$$\mathbb{B}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds \ \ t \in \mathbb{R}_+ \qquad (4.5)$$

$$\mathbb{C}x(t) = K\left(t, x(\mu(t))\right) \ \ t \in \mathbb{R}_+ \qquad (4.6)$$

Then these equation can be transformed in to operator equation as

$$\mathbb{A}x\mathbb{B}x + \mathbb{C}x = x(t), \quad x \in \mathbb{S} \ \ t \in \mathbb{R}_+ \qquad (4.7)$$

We shall show that the operator $\mathbb{A}, \mathbb{B}$ *and* $\mathbb{C}$ are satisfy all conditions of theorem 2.1. This will be achieved in the series of following steps.

Step I: We show that $\mathbb{A}$ *and* $\mathbb{C}$ is Lipchitz on $\mathbb{X}$.

Let $x, y \in \mathbb{X}$ be arbitrary, then by the hypothesis (H_1) we get,

$$\|\mathbb{A}x(t) - \mathbb{A}y(t)\| = \left|f\left(t, x(\theta(t))\right) - f\left(t, y(\theta(t))\right)\right|$$
$$\leq l(t)\left|x(\theta(t)) - y(\theta(t))\right|$$
$$\leq L|x(t) - y(t)|$$
$$= \phi_{\mathbb{A}}(\|x - y\|)$$

for all $x, y \in \mathbb{X}$, where $\phi_{\mathbb{A}}$ is a - function by $\phi_{\mathbb{A}}(r) = \frac{L_1}{K_1 + r} < r, r > 0$. This shows that $\mathbb{A}$ is a - Lipschitz operator on $\mathbb{X}$ into itself. Similarly

$$\|\mathbb{C}x(t) - \mathbb{C}y(t)\| = |K(t, x(\mu(t))) - f(t, y(\mu(t)))|$$
$$\leq n(t)\left|x(\mu(t)) - y(\mu(t))\right|$$
$$\leq N|x(t) - y(t)|$$
$$= \phi_{\mathbb{C}}(\|x - y\|) \ .$$

for all $x, y \in \mathbb{X}$, where $\phi_{\mathbb{C}}$ is a - function by $\phi_{\mathbb{C}}(r) = \frac{L_2}{K_2 + r} < r, r > 0$. This shows that $\mathbb{C}$ is a - Lipschitz operator on $\mathbb{X}$.

Step II: To show that $\mathbb{B}$ is completely continuous on $\mathbb{X}$.

Let $\{x_n\}$ be a sequence in $\mathbb{S}$ converging to point x.Then by lebsguge dominated converges theorem for all $t \in \mathbb{R}_+$. We obtain,

$$\lim_{n\to\infty} \mathbb{B}x_n(t) = \lim_{n\to\infty} \left\{\frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds\right\}$$

$$\leq \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}} \lim_{n\to\infty} g\left(s, x(\eta(s))\right) ds$$

$$\leq \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds$$

$$\leq \mathbb{B}x(t) \quad \forall\, t \in \mathbb{R}_+.$$

This shows that $\mathbb{B}x_n$ converges monotonically to $\mathbb{B}x$ point wise on$\mathbb{R}_+$. Further since $\mathbb{B}x_n \subset \mathbb{B}(\mathbb{S})$, $\mathbb{B}x_n$is an equicontinuous sequence in $\mathbb{X}$. We have that $\mathbb{B}x_n(t) \to \mathbb{B}x$ uniformly, whence $\mathbb{B}$ is continuous operator in $\mathbb{S}$.

Next we to show that the sequence $\mathbb{B}x_n$ is an equicontinuous sequence of the function $\mathbb{X}$ .Let $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, we obtain,

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)| = \left| \begin{matrix} \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{v(t_2,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds \\ - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds \end{matrix} \right|$$

$$\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{v(t_2,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{v(t_1,s)}{(t_2-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds \right|$$

$$+ \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{v(t_1,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds \right|$$

$$+ \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{1-\alpha}} \left| g\left(s, x(\eta(s))\right) \right| ds +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|v(t_2,s)|}{(t_2-s)^{1-\alpha}} \left| g\left(s, x(\eta(s))\right) \right| ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |v(t_1,s)| \left| [(t_1-s)^{1-\alpha} - (t_2-s)^{1-\alpha}] \right| \left| g\left(s, x(\eta(s))\right) \right| ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{T} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{1-\alpha}} h(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|v(t,s)|}{(t_2-s)^{1-\alpha}} h(s)\, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{T} |v(t_2-s)| |(t_2-s)^{1-\alpha}|\; h(s) ds$$

$$\leq \frac{\|h\|}{\Gamma(\alpha)} \int_0^{T} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{1-\alpha}} ds + \frac{\|h\|}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|v(t,s)|}{(t_2-s)^{1-\alpha}} ds$$

$$+ \frac{\|h\|}{\Gamma(\alpha)} \int_0^{T} |v(t_1,s)| \left| [(t_1-s)^{1-\alpha} - (t_2-s)^{1-\alpha}] \right| ds$$

$$\longrightarrow 0 \;\; as \;\; t_1 \to t_2, \forall\, n \in N.$$

This shows that the convergence is uniform by using the property of uniform convergence that is uniform convergence imply continuity.

Hence $\mathbb{B}$ is continuous on $\mathbb{S}$.

Step III: Next than we prove the set $\mathbb{B}(\mathbb{S})$ is uniformly bounded in $\mathbb{S}$,

Let $x \in \mathbb{S}$ be any point. Then we have,

$$|\mathbb{B}x(t)| \le \left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds\right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|v(t,s)|}{(t-s)^{1-\alpha}}\, h(s)\, ds$$

$$\le \frac{V\|h\|}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\, ds$$

$$\le \frac{V\|h\|}{\Gamma(\alpha)}\frac{T^\alpha}{\alpha}$$

$$\le \frac{V\|h\|T^\alpha}{\Gamma(\alpha+1)}$$

$for\ all\ \ t \in \mathbb{R}_+$ Thus $\|\mathbb{B}x\| \le \frac{V\|h\|T^\alpha}{\Gamma(\alpha+1)}$,

Now we will show that $\mathbb{B}(\mathbb{S})$ is equicontinuous set in $\mathbb{X}$.

Let $t_1, t_2 \in \mathbb{R}_+$ with $t_1 > t_2$, and $x \in \mathbb{S}$ we obtain,

$$|\mathbb{B}x(t_2) - \mathbb{B}x(t_1)| = \left|\begin{matrix} \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{v(t_2,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds \\ -\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds \end{matrix}\right|$$

$$\le \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{v(t_2,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{v(t_1,s)}{(t_2-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds\right|$$

$$+ \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{v(t_1,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds\right|$$

$$+ \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{1-\alpha}} g\left(s, x(\eta(s))\right) ds\right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{1-\alpha}} \left|g\left(s, x(\eta(s))\right)\right| ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{|v(t_2,s)|}{(t_2-s)^{1-\alpha}} \left|g\left(s, x(\eta(s))\right)\right| ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1} |v(t_1,s)|\left|[(t_1-s)^{1-\alpha} - (t_2-s)^{1-\alpha}]\right| \left|g\left(s, x(\eta(s))\right)\right| ds$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^{T} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{1-\alpha}} h(s) ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{|v(t,s)|}{(t_2-s)^{1-\alpha}} h(s)\, ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^{T} |v(t_2-s)||(t_2-s)^{1-\alpha}|\ h(s) ds$$

$$\le \frac{\|h\|}{\Gamma(\alpha)}\int_0^{T} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{1-\alpha}} ds + \frac{\|h\|}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{|v(t,s)|}{(t_2-s)^{1-\alpha}} ds$$

$$+ \frac{\|h\|}{\Gamma(\alpha)}\int_0^{T} |v(t_1,s)|\left|[(t_1-s)^{1-\alpha} - (t_2-s)^{1-\alpha}]\right| ds$$

$$\longrightarrow 0 \ \ as \ \ t_1 \to t_2.$$

Therefore by Arzela-Ascoli theorem that $\mathbb{B}$ is completely continuous on operator $\mathbb{S}$.

Step IV: we show that hypothesis (c) of theorem (2.3) is satisfied,

Let $x \in \mathbb{X}$ be a fixed element such that $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x, \forall\, y \in \mathbb{S} \Longrightarrow x \in \mathbb{S}$ then, we have

$$x(t) = \mathbb{A}x(t)\mathbb{B}y(t) + \mathbb{C}x(t), \quad t \in \mathbb{R}_+ \ \forall\, x, y \in \mathbb{S}$$

$$|x(t)| \le |\mathbb{A}x(t)||\mathbb{B}y(t)| + |\mathbb{C}x(t)|$$

$$\le \left|f\left(t, x(\theta(t))\right)\right|\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}}\, g\left(s, x(\eta(s))\right)\, ds\right| + \left|K\left(t, x(\mu(t))\right)\right|$$

$$\le \left[\left|f\left(t, x(\theta(t))\right) - f(t,0)\right|\right] \times \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|v(t,s)|}{(t-s)^{1-\alpha}}\left|g\left(s, x(\eta(s))\right)\right| ds$$

$$+ \left|K\left(t, x(\mu(t))\right) - K(t,0)\right|$$

$$\le \left[\left|f\left(t, x(\theta(t))\right) - f(t,0)\right|\right] \times \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|v(t,s)|}{(t-s)^{1-\alpha}} h(s)\, \|x\|\, ds$$

$$+ \left|K\left(t, x(\mu(t))\right) - K(t,0)\right|$$

$$\le [F] \times \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|v(t,s)|}{(t-s)^{1-\alpha}} h(s) ds + [K]$$

$$\le [F] \times \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|v(t,s)|}{(t-s)^{1-\alpha}} h(s) ds + [K]$$

$$\le [F]\frac{V\|h\|T^{\alpha}}{\Gamma(\alpha+1)} + [K]$$

On taking supremum over $t \in \mathbb{R}_+$,

$$\|x\| \le [F]\frac{V\|h\|T^{\alpha}}{\Gamma(\alpha+1)} + [K]$$

Which implies that $x \in \mathbb{S}$ and so, hypothesis (c) of theorem (2.3) is satisfied.

Step V: we show that hypothesis (d) of theorem (2.3) is satisfied,

Let $\mathbb{M}\phi_{\mathbb{A}}(r) + \phi_{\mathbb{C}}(r) \le r$, for $r > 0$
Now the operator $\mathbb{B}$ is bounded by Step III and continuous step V. Therefore, the number $\mathbb{M} = sup\{\|\mathbb{B}x\| : x \in \mathbb{S}\}$ exists in view of compactness of $\overline{\mathbb{B}(\mathbb{S})}$

and $\mathbb{M} \le \frac{V\|h\|T^{\alpha}}{\Gamma(\alpha+1)}$. Therefore, by equation (4.1)

$$\mathbb{M}\phi_A(r) + \phi_{\mathbb{C}}(r) \le \mathbb{M}\frac{L_1 r}{K_1 + r} + \frac{L_2 r}{K_2 + r} < r, \text{ for } r > 0.$$

For $r > 0$. Thus all the conditions of theorem (2.3) are satisfied. Hence apply it to operator equation (4.7) and conclude that the NFQIE (1.1) has a solution on $\mathbb{R}_+$. This is complete proof.

Step VI: Now to show that locally attractivity on $\mathbb{R}_+$. then we have

$$|x(t) - y(t)| = \left| \begin{array}{l} \left\{\left[f\left(t,x(\theta(t))\right)\right]\frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}} g\left(s,x(\eta(s))\right) ds + K\left(t,x(\mu(t))\right)\right\} \\ -\left\{\left[f\left(t,y(\theta(t))\right)\right]\frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}} g\left(s,y(\eta(s))\right) ds + K\left(t,y(\mu(t))\right)\right\} \end{array} \right|$$

$$\leq \left|\left[f\left(t,x(\theta(t))\right)\right]\right| \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}}\, g\left(s,x(\eta(s))\right) ds + \left|K\left(t,x(\mu(t))\right)\right|$$

$$+\left|\left[f\left(t,y(\theta(t))\right)\right]\right| \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}}\, g\left(s,y(\eta(s))\right) ds + \left|K\left(t,y(\mu(t))\right)\right|$$

$$\leq 2[F]_{+}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{1-\alpha}}\, g\left(s,y(\eta(s))\right) ds + 2[G]$$

$$\leq 2[F]\frac{V\|h\|T^{\alpha}}{\Gamma(\alpha+1)} + 2[G] \ \forall\, t \in \mathbb{R}_+$$

$$\leq 2F\frac{V\|h\|T^{\alpha}}{\Gamma(\alpha+1)} + 2K \ \ \forall t \in \mathbb{R}_+$$

Since $\lim_{t\to\infty} v(t) = 0$, hypothesis ($H_2$) and for $\epsilon > 0$ there is real number $T > 0$ such that $v(t) \leq \frac{\|h\|T^{\alpha}}{\Gamma(\alpha+1)}$ for all $t \geq T$. then for inequalities it follows that

$|x(t) - y(t)| < \epsilon$ for all $t \geq T$ this is complete proof.

It follows that all condition are satisfied. Thus by theorem above problem has locally attractive solution on $\mathbb{R}_+$.

V. Conclusion:

In this paper we have studied the solutions for the nonlinear functional integral equation of fractional order. We need the additionally assumptions on positivity on the nonlinearities involved integral equations of fractional order to require characterization of attractivity of the given solutions. Finally we conclude that this paper we mention to represent of attractivity results using some arguments with appropriate modification and some results in the directions of Dhages theorem.

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