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The Drin Sequence

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Define the *Drin sequence*, $\{D_n\}$, of odd numbers, to be 1, **3**, 7, 13, **21**, 31, 43, **57**, 73, 91, **111**, …, where the differences are 2, 4, 6, 8, 10, 12, …

Theorem 1: The Drin sequence, mod 3, is 1, **0**, 1, 1, **0**, 1, 1, **0**, 1, 1, **0**, 1 …

Proof: The differences, 2, 4, 6, 8, 10, 12, 14, 16, 18,… are 2, 1, 0, 2, 1, 0, 2, 1, 0, mod 3, a periodic sequence with period, 2, 1, 0. Since $D_1 = 1$, we have, using the periodicity of the increments, mod 3, $D_2 = 1 + 2 = 0$ (mod 3), $D_3 = 1 + 2 + 1 = 1$ (mod 3), $D_4 = 1 + 2 + 1 + 0 = 1$ (mod 3), $D_5 = 1 + 2 + 1 + 0 + 2 = 0$ (mod 3), … ■

Corollary 1.1: As the bold **0**'s in Theorem 1 indicate, the corresponding terms of $\{D_n\}$ in bold font namely, D_{3m-1}, are divisible by 3.

Corollary 1.2: 6 divides the difference between consecutive bolded numbers in $\{D_n\}$.

Proof: The difference between consecutive bolded numbers in $\{D_n\}$ is the sum of three consecutive even numbers, $2k + (2k + 2) + (2k + 4)$, which equals $6k + 6$. ■

Corollary 1.3: The members of $\{D_n\}$ that are not in bold font equal 1 (mod 3).

Corollary 1.4: The difference of *consecutive* nonbolded numbers of $\{D_n\}$ is a multiple of 6.

Theorem 2: $D_n = n^2 - n + 1$.

Proof: The second differences of D_n are constant, namely, 2, so the equation of the sequence is quadratic [1]. Let $D_n = an^2 + bn + c$. Since $D_1 = 1$, $D_2 = 3$, and $D_3 = 7$, we obtain the equations: $a + b + c = 1$, $4a + 2b + c = 3$, and $9a + 3b + c = 7$, whose solution is $a = c = 1$, $b = -1$. ■

Corollary 2.1: $D_{n+1} - D_n = 2n$.

Proof: $D_{n+1} - D_n = [(n + 1)^2 - (n + 1) + 1] - (n^2 - n + 1) = [n^2 + 2n + 1 - n] - n^2 + n - 1 = 2n$. ■

The reader is reminded that the n-th *oblong* number is given by, $O_n = n(n + 1)$. [1]

Corollary 2.2: $D_n = O_{n-1} + 1$.

Proof: $D_n - 1 = n^2 - n = n(n - 1) = O_{n-1}$. ■

Theorem 3: $D_n D_{n+1} = D_{n^2+1}$.

Proof: $D_n D_{n+1} = (O_{n-1}+1)(O_n+1) = O_{n-1}O_n + O_{n-1} + O_n + 1 =$ $(n-1)n \cdot n(n+1) + (n-1)n + n(n+1) + 1 =$ $n^2(n^2 - 1) + 2n^2 + 1 = n^4 - n^2 + 2n^2 + 1 = n^4 + n^2 + 1 = n^2(n^2 + 1) + 1 = O_{n^2} + 1 = D_{n^2+1}$. ■

Theorem 4: $\sum_{n=1}^{k} D_n = \frac{k\left(k^2+2\right)}{3}$.

Proof: $(1^2 - 1 + 1) + (2^2 - 2 + 1) + (3^2 - 3 + 1) + \ldots + (k^2 - k + 1) = \frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2} + k =$

$$\frac{k(k+1)(2k+1) - 3k(k+1) + 6k}{6} = \frac{k\left((k+1)(2k+1) - 3(k+1) + 6\right)}{6} = \frac{k\left(2k^2 + 3k + 1 - 3k - 3 + 6\right)}{6} =$$

$\frac{k\left(k^2+2\right)}{3}$. ■

The Drin sequence is the middle column of the following array of the natural numbers. The last entry of the n-th row is n^2. The $(n - 1)$-st entry of the n-th row is O_{n-1}.

1

2 **3** 4

5 6 **7** 8 9

10 11 12 **13** 14 15 16

⋮

Reference

[1] M.Lewinter, J.Meyer, *Elementary Number Theory with Programming*, Wiley & Sons. 2015.