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A Reliable Iterative Decomposition Method for Solving Integro-Differential Equations

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***ABSTRACT**: An iterative decomposition technique is derived and applied to solve rth order linear and nonlinear integro-differential equations. The solution was obtained by decomposition of the assumed series solution for the integro-differential equations considered. The initial approximation was obtained by evaluating the source term and subsequent approximations were obtained by applying the nonlinear operator on the sum of the previous solutions. Numerical examples showed that the technique is accurate, simple and efficient compared to other methods in literature.*

***KEYWORDS**: Iterative decomposition method, nonlinear operator, approximate solution, initial value problem, Integro-Differential Equations.*

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I. INTRODUCTION

Ordinary Integro-differential equations abound in many branches of linear and nonlinear applied problems. Models arising from mechanics, astronomy, economics, engineering, sciences and other related fields are usually functional equations including integro-differential equations. Special usage of integro-differential equations is visible in the mathematical modelling on spatio-temporal development of epidemics [1].

Generally, it is impossible to get an analytic answer for such equations [2, 3]. Because of that, various numerical methods have been devoted to finding the approximate solutions to such equations [2]. The numerical solution of this class of functional equations is discussed by a large number of authors. A few of these methods are as follows: differential quadrature method based on modified cubic B-splines [3], non-polynomial splines for solving system of second order boundary value problems [4], quartic trigonometric B-spline algorithm for numerical solution of the regularized long wave equation [5], modified Laplace Adomian decomposition method [6], modified variational iteration technique [7], Legendre wavelets operational method [8], single term Walsh series technique [9], Projection method [10] and Modified projection-iterative method [11].

II. CONVERSION OF rth ORDER IDEs TO SYSTEM OF IDEs

Consider the rth order linear initial value problem of integro-differential equation:

$$y^{(r)}(x) = h(x) + f(x)y(x) + \lambda \int_0^x k(x,\ t)dt \qquad (1)$$

With initial conditions

$$y(0) = \alpha_0,\ y'(0) = \alpha_1, . . . , y^{(r-1)}(0) = \alpha_{r-1}. \qquad (2)$$

Consider the transformation

$$\begin{aligned} y(x) &= y_1(x), \\ y'(x) &= y_2(x), \\ y''(x) &= y_3(x), \\ &\ \ \vdots \\ y^{(r-1)}(x) &= y_r(x). \end{aligned} \qquad (3)$$

Re-writing the equations in (3) as a system of differential equations gives

$$\frac{dy_1}{dx} = y_2(x),$$

$$\frac{dy_2}{dx} = y_3(x),$$

$$\vdots$$

$$\frac{dy_r}{dx} = h(x) + f(x)y_1(x) + \lambda \int_0^x k(x,\ t)y_1(t)dt \tag{4}$$

with

$$y_1^{(0)} = \alpha_0,\ \ y_2^{(0)} = \alpha_1, \ldots, y_r^{(0)} = \alpha_{r-1}. \tag{5}$$

Equations in (4) are written as a system of equations with the Lagrange multiplier, λ given as

$$\lambda_i = \pm 1,\ 2, .\ .\ ., \ r. \tag{6}$$

$$y_1^{(b+1)}(x) = y_1^{(0)}(x) + \int_0^x y_2^{(b)}(s)ds,$$

$$y_2^{(b+1)}(x) = y_2^{(0)}(x) + \int_0^x y_3^{(b)}(s)ds,$$

$$y_3^{(b+1)}(x) = y_3^{(0)}(x) + \int_0^x y_4^{(b)}(s)ds,$$

$$\vdots$$

$$y_r^{(b+1)}(x) = y_1^{(0)}(r) + \int_0^x \left[h(s) + f(s)y_1^{(b)}(s) + \lambda \int_0^s k(s,t)y_1^{(b)}(t) \right] ds \tag{7}$$

III. SOLUTION TECHNIQUE

Consider the general rth order integro-differential equation

$$y^{(r)}(x) + f(x)y(x) + \int_{p(x)}^{q(x)} k(x,t)y^{(b)}\, y^{(m)}dt = h(x) \tag{8}$$

with

$$y(a) = \alpha_0,\ \ y'(a) = \alpha_1,\ \ y''(a) = \alpha_2 \ldots, y^{(r-1)}(a) = \alpha_{r-1} \tag{9}$$

where $\alpha_i : i = 0,\ 1,\ 2, .\ .\ .\ .\ , r-1$ are real constants, $m,\ n,\ b$ are integers with $b \le m \le r$, the functions $f(x)$, $h(x)$ and $k(x,t)$ are given and $y(x)$ is the unknown function to be determined.

Therefore, consider the following general nonlinear system

$$y = f + N(y) \tag{10}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function.

The assumed series solution of (10) is given by

$$y = \sum_{j=0}^{\infty} y_j \tag{11}$$

The nonlinear operator, N is defined as

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$$N\left(\sum_{j=0}^{\infty} y_j\right) = N(y_0) + \sum_{j=1}^{\infty}\left\{N\left(\sum_{k=0}^{\infty} y_k\right) - N\left(\sum_{k=0}^{j-1} y_k\right)\right\} \tag{12}$$

Using equations (11) and (12), the nonlinear system (10) is written as

$$\sum_{j=0}^{\infty} y_j = f + N(y_0) + \sum_{j=1}^{\infty}\left\{N\left(\sum_{k=0}^{\infty} y_k\right) - N\left(\sum_{k=0}^{j-1} y_k\right)\right\} \tag{13}$$

and the recurrence relation for the problem is derived as follows:

$$y_0 = f$$
$$y_1 = N(y_0)$$
$$y_2 = N(y_0 + y_1) - N(y_0)$$
$$y_3 = N(y_0 + y_1 + y_2) - N(y_0 + y_1)$$
$$\vdots$$
$$y_{j+1} = N\big(y_0 + y_1 + y_2 + \ldots + y_j\big) - N\big(y_0 + y_1 + y_2 + \ldots + y_{j-1}\big) \tag{14}$$

Hence, the jth term approximate solution for (10) is given as

$$y = y_o + y_1 + y_2 + \ldots + y_{j-1} \tag{15}$$

IV. NUMERICAL EXAMPLES

The error obtained in this work is defined as

$$Error = |y(x) - y_{iID}| : jID = 0,1,2,3,\ldots, j-1 \tag{16}$$

where $y(x)$ is the exact solution for the problem considered and y_{jID} is the approximate solution obtained using the iterative decomposition technique discussed.

All computations and programmes are carried out with the aid of MATLAB soft-ware.

Example 1: Consider the first order linear Volterra integro-differential equation

$$y'(x) + y(x) = (x^2 + 2x + 1)e^{-x} + 5x + 8 - \int_0^x ty(t)dt : y(0) = 10. \tag{17}$$

The exact solution is

$$y(x) = 10 - xe^{-x}.$$

Example 2: Consider the second order linear Fredholm integro-differential equation

$$y''(x) = x - sinx - \int_0^{\frac{\pi}{2}} xty(t)dt : y(0) = 0,\ y'(0) = 1. \tag{18}$$

The exact solution is

$$y(x) = sinx.$$

Example 3: Consider the second order nonlinear Fredholm integro-differential equation

$$y''(x) = 10 - \frac{146}{35}x + \frac{1}{2}\int_{-1}^{1} xty^2(t)dt : y(0) = 1,\ y'(0) = 0. \tag{19}$$

The exact solution is

$$y(x) = 1 + 5x^2 - x^3.$$

Tables of Results

TABLE 1
Numerical Results for Example 1: Comparison between the absolute errors in the Polynomial Collocation Approach and the present method

x	Exact Solution	Ajileye and Aminu (2022)	Error	Present Method	Error
0.0	10.00000000	9.999999994	6.00E-09	10.0000000	0.00000
0.1	9.909516258	9.909509989	6.27E-06	9.9095162567	1.27E-09
0.2	9.836253849	9.836242518	1.13E-05	9.8362538284	2.06E-08
0.3	9.777754534	9.777745550	8.98E-06	9.7777545087	2.53E-08
0.4	9.731871982	9.731868389	3.59E-06	9.7318719548	2.72E-08
0.5	9.696734670	9.696733792	8.78E-07	9.6967344070	2.63E-07
0.6	9.670713018	9.670710096	2.02E-06	9.6707127850	2.33E-07
0.7	9.652390287	9.652383354	6.93E-06	9.6523883270	1.96E-07
0.8	9.640536829	9.640529449	7.38E-06	9.6405366880	1.41E-07
9.9	9.634087306	9.634086223	1.08E-06	9.6340871990	1.07E-07
1.0	9.632120559	9.632125613	5.05E-06	9.6321194790	1.08E-06

TABLE 2
Numerical Results for Example 2: Comparison between the absolute errors in the ADM and Present method

x	Exact Solution	MOHD et.al (2017)	Error	Present Method	Error
0.0	0.0000000	0.0000000	0.000000	0.0000000	0.000000
0.1	0.0998334	0.0998329	5.00E-07	0.0998332	2.00E-07
0.2	0.1986690	0.1986650	4.00E-06	0.1986670	2.08E-07
0.3	0.2955200	0.2955060	1.40E-05	0.2955160	4.00E-06
0.4	0.3894180	0.3893840	3.40E-05	0.3894150	3.00E-06
0.5	0.4794260	0.4793580	6.80E-05	0.4794200	6.00E-06
0.6	0.5646420	0.5645260	1.16E-04	0.5646350	7.00E-06
0.7	0.6442180	0.6440330	1.85E-04	0.6442100	8.00E-06
0.8	0.7173560	0.7170800	2.76E-04	0.7173460	1.00E-05
0.9	0.7833270	0.7829330	3.94E-04	0.7833150	1.26E-05
1.0	0.8414710	0.8409310	5.40E-04	0.8414520	1.90E-05

TABLE 3
Numerical Results for Example 3: Comparison between the absolute errors in the cubic spline collocation method and the Present method

x	Exact Solution	Taiwo and Gegele (2014)	Error	Present Method	Error
0.0	1.000	1.000000000	1.0000000	1.000000000	0.000000
0.1	1.049	1.049000060	6.008E-08	1.049000030	3.000E-08
0.2	1.192	1.192000079	7.918E-08	1.192000056	5.600E-08
0.3	1.423	1.423000843	8.432E-07	1.423000827	8.270E-07
0.4	1.736	1.736000688	6.884E-07	1.736000679	6.790E-07
0.5	2.125	2.125000572	5.718E-07	2.125000563	5.630E-07
0.6	2.584	2.584000562	5.623E-07	2.584000554	5.540E-07
0.7	3.107	3.107000401	4.009E-07	3.107003521	3.521E-07
0.8	3.688	3.688000293	2.929E-07	3.688000285	2.850E-07
0.9	4.321	4.321000289	2.887E-07	4.321000272	2.720E-07
1.0	5.000	5.000000200	1.999E-07	5.000000150	1.500E-07

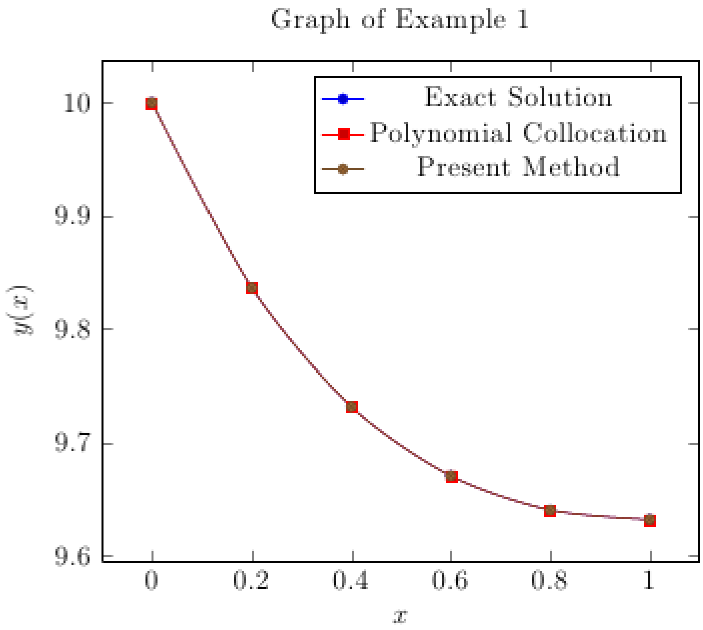

Figure 1: The behaviour of the exact solution compared with the solutions by the Polynomial Collocation Approach and the present method.

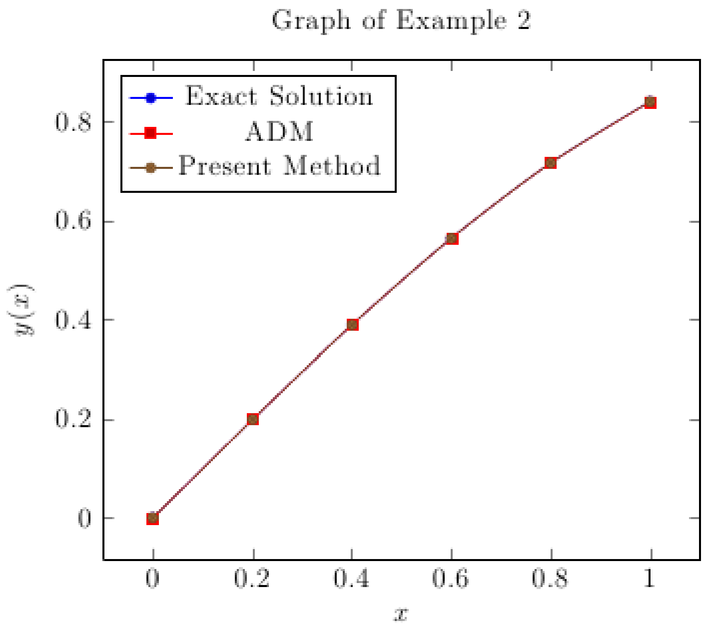

Figure 2: The behaviour of the exact solution compared with the solutions by ADM and the Present Method.

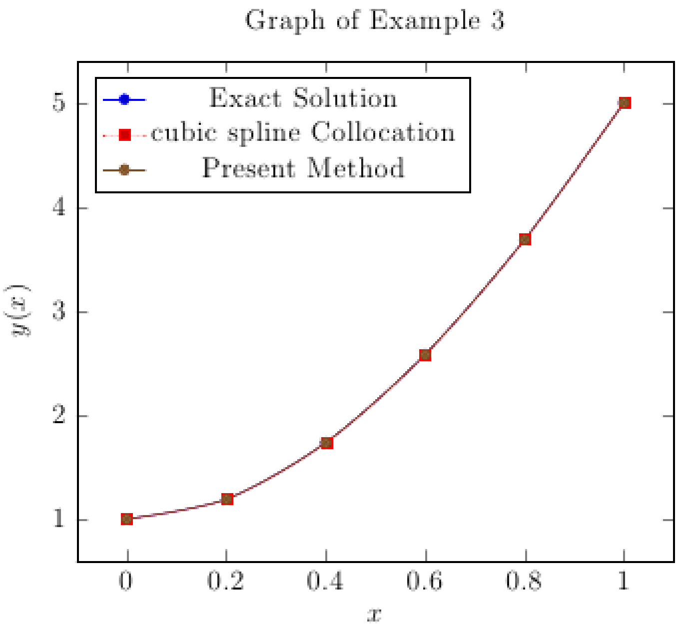

Figure 3: The behaviour of the exact solution compared with the solutions by cubic spline collocation method and Present Method.

V. CONCLUSION

In this paper, a reliable iterative decomposition method for solving linear and nonlinear integro-differential equations is derived and implemented. Three equations of first and second orders are considered and the results obtained are presented graphically in Figures 1 - 3. The results obtained by the present method when compared with the exact solution and results by other methods in literature are generally better as indicated in Tables 1 - 3.

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