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Markov Triples

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Markov triples, (a, b, c), consist of natural numbers that solve

$$a^2 + b^2 + c^2 = 3abc \qquad (*)$$

a, b, and c are then called Markov numbers. The Markov numbers less than 1000 are 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, and 985. Here are several triples: (1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), and (2, 5, 29). Since squares are 0 or 1 (mod 3), (*) becomes $a^2 + b^2 + c^2 = 0$ (mod 3), we must have either $a^2 = b^2 = c^2 = 0$ (mod 3) or $a^2 = b^2 = c^2 = 1$ (mod 3). Other modular conditions include: $a^2 + b^2 = 0$ (mod c), $a^2 + c^2 = 0$ (mod b), and $b^2 + c^2 = 0$ (mod a). Writing (*) as the quadratic equation in c, $c^2 - 3abc + (a^2 + b^2) = 0$, we require the discriminant, $9a^2b^2 - 4(a^2 + b^2)$, to be a perfect square. Thus most pairs, (a, b), are not part of a Markov triple. All odd Markov numbers have the form $4k + 1$, and all even Markov numbers have the form $32k + 2$. Every other Fibonacci number, f_{2k+1}, that is, 1, 2, 5, 13, 34, … is a Markov number. (*) can be rewritten $3ab - c =$ $3ab - c = \dfrac{a^2 + b^2}{c}$, from which we have $c \mid a^2 + b^2$. (By symmetry, we also have $a \mid b^2 + c^2$ and $b \mid a^2 + c^2$, of course.)

Lemma 1: If we take two numbers from any triple, we obtain a quadratic equation in the third number, leading to two triples, namely the one we start with and a new one.

Remark: This implies that (*) has infinitely many solutions in natural numbers.

Example: Given (2, 5, 29), consider the equation satisfied by $(2, 29, x)$, namely $2^2 + 29^2 + x^2 = (3\cdot 2\cdot 29)x$, or $x^2 - 174x + 845 = 0$. This becomes $(x - 169)(x - 5) = 0$, so $x = 5$, yielding the given triple, (2, 5, 29), and $x = 169$, yielding the new triple, (2, 29, 169).

We briefly review ordinary and generalized Pell equations.

Lemma 2: Let (r, t) be the smallest solution in positive integers to the ordinary Pell equation, $x^2 - ky^2 = 1$, and let (x_1, y_1) be the smallest solution in positive integers to the *generalized* Pell equation, $x^2 - ky^2 = d$, where $d \neq 1$. Then the n-th solution, (x_n, y_n), to the generalized Pell equation is given by

$$\boxed{x_n + y_n\sqrt{k} = \left(r + t\sqrt{k}\right)^{n-1}\left(x_1 + y_1\sqrt{k}\right)}$$ See [1]. ■

We use the exponent, $n - 1$, so that when $n = 1$, we have $\left(r + t\sqrt{k}\right)^{0}\left(x_1 + y_1\sqrt{k}\right) = x_1 + y_1\sqrt{k}$.

Example: Let us find infinitely many solutions, (x_n, y_n), to $x^2 - 2y^2 = 2$.

Step 1: $x_1 = 2$ and $y_1 = 1$ solve the generalized Pell equation $x^2 - 2y^2 = 2$.

Step 2: $r = 3$ and $t = 2$ solve the associated ordinary Pell equation $x^2 - 2y^2 = 1$.

Step 3: By Lemma 2, we obtain, for $n = 1, 2, 3, \ldots$, $x_n + y_n\sqrt{2} = \left(3+2\sqrt{2}\right)^{n-1}\left(2+\sqrt{2}\right)$.

Example: When $n = 2$, $x_2 + y_2\sqrt{2} = \left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right) = 10+7\sqrt{2}$. Then $x_2 = 10$ and $y_2 = 7$. When $n = 3$, $x_3 + y_3\sqrt{2} = \left(3+2\sqrt{2}\right)^2\left(2+\sqrt{2}\right) = \left(17+12\sqrt{2}\right)\left(2+\sqrt{2}\right) = 58+41\sqrt{2}$. Then $x_3 = 58$ and $y_3 = 41$.

Theorem 1: There are infinitely many Markov numbers, (a, b, c), for which $a = 1$.

Proof: Setting $a = 1$ in (*) yields $1 + b^2 + c^2 = 3bc \Rightarrow c^2 - 3bc + (b^2 + 1) = 0 \Rightarrow$

$$c = \frac{3b \pm \sqrt{9b^2 - 4\left(b^2+1\right)}}{2} \Rightarrow c = \frac{3b \pm \sqrt{5b^2 - 4}}{2}$$

By guesswork, letting $b = 1$, we have $c = \frac{3 \pm 1}{2}$, yielding the solutions (1, 1, 1) and (1, 1, 2). For additional solutions, we let $5b^2 - 4 = s^2$, in which case,

$$\boxed{c = \frac{3b \pm s}{2}} \qquad (**)$$

Observe that $5b^2 - 4 = s^2$ becomes the generalized Pell Equation, $s^2 - 5b^2 = -4$, with smallest positive solution, $s_1 = b_1 = 1$. The associated ordinary Pell Equation, $s^2 - 5b^2 = 1$, has the smallest positive solution, $r = 9$, $t = 4$. Then

$\boxed{s_n + b_n\sqrt{5} = \left(9+4\sqrt{5}\right)^{n-1}\left(1+\sqrt{5}\right)}$ which has infinitely many solutions. ■

Example: Let $n = 2$. Then $s_2 + b_2\sqrt{5} = \left(9+4\sqrt{5}\right)\left(1+\sqrt{5}\right) = 29+13\sqrt{5}$, so $s_2 = 29$ and $b_2 = 13$. By (**), we obtain two values for c, $\frac{39 \pm 29}{2}$ = 34 and 5, yielding the Markov triples (1, 13, 5) and (1, 13, 34).

Theorem 2: There are infinitely many Markov numbers, (a, b, c), for which $a = 2$.

Proof: Setting $a = 2$ in (*) yields $4 + b^2 + c^2 = 6bc \Rightarrow c^2 - 6bc + (b^2 + 4) = 0 \Rightarrow$

$$c = \frac{6b \pm \sqrt{36b^2 - 4\left(b^2+4\right)}}{2} = 3b \pm \sqrt{9b^2 - \left(b^2+4\right)} = 3b \pm \sqrt{8b^2 - 4} \Rightarrow$$

$$c = 3b \pm 2\sqrt{2b^2 - 1} = 3b \pm 2s$$

(***)

$2b^2 - 1 = s^2$ becomes the generalized Pell Equation, $s^2 - 2b^2 = -1$, with smallest positive solution, $s_1 = b_1 = 1$. The associated ordinary Pell Equation, $s^2 - 2b^2 = 1$, has the smallest positive solution, $r = 3$, $t = 2$. Then

$\boxed{s_n + b_n\sqrt{2} = \left(3+2\sqrt{2}\right)^{n-1}\left(1+\sqrt{2}\right)}$ which has infinitely many solutions. ■

Example: Let $n = 2$. Then $s_2 + b_2\sqrt{2} = \left(3+2\sqrt{2}\right)\left(1+\sqrt{2}\right) = 7+5\sqrt{2}$, so $s_2 = 7$ and $b_2 = 5$. By (***), we obtain two values for c, 15 ± 14 = 29 and 1, yielding the triples (2, 5, 29) and (2, 5, 1).

A Markov triple of the form (a, a, a) satisfies $3a^2 = 3a^3$, whose only solution in natural numbers is $a = 1$, which yields (1, 1, 1). Our next theorem is more general.

Theorem 3: With the exception of (1, 1, 1) and (1, 1, 2), every Markov triple consists of three distinct Markov numbers.

Proof: Assume that $a = b$. Then (*) becomes $2a^2 + c^2 = 3a^2c$, or $c^2 - 3a^2c + 2a^2 = 0$. We have $c = \frac{3a^2 \pm \sqrt{9a^4 - 8a^2}}{2} = a\left(\frac{3a \pm \sqrt{9a^2 - 8}}{2}\right)$. We require $9a^2 - 8 = s^2$, or $(3a)^2 - s^2 = 8$. The only solution to this last equation is $a = s = 1$. ■

Theorem 4: No Markov triple contains 3.

Proof: Let $a = 3$. Then (*) becomes $9 + b^2 + c^2 = 9bc$, which becomes $c^2 - 9bc + (b^2 + 9) = 0$, so $c = \frac{9b \pm \sqrt{81b^2 - 4\left(b^2+9\right)}}{2} = \frac{9b \pm \sqrt{77b^2 - 36}}{2}$. This requires that $77b^2 - 36 = s^2$, which becomes $-1 = s^2$ (mod 7), or $s^2 = 6$ (mod 7). Now the only quadratic resides, mod 7, are 0, 1, 2, 3, and 5. ■

It has been conjectured that two different Markov triples cannot have the same *maximum* element. Thus the existence of the Markov triple, (2, 5, 29), for example, precludes the existence of a different Markov triple having maximum element, 29. The Markov triple, (2, 29, 169), contains 29, but the maximum element is 169.

Theorem 5: $a^2 + b^2 + c^2 = (4k)abc$ has no (positive) integer solutions.

Proof: Observe that this becomes $a^2 + b^2 + c^2 = 0$ (mod 4). Since squares are 0 or 1 mod 4, we must have $a^2 = b^2 = c^2 = 0$ (mod 4) which implies that a, b, and c are even. Let $a = 2A$, $b = 2B$, and $c = 2C$. Then we have $4A^2 + 4B^2 + 4C^2 = (4k)\cdot 8ABC$, or $A^2 + B^2 + C^2 = 8kABC$. Repeating the argument, mod 4, we find that A, B, and C must be even. Continuing in this manner presents an absurd infinite descent. ■

Reference

[1] M.Lewinter, J.Meyer, *Elementary Number Theory with Programming*, Wiley & Sons. 2015.