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The comparison of some types of convergences of double sequences in 2-quasi-normed spaces

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***ABSTRACT**: The main purpose of this paper is to compare two types of convergence of double sequence, whose elements are in $2-$quasi-normed spaces. Firstly, we have introduced a new function in a quasi-normed space, which we have entitled the $2-$quasi-norm, and have seen that every quasi-normed space can be equipped with a $2-$quasi$-$norm. The first results in the comparison of the types of convergence of double sequences are precisely related to the convergence according to quasi$-$norm and $2-$quasi$-$norm. To continue, we are focused on statistical convergence and that one that is related with ideals, and in the end, we have compared each of those types of convergence with their repeated ones. During this paper, we have defined a new type of convergence, which is called or-convergence, that makes possible a one-to-one function between each double sequence with a usually sequence, with one index and it enables that every double sequence will be bounded, a property that in general is not satisfied from Pringsheim's convergent sequences.*

***KEYWORDS**: $2-$quasi$-$normed spaces, Pringsheim's convergence, statistical convergence, $\mathcal{I}-$convergence and double sequences.*

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I. INTRODUCTION

The convergence of double sequence and the spaces constructed by them, have been studied from authors like S. Sarabadan, S. Talebi, A.K. Banerje, R. Mondal, B. Altay and F. Başar et al. in [1], [2], [3], [4], [6] and [7]. They are focused on Pringsheim's convergence and in statistical convergence that related with ideals in the case of double sequences in normed space and $2-$ quasi $-$ normed space. For the $2-$ normed spaces and $2-$quasi$-$normed spaces have worked many authors, that can be found in C. Park paper in [5].

These studies encourage additional investigation into the comparison of the types of convergence for double sequences, when a space is already equipped with a $2-$quasi$-$normed function.

In the first section of this paper, we have constructed a $2-$quasi$-$norm into a quasi$-$normed space, which is generated from quasi$-$norm and then we have studied the relation between the convergence of double sequences linked with quasi$-$norm and $2-$quasi$-$norm. It is worth noting the Proposition 3.1.4 over a sufficient condition of convergence in terms of $2-$quasi$-$norm for double sequences convergent according to quasi$-$norm. The relationship between the statistical convergence and the one according to ideals has then been observed, related with quasi$-$norm or $2-$quasi$-$normed convergence.

Also, we have defined a new type of convergence, that we have called or$-$convergence because, a famous criterion of convergence, the Pringsheim' criterion which is given as:

$\forall\varepsilon > 0$, exists $p \in N$ such that $\forall m, n \geq p$ we have $\|x_{m,n} - x, z\| < \varepsilon$ is replaced with the following condition:

$\forall\varepsilon > 0$, exists $p \in N$ such that ($\forall m \geq p$, and $\forall n \in N$) or ($\forall n \geq p$, and $\forall m \in N$) we have $\|x_{m,n} - x, z\| < \varepsilon$.

For the last condition, we have observed that it guarantees the boundness of the convergent sequence and exists a one-to-one function between every or$-$convergent double sequence and a usually sequence, with one index.

For all these relationships are given some examples constructed in a quasi $-$ normed and $2-$quasi$-$normed spaces like $L_{\frac{1}{2}}([1,2])$as in [9].

The second section talks about the relation between the iterated limits and their double corresponding for each type of convergence mentioned above. Here we are focused on treated the cases when the existence of iterated limits guarantees double ones. To realize that, we are based in the notion of uniformly convergence according to one index, where we have given the definition of iterated statistical limits and that one based on ideals. We also have given an example which satisfies this property.

II. NOTATIONS

In this section we are giving some notations of quasi−normed, quasi−2 −normed and 2 −quasi−normed spaces. Also, we have given some definitions over the well-known types of convergences for the double sequences.

In [6] is defined the 2 −normed function, which satisfies the following conditions:

Definition 2.1 [6] Let X be a real linear space of dimension greater than 1 and let $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following four conditions:

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent in X.
2. $\|x, y\| = \|y, x\|$ for all $x, y \in X$.
3. $\|x, \alpha y\| = |\alpha|\|x, y\|$, for every real number α.
4. $\|x, y + z\| \leq \|x, y\| + \|y, z\|$, for all $x, y, z \in X$.

The function $\|.,.\|$ is called 2 −norm on X and the pair $(X, \|.,.\|)$ is called a linear 2 −normed space.

Also, in [5] is defined the quasi−2 −normed space.

Definition 2.2[5] Let X be a linear space. A quasi−2 −norm is a real valued function on $X \times X$ satisfying three conditions of definition 2.1 (1, 2, and 3) and the condition:

4. There is a constant $K \geq 1$ such that $\|x + y, z\| \leq K\|x, z\| + K\|y, z\|$ for all $x, y, z \in X$.

The pair $(X, \|.,.\|)$ is called quasi− 2 −normed space if $\|.,.\|$ is a quasi−2 −normed on X. The smallest possible K is called the modulus of concavity of $\|.,.\|$.

Definition 2.3 [8]

Let X be a linear space. A function $\|.\|: X \to \mathbb{R}^+$is said to be quasi − norm on X if the following conditions hold:

(i) $\|x\| = 0 \Leftrightarrow x = 0$

(ii) for every $x \in X$ and for every $\forall \lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda| \cdot \|x\|$

(iii) for every $x, y \in X$, $\|x + y\| \leq K(\|x\| + \|y\|)$ where $K \geq 1$ is a constant independent from variables x and y.

The smallest possible K, such that the above conditions hold, is called the modulus of concavity of quasi−norm $\|.\|$.

If the linear space X is equipped with a quasi − norm $\|.\|$, then $(X, \|.\|)$ is called quasi−normed space.

Le X be a quasi−normed space of dimension greater than 1.

Let us construct the function $\|.,.\|: X \times X \to \mathbb{R}^+$, such that:

$$\|x, y\| = \begin{cases} 0 & x \text{ and } y \text{ are linearly dependent} \\ \|x\|\|y\| & \text{elsewhere} \end{cases}$$

This function satisfies the following conditions:

1. For all $x, y \in X, \|x, y\| = \|y, x\|$ because if x, y are linearly dependent, then y, x are also linearly dependent. From this, $\|x, y\| = \|y, x\|$.
2. For all $x, y \in X$, for every $\alpha \in \mathbb{R}$,

$$\|x, \alpha y\| = \|x\|\|\alpha y\| = \|x\||\alpha|\|y\| = |\alpha|\|x, y\|$$

or zero when they are linearly dependent.

If x and y are linearly dependent then exists a constant k such that $y = kx$. Since $\alpha y = \alpha kx$ is linearly dependent with x, we write$\|x, \alpha y\| = \|x, \alpha kx\| = |\alpha|\|x, kx\| = 0$.

This means that for all $x, y \in X$ and for every $\alpha \in \mathbb{R}$, we have that

$$\|x, \alpha y\| = |\alpha|\|x, y\|.$$

3. For all $x, y, z \in X$, exists K≥ 1 such that

$$\|x + y, z\| \leq K\|x, z\| + K\|y, z\|,$$

If $x + y$ and z are linearly dependent, then

$$\|x + y, z\| = 0 \leq K\|x, z\| + K\|y, z\|.$$

In all other cases,

$$\|x + y, z\| = \|x + y\|\|z\| \leq K(\|x\| + \|y\|)\|z\| = K(\|x\|\|z\| + \|y\|\|z\|) = K\|x, z\| + K\|y, z\|.$$

So, the conditions 1, 2, 3 and 4 of quasi−2 −norm function hold.

The function $\|.,.\|$ defined as above we called 2 −quasi−norm in X.

We have obtained that:

Proposition 2.4 LeX be a quasi−normed space of dimension greater than 1. The quasi−norm on X generates a 2 −quasi−norm on X.

This means that every quasi − normed space $(X, \|.\|)$ of dimension greater than 1 will equiped with a quasi−2 −norm (like 2 −quasi−norm defined as above).

Let $(X, \|.\|)$ be a quasi−normed space of dimension greater than 1. Also, this space can be equiped with a 2 −quasi−norm as above.
The convergence of a double sequence was firstly introduced by Mohammad Mursaleen.
A double sequence$x = (x_{j,k})_{\mathrm{j,k}\in\mathbb{N}}$ of real numbers is said to be convergent in the Pringsheim's sense if for every $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that for all $j, k \geq n_0$ implies that $|x_{j,k} - l| < \varepsilon$.

Assume $(x_{m,n})$ is a real numbers double sequence that is p −convergent (p −convergent is the short for Pringsheim convergent). The functor $p - lim x_{m,n}$ yielding a real number α which is defined as follows:
Definition 2.5 [6] Let us consider a real number $\varepsilon > 0$. Then there exists a natural number $n(\varepsilon)$ such that for every natural numbers m, n such that $m, n \geq n(\varepsilon)$ holds $|x_{m,n} - \alpha| < \varepsilon$.

We say that $(x_{m,n})$ is convergent in the first coordinate if and only if:
Definition 2.6 [6] Let us consider an element $m \in \mathbb{N}$. Then curry'$(x_{m,n}, m)$ is convergent.

We say that $x_{m,n}$ is convergent in the second coordinate if and only if:
Definition 2.7 [6] Let us consider an element $n \in \mathbb{N}$. Then curry $(x_{m,n}, n)$ is convergent.

The limit in the first coordinate of $x_{m,n}$ yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by:
Definition 2.8 [6] Let us consider an element $m \in \mathbb{N}$. Then $\alpha(m) = \lim curry'(x_{m,n}, m)$.

The limit in the second coordinate of $x_{m,n}$ yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by:
Definition 2.9 [6] Let us consider an element $n \in \mathbb{N}$. Then $\alpha(n) = \lim curry(x_{m,n}, n)$.

Assume that the lim in the first coordinate of $x_{m,n}$ is convergent. The first coordinate major iterated lim of $x_{m,n}$ yielding a real number is defined by:
Definition 2.10 [6]Let $\varepsilon > 0$ be a real number. Then there existsa natural number M such that for every natural number m such that $m > M$ holds |(the lim in the first coordinate of $x_{m,n})(m) - \alpha| < \varepsilon$.

Assume that the lim in the second coordinate of $x_{m,n}$ is convergent. The second coordinate major iterated lim of $x_{m,n}$ yielding a real number is defined by:
Definition 2.11 [6] Let $\varepsilon > 0$ be a real number. Then there exists a natural number N such that for every natural number n such that $n > N$ holds |(the lim in the second coordinate of $x_{m,n})(n) - \alpha| < \varepsilon$.

The limits of definition 2.10 and 2.11 are called iterated limits.

Let $x_{m,n}$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. We say that $x_{m,n}$ is uniformly convergent in the first coordinate if and only if:
Definition 2.12 [6] (i) $x_{m,n}x$ is convergent in the first coordinate, and
(ii) for every $\varepsilon > 0$, there exists a natural number M such that for every natural number m such that $m \geq M$ for every natural number m, $|x_{n,m} -$ (the lim in the first coordinate of $x_{m,n})(n)| < \varepsilon$.
We say that $x_{m,n}$ is uniformly convergent in the second coordinate if and only if:
Definition 2.13 [6] (i) $x_{m,n}$ is convergent in the second coordinate, and
(ii) for every $\varepsilon > 0$, there exists a natural number N such that for every natural number n such that $n \geq N$ for every natural number N, $|x_{n,m} -$ (the lim in the second coordinate of $x_{m,n})(m)| < \varepsilon$.

Similarly, we define above concepts for double sequences in a quasi−normed and 2 −quasi−normed spaces.

Definition 2.14 [6, 7] Let $(x_{m,n})$ be a double sequence and let $\alpha_j = sup\{x_{m,n}: m, n \geq j\}$ for each j. The Pringsheim limit superior of $(x_{m,n})$ is defined as follows:
i) If $\alpha_j = +\infty$ for each j, then $lim\ sup\ x_{m,n} = +\infty$.
ii) If $\alpha_j < +\infty$ for some j, then $lim\ sup\ x_{m,n} = \inf_j \{\alpha_j\}$.

In the same way as limit superior, is defined also the Pringsheim limit inferior of $x_{m,n}$.
If $x = x_{m,n}$ and $y = y_{m,n}$ are double sequences, then ([7])
i) $lim\ inf\ x_{m,n} \leq lim\ sup x_{m,n}$
ii) $lim\ x_{m,n} = s$ if and only if $lim\ inf\ x_{m,n} = lim\ sup x_{m,n} = s$
iii) $lim\ sup(-x_{m,n}) = -\ lim\ \ inf\ x_{m,n}$
iv) $lim\ sup(x_{m,n} + y_{m,n}) \leq lim\ sup\ x_{m,n} + lim\ sup\ y_{m,n}$
v)$lim\ inf(x_{m,n} + y_{m,n}) \geq lim\ inf\ x_{m,n} + lim\ inf\ y_{m,n}$.

Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X (where X is a normed space) is said to be statistically convergent to $l \in X$ ([1]), if the set $A(\varepsilon) = \{n \in \mathbb{N}: \|x_n - l\| \geq \varepsilon\}$ has natural density zero for each$\varepsilon > 0$. In other words, for each $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{n} card(\{k \leq n: \|x_k - l\| \geq \varepsilon\}) = 0$.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a set of pairs of positive integers and let $A(m,n)$ be the of numbers of (j,k) in A such that $j \leq m$ and $k \leq n$. Then the two−dimensional concept of natural density can be defined as follows:
Definition 2.15 [1] The lower asymptotic density of a set $A \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{d_2}(A) = \liminf_{m,n} \frac{A(m,n)}{mn}.$$

If the sequence $\left(\frac{A(m,n)}{mn}\right)_{n,m\in\mathbb{N}}$ has a limit in Pringsheim's sense, then we say that $A \subseteq \mathbb{N} \times \mathbb{N}$ has a double natural density and is defined as $d_2(A) = \lim_{m,n\to\infty} \frac{A(m,n)}{mn}$.
Let Y be an arbitrary set
Definition 2.16 [1] A family $\mathcal{I} \subseteq \mathcal{P}(Y)$ of subsets nonempty set Y is said to be ideal in Y if:
i) $\phi \in \mathcal{I}$
ii) $A, B \in \mathcal{I}$ implies that $A \cup B \in \mathcal{I}$
iii) $A \in \mathcal{I}$, $B \subseteq A$ implies that $B \in \mathcal{I}$.
$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.
Definition 2.17 [1] Let $Y \neq \phi$. A nonempty family F of subsets of Y is said to be a filter in Y provided:
i) $\phi \in F$
ii) $A, B \in F$ implies that $A \cap B \in F$
iii) $A \in F$, $A \subseteq B$ implies that $B \in F$.
Definition 2.18 [1] A nontrivial ideal $\mathcal{I}$ in Y is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in Y$.
Definition 2.19 [1] A nontrivial ideal $\mathcal{I}$ in $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$.
It is evident that a strongly admissible ideal is admissible also.
Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$ −convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N}: \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$

Definition 2.20 [1] A sequence $(x_n)_{n\in\mathbb{N}}$ in a 2 −normed space $(X, \|.,.\|)$ is said to be convergent to x in X if $\lim_{n\to\infty} \|x_n - x, z\| = 0$ for every $z \in X$.
This can be written by the formula

$$(\forall z \in X)(\forall\varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0), \|x_n - x, z\| < \varepsilon$$

We write it as $x_n \xrightarrow{\|.,.\|_X} x$.
Similarly, we define the sequence convergence in a 2 −quasi−normed space.

Definition 2.21 [1] A double sequence $x = (x_{j,k})_{j,k\in\mathbb{N}}$ of all elements of X(where X is a metric space) is said to be $\mathcal{I}$ −convergent to $l \in X$ if for every $\varepsilon > 0$ we have $A(\varepsilon) \in \mathcal{I}$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N}: \rho(x_{mn}, l) \geq \varepsilon\}$ and we write it as $\mathcal{I} - \lim_{n,m\to\infty} x_{mn} = l$.
Similarly, we define the sequence $\mathcal{I}$ −convergence in a 2 −quasi−normed space.

Definition 2.22 [1] A double sequence $x = (x_{j,k})_{j,k\in\mathbb{N}}$ of all elements of X (where X is a 2 −quasi−normed space) is said to be $\mathcal{I}$ −convergent to $l \in X$ if for every $z \in X$ and $\varepsilon > 0$ we have $A(\varepsilon) \in \mathcal{I}$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N}: \|x_n - l, z\| \geq \varepsilon\}$ and we write it as $\mathcal{I} - \lim_{n,m\to\infty} x_{mn} = l$.

III. MAIN RESULTS

III.1 Comparison of types of convergence

Now let us study the comparison of different types of convergence.
Let $(X,\|.\|)$ be a quasi−normed space of dimension greater than 1 and $\|.,.\|$ the 2 −quasi−norm generates by the quasi−norm $\|.\|$.
Definition 3.1.1 A double sequence $x=(x_{m,n})_{m,n\in\mathbb{N}}$ is said to be convergent in the Pringsheim's sense if for each $\varepsilon>0$ there exists a positive integer $n_0=n_0(\varepsilon)$ such that for all $m,n\geq n_0$, implies that $\|x_{m,n}-x\|<\varepsilon$.
We can easily see that:
If a double sequence $x=(x_{m,n})_{m,n\in\mathbb{N}}$ converges to x in a 2 − quasi − normed space $(X,\|.,.\|)$, then it is convergent to x in the Pringsheim's sense for the quasi−normed space $(X,\|.,.\|)$. To prove this let us take, $z\in X$ such that $\|z\|=1$, and from the definition of a double sequence, convergent in a 2 −quasi−normed space the above definition holds.
Thus we have verified that:
Proposition 3.1.2 The convergence in 2 −quasi−normed space implies the convergence in quasi-normed space.

In general, the convergence of a double sequence in a quasi-normed space does not implies the convergence in a 2 −quasi−normed space X.
Example 3.1.3 Let $L_{\frac{1}{2}}([1,2])$ be the quasi−normed space of functions such that $\left(\int_1^2\sqrt{|f|}dx\right)^2<+\infty$ and let $f_{mn}(x)=\frac{1}{x^{2mn}}$ be a double sequence in $l_{\frac{1}{2}}[1,2]$. The following equations hold:

$$\|f_{mn}(x)\|_{\frac{1}{2}}=\left(\int_1^2\sqrt{|f_{mn}|}dx\right)^2=\left(\int_1^2\frac{1}{x^{mn}}dx\right)^2=\left(\frac{x^{1-mn}}{1-mn}|_1^2\right)^2=\left[\frac{1}{1-mn}(2^{1-mn}-1)\right]^2=\frac{\left(\frac{1}{2^{mn}}-1\right)^2}{(1-mn)^2}\xrightarrow[m,n\to\infty]{}0. \quad (1)$$

So, for $f_{mn}(x)\xrightarrow{\|\cdot\|_{\frac{1}{2}}}f(x)=0, \forall\, x\in[1,2]$.
Also, for $g_{m,n}(x)=x^{2mn}$,

$$\|f_{m,n},g_{mn}\|_{\frac{1}{2}}=\|f_{m,n}\|_{\frac{1}{2}}\|g_{mn}\|_{\frac{1}{2}}=\frac{\left(\frac{1}{2^{mn}}-1\right)^2}{(1-mn)^2}\left(\int_1^2\sqrt{x^{2mn}}dx\right)^2=\frac{\left(\frac{1}{2^{mn}}-1\right)^2}{(1-mn)^2}\left(\frac{x^{mn+1}}{mn+1}|_1^2\right)^2 \quad (2)$$

$$=\frac{\left(\frac{1}{2^{mn}}-1\right)^2}{(1-mn)^2}\frac{1}{(mn+1)^2}(2^{mn+1}-1)^2=\frac{2^{mn}\left(2-\frac{1}{2^{mn}}\right)^2\left(\frac{1}{2^{mn}}-1\right)^2}{(mn)^2\left(\frac{1}{mn}-1\right)^2\left(\frac{1}{mn}+1\right)^2}\xrightarrow[m,n\to\infty]{}\infty.$$

From (2) we see that exists the functions $g_{mn}(x)$ for m and n sufficiently large numbers, such that $\|f_{m,n}-f_{mn}\|_{\frac{1}{2}}\xrightarrow[m,n\to\infty]{}\infty$, and this implies that that $\|f_{m,n}\|_{\frac{1}{2}}$ do not converge to $f(x)=0$ in 2 −quasi−normed generated by the quasi−norm of $L_{\frac{1}{2}}([1,2])$.

The following proposition holds:
Proposition 3.1.4 If a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ converges to x, in a quasi − normed space $(X,\|.\|)$ and $\lim_{n\to\infty}\|x_{m,n}-x,y\|=0$ for every $y\in Y$, where Y is dense in X, then $(x_{m,n})_{m,n\in\mathbb{N}}$ converge to x in 2 −quasi−normed space $(X,\|.,.\|)$ generated by the quasi−norm.
Proof
For every $z\in X$, $\exists(y_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}y_n=z$. That means:

$(\forall\varepsilon>0)(\exists n_0\in\mathbb{N})(\forall n\geq n_0), \|y_n-z\|<\frac{\varepsilon}{\sqrt{2K}}$, (where K is the modulus of concavity of quasi−norm)

Also, $\lim_{m,n\to\infty}x_{m,n}=x$ if and only if $(\forall\varepsilon>0)(\exists n_1\in\mathbb{N})(\forall m,n\geq n_1)$, $\|x_{m,n}-x\|<\frac{\varepsilon}{\sqrt{2K}}$.
Take a $y_{n'}\in Y$ for $n'\geq n_0$ and we see:

$$\begin{aligned}\|x_{m,n}-x,z\|&=\|x_{m,n}-x\|\|z\|=\|x_{m,n}-x\|\|-y_{n'}+z+y_{n'}\|\\&\leq K\|x_{m,n}-x\|\|y_{n'}-z\|+K\|x_{m,n}-x\|\|y_{n'}\|\\&=K\|x_{m,n}-x\|\|y_{n'}-z\|+K\|x_{m,n}-x,y_{n'}\|.\end{aligned}$$

For $y_{n'} \in Y$ we have: $\lim_{m,n\to\infty} \|x_{m,n} - x, y_{n'}\| = 0$ iff $(\exists n_2 \in \mathbb{N})(\forall m,n \geq n_2), \|x_{m,n} - x, y_{n'}\| < \frac{\varepsilon}{2K}$.
If we take $p = max\{n_1, n_2\}$ we obtain that:

$$\|x_{m,n} - x, z\| \leq K\|x_{m,n} - x\|\|y_{n'} - z\| + K\|x_{m,n} - x, y_{n'_n}\| < K\frac{\varepsilon}{\sqrt{2K}}\frac{\varepsilon}{\sqrt{2K}} + K\frac{\varepsilon}{2K} < \varepsilon, \text{ for every } m,n \geq p.$$

This means that $\lim_{m,n\to\infty} \|x_{m,n} - x, z\| = 0$, for every $z \in X$. This completes the proof.

Corollary 3.1.5 The Pringsheim's convergence of a double sequence in a quasi−normed subspace Y of X is equivalent with the convergence in 2 −quasi−norm generated by quasi-norm of X if Y is dense in X.

The double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in a quasi−normed space is called statistically convergent in $x \in X$, if the set $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N}: \|x_{m,n} - x\| \geq \varepsilon\}$ has natural density zero for each $\varepsilon > 0$. In other words, for each

$$\varepsilon > 0, \lim_{m,n\to\infty} \frac{1}{mn} card\{k \leq m, l \leq n: \|x_{m,n} - x\| \geq \varepsilon\} = 0.$$

The following proposition holds:

Proposition 3.1.6 Every double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ that converges to x in a quasi − normed space is statistically convergent to x.

Proof If $(x_{m,n})_{m,n\in\mathbb{N}}$ converges to x in a quasi−normed space then we have:
$\forall\varepsilon > 0, \exists p \in \mathbb{N}$ such that for every $m,n \geq p$, $\|x_{m,n} - x\| < \varepsilon$. Let us have $\{k \leq m, l \leq n: \|x_{m,n} - x\| \geq \varepsilon\}$.
We can see that:
If $m,n < p$, then $cardA(\varepsilon) = mn$;
If $m \geq p$ and $n < p$, then $cardA(\varepsilon) = mn$;
If $m < p$ and $n \geq p$, then $cardA(\varepsilon) = mn$.
If $m,n \geq p$, then $cardA(\varepsilon) = mn - (m-p+1)(n-p+1)$.
So,

$$\frac{cardA(\varepsilon)}{mn} = \begin{cases} 1, & if\ m < p\ \ or\ \ n < p \\ \frac{mn - (m-p+1)(n-p+1)}{mn}, & if\ \ m,n \geq p. \end{cases}$$

$$\lim_{m,n\to\infty} \frac{cardA(\varepsilon)}{mn} = \lim_{m,n\to\infty} \frac{mn - (m-p+1)(n-p+1)}{mn},$$

because, if $m,n \to \infty$ then $m,n \geq p$ for every $p \in \mathbb{N}$.
Thus

$$\lim_{m,n\to\infty} \frac{cardA(\varepsilon)}{mn} = \lim_{m,n\to\infty} 1 - \frac{(m-p+1)(n-p+1)}{mn} = 0$$

This completes the proof.

Now, let us give an example of double sequence that is statistically convergent to x, but it is not convergent to x according to quasi−norm on X.

Example 3.1.7 Let $(x_{m,n})_{m,n\in\mathbb{N}}$ be the double sequence that is given by the following table

$$\begin{pmatrix} 1 & 0 & 0\cdots \\ 0 & 1 & 0\cdots \\ 0 & 0 & 1\cdots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

where 1 is denote an element $x \in X$ such that $\|x\|$=1 and 0 is denote the element 0 of X space.
The double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically convergent to 0, because for every $\varepsilon > 0$ can see that:

$$card\{k \leq m, l \leq n: \|x_{m,n}\| \geq \varepsilon\} = min(m,n).$$

So,

$$\lim_{m,n\to\infty} \frac{cardA(\varepsilon)}{mn} = \lim_{m,n\to\infty} \frac{min(m,n)}{mn}.$$

This implies that

$$0 \leq \frac{min(m,n)}{mn} < \frac{m}{mn} = \frac{1}{n} \xrightarrow[n\to\infty]{} 0\ or\ \frac{n}{mn} = \frac{1}{m} \xrightarrow[m\to\infty]{} 0.$$

Thus,

$$\lim_{m,n\to\infty} \frac{cardA(\varepsilon)}{mn} = 0,$$

And

$$(x_{m,n}) \underset{s}{\rightarrow} 0.$$

On the other hand, the double sequence $(x_{m,n})$ does not converge to 0 according to quasi-norm of X, because for all $p \in \mathbb{N}, \exists m, n \geq p$ such that $\|x_{m,n}\| = 1$.

In the same way with theorem 3.10 in [2], we can proof the following proposition.

Proposition 3.1.8 Let $(X, \|.,.\|)$ be a $2-$quasi$-$normed space. A double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically convergent to $x \in X$ if and only if there exists a subset $A = \{(m,n) \in \mathbb{N} \times \mathbb{N}: d_2(A_{mn}) = 1\}$ and $x_{m,n} \overset{\|.,.\|}{\longrightarrow} x$ for $(m,n) \in A$.

Put $\mathcal{J}_d = \{A \subset \mathbb{N} \times \mathbb{N}: d_2(A) = 0\}$. Then $\mathcal{J}_d$ is an admissible ideal in $\mathbb{N} \times \mathbb{N}$ and $\mathcal{J}_{d_2}-$convergence becomes statistical convergence (Remark 3.3 [1])

Definition 3.1.9 [1] A double sequence $x = (x_{j,k})_{j,k\in\mathbb{N}}$ in a $2-$normed space $(X, \|.,.\|)$ is said to be $\mathcal{J}_2-$convergent to $l \in X$ if for all $\varepsilon > 0$ and nonzero $z \in X$, the set $A(\varepsilon) = \{(j,k): \|x_{j,k} - l, z\| \geq \varepsilon\} \in \mathcal{J}_2$. In this case we can write it as $\mathcal{J}_2 - \lim_{j,k} x_{j,k} = l$.

We can easily see that $\mathcal{J}_d$ is an admissible ideal in $\mathbb{N} \times \mathbb{N}$, from definition of $d_2(A)$ and the cardinality properties, $\mathcal{J}_{d_2}-$convergence becomes statistical convergence, because of the definition of this type of convergence and the fact that $d_2(A) = 0$ for every $A \in \mathcal{J}_{d_2}$.

So, the remark 3.3 [1] is also valid in a $2-$quasi$-$normed space $(X, \|.,.\|)$.

If $\mathcal{J}$ is the ideal $\mathcal{J}_0 = \{A \subset \mathbb{N} \times \mathbb{N}: \exists m(A) \in \mathbb{N} \mid i,j \geq m(A), \textit{implies that } (i,j) \notin \mathbb{N} \times \mathbb{N} - A\}$, then $\mathcal{J}_2-$convergence coincide with $2-$ quasi$-$normed convergence (similar as Remark 3.4 of [1]).

If $x = (x_{j,k})_{j,k\in\mathbb{N}}$ is $\mathcal{J}_2-$convergent then $(x_{j,k})_{j,k\in\mathbb{N}}$ does not need to be $2-$ norm convergent (and also $2-$quasi $-$normed convergent).

Example 3.1.10 In [1] is an example of $\mathcal{J}_2-$convergent double sequence, that is not $2-$normed convergent. It is a special case of $2-$quasi$-$normed convergent.

Let give another definition of double sequence in a $2-$quasi$-$normed space.

Definition 3.1.11 The double sequence $(x_{m,n})$ is called convergent to $x \in X$ if for every $y \in X$ and for every $\varepsilon > 0$, exists $p \in \mathbb{N}$ such that for $m \geq p$ or $n \geq p$, $\|x_{mn} - x, y\| < \varepsilon$.

In this case, we can write $or - \lim_{m,n\to\infty} x_{mn} = x$.

It is clear that, $or - \lim_{m,n\to\infty} x_{mn} = x$ implies the convergence in Pringsheim sense. Also, we can write the double sequence (x_{mn}) in a table and we can find a usual sequence y_n that converges to x.

$$\begin{pmatrix} x_{11} & x_{12} & \ldots x_{1n} \ldots \\ x_{21} & x_{22} & \ldots x_{2n} \ldots \\ \ldots & \ldots & \ldots \ldots \ldots \\ & & \end{pmatrix},$$

$$x_{11}, x_{12}, x_{22}, x_{21}, \ldots, x_{1p}, x_{\varepsilon p}, \ldots, x_{pp}, x_{p,p-1}, \ldots x_{p1}, \ldots$$

For all $n > p^2, y_n = x_{ij}$ where $i > p$ or $j > p$, and$\|y_n - x\| < \varepsilon$, this implies that $y_n \underset{\|.,.\|}{\longrightarrow} x$.

We can see easily that:

Furthermore, an $or-$convergent double sequence is bounded. Also exists an one-to-one function between each $or-$convergent double sequence with a usually sequence.

III.2 Comparison of limits and their iterated limits

Let us see the comparison of these types of limits and their iterated limits, $\lim_{m\to\infty} (\lim_{n\to\infty} x_{mn})$ and $\lim_{n\to\infty} \left(\lim_{m\to\infty} x_{mn}\right)$.

The following proposition holds:

Proposition 3.2.1 If the double sequence $(x_{mn})_{m,n\in\mathbb{N}}$ is Pringsheim convergent to x in the quasi$-2-$normed ($2-$quasi$-$normed) space X and for every $m\in\mathbb{N}$ exists $\lim_{n\to\infty} x_{mn}$, then exists the iterated limits, $\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right)=x$.
Proof The double limit, $\lim_{m,n\to\infty} x_{mn}=x$ if and only if for every $z\in X$, for each $\frac{\varepsilon}{2K}>0$, exists $p\in\mathbb{N}$ such that for every $m,n\geq p$, $\|x_{mn}-x,z\|<\frac{\varepsilon}{2K}$ (where K is modulus of concavity of quasi$-$norm).
For all $m\in\mathbb{N}$, let us denote $y_m=\lim_{n\to\infty} x_{mn}$. This means that for every $z\in X$, for each $\frac{\varepsilon}{2K}>0$, exists $p_1\in\mathbb{N}$ such that for every $n\geq p_1$, $\|x_{mn}-y_m,z\|<\frac{\varepsilon}{2K}$. Now we can write the following inequalities:

$$\|y_m-x,z\|\leq K(\|x_{mn}-y_m,z\|+\|x_{mn}-x,z\|)<K\left(\frac{\varepsilon}{2K}+\frac{\varepsilon}{2K}\right)<\varepsilon,$$

for all $m,n\geq max\{p,p_1\}$.
So, for every $m\geq max\{p,p_1\}$, $\|y_m-x,z\|<\varepsilon$, and this means that $\lim_{m\to\infty} y_m=x$.
This completes the proof.
Corollary 3.2.2 If the double sequence (x_{mn}) is Pringsheim's convergent to x in a quasi $-2-$ normed ($2-$ quasi $-$ normed) space X and exist the iterated limits, $\lim_{m\to\infty}(\lim_{n\to\infty} x_{mn})$ and $\lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right)$, then the following equality folds:

$$x=\lim_{m,n\to\infty} x_{mn}=\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right).$$

The proof is immediate from the above proposition.

Let us suppose that the double sequence $(x_{mn})_{m,n\in\mathbb{N}}$ is statistically convergent to $x\in X$. So, $\lim_{m,n\to\infty}\frac{d_2(A_{mn})}{mn}=0$, where $d_2(A_{mn})=card\{(i,j); i\leq m, j\leq n\colon \|x_{ij}-x,z\|\geq\varepsilon\}$, for all $z\in X$ and $\varepsilon>0$.
For every $n\in\mathbb{N}$, let us denote $d(A_n)=card\{i\in\mathbb{N}; i\leq m, j=n\colon \|x_{ij}-x,z\|\geq\varepsilon\}$.
We can easily see that

$$d_2(A_{mn})=\sum_{i=1}^{n} d(A_i).$$

So, for every $n\in\mathbb{N}$, exist $\alpha_n=\lim_{m\to\infty}\frac{d_2(A_{mn})}{mn}$, then $0\leq\lim_{m\to\infty}\frac{d(A_n)}{mn}\leq\alpha_n$. This implies that

$$0\leq\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{d(A_n)}{mn}\right)\leq\lim_{n\to\infty}\alpha_n=0.$$

Definition 3.2.3 The iterated statistical limit $(s)\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right)$ is called the iterated Pringsheim limit

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{d(A_n)}{mn}\right).$$

The following proposition holds:
Proposition 3.2.4 If the double sequence $(x_{mn})_{m,n\in\mathbb{N}}$ is statistically convergent to x in a quasi$-2-$normed ($2-$quasi$-$normed) space X and for every $m\in\mathbb{N}$ exist $\lim_{n\to\infty}\frac{d_2(A_{mn})}{mn}$, then exist the iterated statistical limit $(s)\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right)=x$.
Corollary 3.2.5 If the double sequence $(x_{mn})_{m,n\in\mathbb{N}}$ is statistically convergent to $x\in X$ and exists the iterated statistical limits $(s)\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right)$ and $(s)\lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right)$, then the equality $x=(s)\lim_{m,n\to\infty} x_{mn}=(s)\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right)=(s)\lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right)$, holds.

Let us now see what happens with $\mathcal{J}-$convergence. Let $\mathcal{J}\subseteq\mathcal{P}(Y)$ be an ideal in Y.
Proposition 3.2.6 The family $\mathcal{J}_2\subseteq\mathcal{P}(Y\times Y)=\mathcal{P}(Y^2)$, such that for every $A\times B\in\mathcal{J}_2, A,B\in\mathcal{J}$ is an ideal in Y^2.
Proof
i) $\phi\times\phi=\phi\in\mathcal{J}_2$, because $\phi\in\mathcal{J}$.
ii)$A_1\times B_1, A_2\times B_2\in\mathcal{J}_2$ implies that $A_1,A_2,B_1,B_2\in\mathcal{J}$, and this implies that

$$(A_1\times B_1)\cup(A_2\times B_2)=(A_1\cup A_2)\times(B_1\cup B_2)\in\mathcal{J}_2.$$

iii) $A_1\times B_1\in\mathcal{J}$; $A_2\times B_2\subseteq A_1\times B_1$ implies that $A_2\times B_2\in\mathcal{J}_2$, because from definition 2.16 iii), $A_2,B_2\in\mathcal{J}$.

Definition 3.2.7 A double sequence (x_{ij}) in a quasi$-2-$normed ($2-$quasi$-$normed) space is said to be $\mathcal{J}_2-$convergent to $x\in X$, if for all $\varepsilon>0$ and non-zero $z\in X$, the set $A(\varepsilon)=\{(i,j)\colon \|x_{ij}-x,z\|\geq\varepsilon\}\in\mathcal{J}_2$.

If $\mathcal{I}$ is an admissible ideal, then:

for all $i \in \mathbb{N}$ and $i \in A(\varepsilon), A(\varepsilon_i) = \{j: \|x_{ij} - x, z\| \geq \varepsilon\} \in \mathcal{I}$
and for all $j \in A(\varepsilon), A(\varepsilon_j) = \{i: \|x_{ij} - x, z\| \geq \varepsilon\} \in \mathcal{I}$.

Definition 3.2.8 Consider the double sequence $(x_{mn})_{m,n\in\mathbb{N}}$. Suppose that there exists the $\mathcal{I} - \lim_{n\to\infty} x_{mn} = y_m \in X$ for every fixed $m \in \mathbb{N}$, and also exists $\mathcal{I} - \lim_{m\to\infty} y_m = y \in X$. The element $y \in X$ is called $\mathcal{I}_2$ −iterrated limit of x_{mn} and is denoted by $\mathcal{I}_2 - \lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = y$.

It is clear from the above definitions that if the double sequence (x_{mn}) is $\mathcal{I}_2$ −convergent to $x \in X$ then there exists $\mathcal{I}_2 - \lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = \mathcal{I}_2 - \lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right) = x$.

To answer the question, “When the existence of the iterated limits brings the Pringsheim’s convergence or the statistical convergence?”, the following proposition holds:

Proposition 3.2.9 If there exists the Pringsheim’s iterated limit, $\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = x$ and the limit $\lim_{n\to\infty} x_{mn}$ is uniformly according to $m \in \mathbb{N}$, then the double sequence (x_{mn}) is Pringsheim’s convergent to x.

Proof From the fact that $\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = x$, we can write:

For every $m \in \mathbb{N}$, denote $y_m = \lim_{n\to\infty} x_{mn}$ and so, for all $z \in X$ and for every $\frac{\varepsilon}{2} > 0$, there exists $p_1 \in \mathbb{N}$, such that for all $n \geq p_1$ and for all $m \in \mathbb{N}$, $\|x_{mn} - y_m, z\| < \frac{\varepsilon}{2K}$. Also, $\lim_{m\to\infty} y_m = x$, and this implies that for all $z \in X$, for every $\frac{\varepsilon}{2} > 0$, exists $p_2 \in \mathbb{N}$, such that for all $m \geq p_2$ and for all $m \in \mathbb{N}$, $\|y_m - x, z\| < \frac{\varepsilon}{2K}$, thus for all $m, n \geq max\{p_1, p_2\}$; $\|x_{mn} - x, z\| < K(\|x_{mn} - y_m, z\| + \|y_m - x, z\|) < K\left(\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K}\right) = \varepsilon$.

This completes the proof.

In a similar way we can prove the same proposition in case of real double sequences. So, the following proposition holds:

Proposition 3.2.10 If there exists the iterated statistical limit, $(s)\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = x$ (or$(s)\lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right) = x$) and it is uniformly in $m \in \mathbb{N}$ (in $n \in \mathbb{N}$), then the double sequence (x_{mn}) is statistically convergent to x.

The proof is immediate from the existence of $\lim_{m\to\infty}\left(\lim_{n\to\infty} \frac{d_2(A_{mn})}{mn}\right) = 0$, the limit $\lim_{n\to\infty} \frac{d_2(A_{mn})}{mn}$ is uniformly in $m \in \mathbb{N}$ and from the above proposition.

If $\mathcal{I} = \mathcal{I}_0$ then $\mathcal{I}_2 -$ converge coincide with usual convergence. So, we have the following proposition.

Proposition 3.2.11 If there exists $\mathcal{I}_2 - \lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = x$ and $\mathcal{I} - \lim_{n\to\infty} x_{mn}$ is uniformly in $m\in\mathbb{N}$, then the double sequence (x_{mn}) is $\mathcal{I}_2$ −convergent to x.

In general, if there excist the Pringsheim iterated limits ((s) iterated limits), it is not enough for the existence of the Pringsheim limit of the double sequence ((s) limit).

Example 3.2.12

Let

$$x_{mn} = \begin{cases} x & m < n \\ -x & m > n \\ 0 & m = n \end{cases}$$

be a double sequence in a 2 −quasi−normed space X.

It can be written as follows:

$$\begin{pmatrix} 0 & x & x\cdots \\ -x & 0 & x\cdots \\ -x & -x & 0\cdots \\ \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = \lim_{m\to\infty}(x) = x$$

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right) = \lim_{n\to\infty}(-x) = -x$$

So, the Pringsheim limit $\lim_{m,n\to\infty}(x_{mn})$ does not exists.

To find: $(s)\ \lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right)$.

$$\forall\, m\in\mathbb{N}, d(A_m) = card\{n: \|x_{mn}-x, z\| \geq \varepsilon\} = m \text{ implies that}$$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{d(A_{mn})}{mn}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{\sum_{k=1}^{m} k}{mn}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{\frac{(1+m)m}{2}}{mn}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{1+m)}{2n}\right) = 0$$

So, $(s)\ \lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = x$ in similar way, we see that $(s)\ \lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right) = -x$.
Thus, does not exist the $(s)\ \lim_{m,n} x_{mn}$.

IV. CONCLUSIONS

In this paper we have obtained some conclusions regarding the comparison of different types of limits of double sequences and between repeated and double limits of the same type.
We are listing them as follows:

1. The convergence in $2-$quasi$-$normed space implies the convergence in quasi$-$normed space.
2. The Pringsheim's convergence of a double sequence in a quasi$-$normed subspace Y of X is equivalent with the convergence in $2-$quasi$-$norm generated by quasi$-$norm of X if Y is dense in X.
3. Every double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ that converges to x in a quasi$-$normed space is statistically convergent to x.
4. $\mathcal{I}_{d_2}-$convergence becomes statistical convergence.
5. If the double sequence (x_{mn}) is Pringsheim's (statistically) convergent to x in a quasi$-2-$normed ($2-$ quasi $-$ normed) space X and exist the iterated limits, $\lim_{m\to\infty}(\lim_{n\to\infty} x_{mn})$ and $\lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right)$,$\left((s)\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) \text{ and } (s)\lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right)\right)$then the following equality holds:
$$x = \lim_{m,n\to\infty} x_{mn} = \lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = \lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right)$$
$$\left(x = (s)\lim_{m,n\to\infty} x_{mn} = (s)\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = (s)\lim_{n\to\infty}\left(\lim_{m\to\infty} x_{mn}\right)\right).$$
6. If there exists $\mathcal{I}_2 - \lim_{m\to\infty}\left(\lim_{n\to\infty} x_{mn}\right) = x$ and $\mathcal{I} - \lim_{n\to\infty} x_{mn}$ is uniformly in $m\in\mathbb{N}$, then the double sequence (x_{mn}) is $\mathcal{I}_2-$convergent to x.

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