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Two-step Hybrid Block Method for the Numerical Solution of Third Order Ordinary Differential Equations

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***ABSTRACT**: A new zero-stable two-step hybrid block method for solving third order initial value problems of ordinary differential equations directly is derived and proposed. In the derivation of the method, the assumed power series solution is interpolated at the initial and the hybrid points while its third ordered derivative is collocated at all the nodal and off-step points in the interval of consideration. The relevant properties of the method were examined and the method was found to be zero-stable, consistent and convergent. A comparison of the results by the method with the exact solutions and other results in literature shows that the method is accurate, simple and effective in solving the class of problems considered.*

***KEYWORDS**: Linear Multistep Method, block method, Hybrid points, Zero-stability, Consistency, and convergence.*

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I. INTRODUCTION

Ordinary Differential Equations initial value problems are commonly found in engineering and scientific processes such as thermodynamics, [1], fluid mechanics, elasticity and quantum mechanics, [2]. Third order differential equations appearing in the field of engineering and science are due to the mathematical formulation of natural phenomena and as such, they do not have closed form solutions, [3, 4]. Since analytical methods that would provide exact solutions to the problems are rare, numerical methods have been proposed by researchers to provide approximate solutions. One of such class of methods is the linear multi-step methods which have been found to be zero stable and have been implemented without the need for either predictors or starting values from other methods, [5 - 7].

Traditionally, higher order differential equations are solved by first reducing them to a system of first order equations. This approach has been found to have serious drawbacks including lengthy time of formulation of the method, time of execution and cost of implementation of the algorithm, [8 - 10]. The two-step hybrid block method is therefore proposed to overcome these setbacks as it is self-starting, easy to apply and produces results that are very close to the exact values of the solution of third order ordinary differential equations.

II. DERIVATION OF THE METHOD

Ordinary Differential Equations initial value problems of the form

$$y''' = f\big(x, y(x), y'(x), y''(x)\big), y(x_0) = \alpha_0, y'(x_0) = \alpha_1, y''(x_0) = \alpha_2 \qquad (1)$$

where $\alpha_i{}'s$ are constants, $f(x, y, y', y'')$ is a given real valued function in the strip $S = [a, b] \subset [-\infty, \infty]$ which is continuous and second variable Lipschitzian over its existence domain for the existence of solution of (1) to be guaranteed are considered in this paper. This section illustrates the derivation of the two-step hybrid block method for solving (1). The solution is approximated in the interval $[x_n, x_{n+1}]$ where the step length is given by $h = x_{n+1} - x_n$. The approximate solution adopted to solve (1) is of the form

$$y(x) = \sum_{j=0}^{c+i-1} a_j x^j \qquad (2)$$

where c, i are numbers of distinct collocation and interpolation points respectively and $a_j{}'s$ are continuous coefficients to be determined.

Substituting the third derivative of (2) into (1), gives

$$\sum_{j=0}^{c+i-1} j(j-1)(j-2)a_j x^{j-3} = f(x,y,y',y''). \qquad (3)$$

Interpolating (2) at $x = x_n, x_{n+\frac{1}{2}}, x_{n+\frac{3}{2}}$ and collocating (3) at $x = x_n, x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}$ lead to the following system of equation

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_7^2 \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 & x_{n+\frac{1}{2}}^7 \\ 1 & x_{n+\frac{3}{2}} & x_{n+\frac{3}{2}}^2 & x_{n+\frac{3}{2}}^3 & x_{n+\frac{3}{2}}^4 & x_{n+\frac{3}{2}}^5 & x_{n+\frac{3}{2}}^6 & x_{n+\frac{3}{2}}^7 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{2}} & 60x_{n+\frac{1}{2}}^2 & 120x_{n+\frac{1}{2}}^3 & 210x_{n+\frac{1}{2}}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{3}{2}} & 60x_{n+\frac{3}{2}}^2 & 120x_{n+\frac{3}{2}}^3 & 210x_{n+\frac{3}{2}}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{2}} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix}. \qquad (4)$$

The continuous implicit hybrid two-step method derived from (2) is of the form

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + \sum v_i \alpha_{vi}(x) y_{n+vi} + h^3[\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_{vi} f_{n+vi}] \qquad (5)$$

where $k = 2$, $vi = [\frac{1}{2}, \frac{3}{2}]$ are hybrid points and α_j, β_j are continuous coefficients to be determined. Therefore, (5) is re-written as

$$y(x) = \alpha_0 y_n + \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} + \alpha_1 y_{n+1} + \alpha_{\frac{3}{2}} y_{n+\frac{3}{2}} + \alpha_2 y_{n+2} + h^3 \left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2}\right] \qquad (6)$$

Setting $\alpha_1 = \alpha_2 = 0$ in (6) leads to

$$y(x) = \alpha_0 y_n + \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} + \alpha_{\frac{3}{2}} y_{n+\frac{3}{2}} + h^3 \left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2}\right]. \qquad (7)$$

The system (4) is then written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} \\ 1 & \frac{3}{2} & \frac{9}{4} & \frac{27}{8} & \frac{81}{16} & \frac{243}{32} & \frac{729}{64} & \frac{2187}{128} \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 12 & 15 & 15 & \frac{105}{8} \\ 0 & 0 & 0 & 6 & 24 & 60 & 120 & 210 \\ 0 & 0 & 0 & 6 & 36 & 135 & 405 & \frac{8505}{8} \\ 0 & 0 & 0 & 6 & 48 & 240 & 960 & 3360 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{2}} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix}. \qquad (8)$$

Solving (8) using Gaussian elimination method leads to the continuous variables $a_i's$ as follows

$$a_0 = y_n$$

$$a_1 = \frac{1}{13440}\left[-35840y_n + 40320y_{n+\frac{1}{2}} - 4480y_{n+\frac{3}{2}} + 207f_n + 1328f_{n+\frac{1}{2}} + 66f_{n+1} + 96f_{n+\frac{3}{2}} - 17f_{n+2}\right]$$

$$a_2 = \frac{1}{1440}\left[1920y_n - 2880y_{n+\frac{1}{2}} + 960y_{n+\frac{3}{2}} - 117f_n - 361f_{n+\frac{1}{2}} + 30f_{n+1} - 39f_{n+\frac{3}{2}} + 7f_{n+2}\right]$$

$$a_3 = \frac{1}{6}f_n$$

$$a_4 = \frac{1}{144}\left[-25f_n + 48y_{n+\frac{1}{2}} - 36f_{n+1} + 16f_{n+\frac{3}{2}} - 3f_{n+2}\right]$$

$$a_5 = \frac{1}{360}\left[35f_n - 104f_{n+\frac{1}{2}} + 114f_{n+1} - 56f_{n+\frac{3}{2}} + 11f_{n+2}\right]$$
$$a_6 = \frac{1}{180}\left[-5f_n + 18f_{n+\frac{1}{2}} - 24f_{n+1} + 14f_{n+\frac{3}{2}} - 3f_{n+2}\right]$$
$$a_7 = \frac{1}{315}\left[f_n - 4f_{n+\frac{1}{2}} + 6f_{n+1} - 4f_{n+\frac{3}{2}} + f_{n+2}\right]$$

The expressions for the $a_i's$ are then written in terms of the parameters $\alpha_j's$ and $\beta_j's$ as the following functions of t.

$$\alpha_0(t) = \left(1 - \frac{8}{3}t + \frac{4}{3}t^2\right) y_n$$

$$\alpha_{\frac{1}{2}}(t) = (3t - 2t^2)y_{n+\frac{1}{2}}$$

$$\alpha_{\frac{3}{2}}(t) = \left(-\frac{1}{3}t + \frac{2}{3}t^2\right) y_{n+\frac{3}{2}}$$

$$\beta_0(t) = \left(\frac{69}{4480}t - \frac{13}{160}t^2 + \frac{1}{6}t^3 - \frac{25}{144}t^4 + \frac{7}{72}t^5 - \frac{1}{36}t^6 + \frac{1}{315}t^7\right) f_n$$

$$\beta_{\frac{1}{2}}(t) = \left(\frac{83}{840}t - \frac{361}{1440}t^2 + \frac{1}{3}t^4 - \frac{13}{45}t^5 + \frac{1}{10}t^6 - \frac{4}{315}t^7\right) f_{n+\frac{1}{2}}$$

$$\beta_1(t) = \left(\frac{11}{2240}t + \frac{1}{48}t^2 - \frac{1}{4}t^4 + \frac{19}{60}t^5 - \frac{2}{15}t^6 + \frac{2}{105}t^7\right) f_{n+1}$$

$$\beta_{\frac{3}{2}}(t) = \left(\frac{1}{140}t - \frac{13}{480}t^2 + \frac{1}{9}t^4 - \frac{7}{45}t^5 + \frac{7}{90}t^6 - \frac{4}{315}t^7\right) f_{n+\frac{3}{2}}$$

$$\beta_2(t) = \left(-\frac{17}{13440}t + \frac{7}{1440}t^2 - \frac{1}{48}t^4 + \frac{11}{360}t^5 - \frac{1}{60}t^6 + \frac{1}{315}t^7\right) f_{n+2}$$

The parameters $\alpha_j(t)$, $\beta_j(t)$ are evaluated at $t = \frac{1}{2}$, 1, $\frac{3}{2}$ and $t = 2$. The values obtained are then substituted into (7) to obtain the implicit hybrid block method as follows:

$$y_{n+1} = -\frac{1}{3}y_n + y_{n+\frac{1}{2}} + \frac{1}{3}y_{n+\frac{3}{2}} - h^3\left[\frac{1}{5760}f_n + \frac{29}{1440}f_{n+\frac{1}{2}} + \frac{7}{320}f_{n+1} - \frac{1}{1440}f_{n+\frac{3}{2}} + \frac{1}{5760}f_{n+2}\right] \qquad (9)$$

and

$$y_{n+2} = y_n - 2y_{n+\frac{1}{2}} + 2y_{n+\frac{3}{2}} + h^3\left[\frac{1}{960}f_n + \frac{7}{120}f_{n+\frac{1}{2}} + \frac{21}{160}f_{n+1} + \frac{7}{120}f_{n+\frac{3}{2}} + \frac{1}{960}f_{n+2}\right]. \qquad (10)$$

Differentiating (7) once with

$$hy'(x) = \alpha_0' y_n + \alpha_{\frac{1}{2}}' y_{n+\frac{1}{2}} + \alpha_{\frac{3}{2}}' y_{n+\frac{3}{2}} + h^3\left[\beta_0' f_n + \beta_{\frac{1}{2}}' f_{n+\frac{1}{2}} + \beta_1' f_{n+1} + \beta_{\frac{3}{2}}' f_{n+\frac{3}{2}} + \beta_2' f_{n+2}\right]. \qquad (11)$$

The derivatives of the parameters $\alpha_j's$ and $\beta_j's$ are written as the following functions of t.

$$\alpha_0'(t) = -\frac{8}{3} + \frac{8}{3}t$$

$$\alpha_{\frac{1}{2}}'(t) = 3 - 4t$$

$$\alpha_{\frac{3}{2}}'(t) = -\frac{1}{3} + \frac{4}{3}t$$

$$\beta_0'(t) = \frac{69}{4480} - \frac{13}{80}t + \frac{1}{2}t^2 - \frac{25}{36}t^3 + \frac{35}{72}t^4 - \frac{1}{6}t^5 + \frac{1}{45}t^6$$

$$\beta_{\frac{1}{2}}'(t) = \frac{83}{840} - \frac{361}{720}t + \frac{4}{3}t^3 - \frac{13}{9}t^4 + \frac{3}{5}t^5 - \frac{4}{45}t^6$$

$$\beta_1'(t) = \frac{11}{2240} + \frac{1}{24}t - t^3 + \frac{19}{12}t^4 - \frac{4}{5}t^5 + \frac{2}{15}t^6$$

$$\beta_{\frac{3}{2}}'(t) = \frac{1}{140} - \frac{13}{240}t + \frac{4}{9}t^3 - \frac{7}{9}t^4 + \frac{7}{15}t^5 - \frac{4}{45}t^6$$

$$\beta_2'(t) = -\frac{17}{13440} + \frac{7}{720}t - \frac{1}{12}t^3 + \frac{11}{72}t^4 - \frac{1}{10}t^5 + \frac{1}{45}t^6$$

The derivatives $\alpha_j'(t)$ and $\beta_j'(t)$ at $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, the values obtained are then substituted into (11) to obtain the following implicit hybrid block scheme which are used together with the main method in (9) and (10) for the numerical solution of (1).

$$hy_n' = -\frac{8}{3}y_n + 3y_{n+\frac{1}{2}} - \frac{1}{3}y_{n+\frac{3}{2}} + h^3\left[\frac{69}{4480}f_n + \frac{83}{840}f_{n+\frac{1}{2}} + \frac{11}{2240}f_{n+1} + \frac{1}{140}f_{n+\frac{3}{2}} - \frac{17}{13440}f_{n+2}\right], \quad (12)$$

$$hy_{n+\frac{1}{2}}' = -\frac{4}{3}y_n + y_{n+\frac{1}{2}} + \frac{1}{3}y_{n+\frac{3}{2}} + h^3\left[-\frac{43}{20160}f_n - \frac{293}{5040}f_{n+\frac{1}{2}} - \frac{13}{560}f_{n+1} + \frac{1}{5040}f_{n+\frac{3}{2}} - \frac{1}{20160}f_{n+2}\right], \quad (13)$$

$$hy_{n+1}' = -y_{n+\frac{1}{2}} + y_{n+\frac{3}{2}} + h^3\left[\frac{1}{8064}f_n - \frac{13}{5040}f_{n+\frac{1}{2}} - \frac{247}{6720}f_{n+1} - \frac{13}{5040}f_{n+\frac{3}{2}} + \frac{1}{8064}f_{n+2}\right], \quad (14)$$

$$hy_{n+\frac{3}{2}}' = \frac{4}{3}y_n - 3y_{n+\frac{1}{2}} + \frac{5}{3}y_{n+\frac{3}{2}} + h^3\left[\frac{3}{2240}f_n + \frac{131}{1680}f_{n+\frac{1}{2}} + \frac{17}{112}f_{n+1} + \frac{11}{560}f_{n+\frac{3}{2}} - \frac{1}{1344}f_{n+2}\right], \quad (15)$$

and

$$hy_{n+2}' = \frac{8}{3}y_n - 5y_{n+\frac{1}{2}} + \frac{7}{3}y_{n+\frac{3}{2}} + h^3\left[\frac{61}{40320}f_n + \frac{41}{252}f_{n+\frac{1}{2}} + \frac{159}{448}f_{n+1} + \frac{641}{2520}f_{n+\frac{3}{2}} + \frac{733}{40320}f_{n+2}\right]. \quad (16)$$

Differentiating (7) twice with $x = x_n + th$ such that $\frac{dt}{dx} = \frac{1}{h}$ to have

$$h^2y''(x) = \alpha_0''y_n + \alpha_{\frac{1}{2}}''y_{n+\frac{1}{2}} + \alpha_{\frac{3}{2}}''y_{n+\frac{3}{2}} + h^3\left[\beta_0''f_n + \beta_{\frac{1}{2}}''f_{n+\frac{1}{2}} + \beta_1''f_{n+1} + \beta_{\frac{3}{2}}''f_{n+\frac{3}{2}} + \beta_2''f_{n+2}\right]. \quad (17)$$

The second derivatives of the parameter $\alpha_j's$ and $\beta_j's$ are written as the following functions of t.

$$\alpha_0''(t) = -\frac{8}{3}$$

$$\alpha_{\frac{1}{2}}''(t) = -4$$

$$\alpha_{\frac{3}{2}}''(t) = \frac{4}{3}$$

$$\beta''_0(t) = -\frac{13}{80} + t - \frac{25}{12}t^2 + \frac{35}{18}t^3 - \frac{5}{6}t^4 + \frac{2}{15}t^5$$

$$\beta''_{\frac{1}{2}}(t) = -\frac{361}{720} + 4t^2 - \frac{52}{9}t^3 + 3t^4 - \frac{8}{15}t^5$$

$$\beta''_1(t) = \frac{1}{24} - 3t^2 + \frac{19}{3}t^3 - 4t^4 + \frac{4}{5}t^5$$

$$\beta''_{\frac{3}{2}}(t) = -\frac{13}{240} + \frac{4}{3}t^2 - \frac{28}{9}t^3 + \frac{7}{3}t^4 - \frac{8}{15}t^5$$

$$\beta''_2(t) = \frac{7}{720} - \frac{1}{4}t^2 + \frac{11}{18}t^3 - \frac{1}{2}t^4 + \frac{2}{15}t^5$$

The second derivatives $\alpha'_j(t)$ and $\beta'_j(t)$ at $t = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2$, the values obtained are then substituted into (17) to obtain the following implicit hybrid block scheme which are used together with the main method in (9) and (10) for the numerical solution of (1).

$$h^2y''_n = \frac{8}{3}y_n - 4y_{n+\frac{1}{2}} + \frac{4}{3}y_{n+\frac{3}{2}} + h^3\left[-\frac{13}{80}f_n - \frac{361}{720}f_{n+\frac{1}{2}} + \frac{1}{24}f_{n+1} - \frac{13}{240}f_{n+\frac{3}{2}} + \frac{7}{720}f_{n+2}\right], \quad (18)$$

$$h^2y''_{n+\frac{1}{2}} = \frac{8}{3}y_n - 4y_{n+\frac{1}{2}} + \frac{4}{3}y_{n+\frac{3}{2}} + h^3\left[\frac{17}{1440}f_n - \frac{19}{360}f_{n+\frac{1}{2}} - \frac{17}{120}f_{n+1} + \frac{7}{360}f_{n+\frac{3}{2}} - \frac{1}{288}f_{n+2}\right], \quad (19)$$

$$h^2y''_{n+1} = \frac{8}{3}y_n - 4y_{n+\frac{1}{2}} + \frac{4}{3}y_{n+\frac{3}{2}} + h^3\left[-\frac{1}{720}f_n + \frac{3}{16}f_{n+\frac{1}{2}} + \frac{7}{40}f_{n+1} - \frac{23}{720}f_{n+\frac{3}{2}} + \frac{1}{240}f_{n+2}\right], \quad (20)$$

$$h^2y''_{n+\frac{3}{2}} = \frac{8}{3}y_n - 4y_{n+\frac{1}{2}} + \frac{4}{3}y_{n+\frac{3}{2}} + h^3\left[\frac{1}{160}f_n + \frac{49}{360}f_{n+\frac{1}{2}} + \frac{59}{120}f_{n+1} + \frac{5}{24}f_{n+\frac{3}{2}} - \frac{13}{1440}f_{n+2}\right], \quad (21)$$

and

$$h^2y''_{n+2} = \frac{8}{3}y_n - 4y_{n+\frac{1}{2}} + \frac{4}{3}y_{n+\frac{3}{2}} + h^3\left[-\frac{1}{144}f_n + \frac{151}{720}f_{n+\frac{1}{2}} + \frac{37}{120}f_{n+1} + \frac{473}{720}f_{n+\frac{3}{2}} + \frac{119}{720}f_{n+2}\right]. \quad (22)$$

III. ANALYSIS OF THE PROPOSED METHOD

In this section, the main properties of the two-step hybrid block method for solving third order initial value problems are presented. The properties include the order and error constant, zero stability, interval of absolute stability, consistency and convergence of the method.

Consider the linear operator L associated with the implicit hybrid block method (9) – (22) defined as

$$L[y(x_n) : h] = \sum_j [\alpha_j y(x_n + jh) - h^3\beta_j y'''(x_n + jh)] \quad (23)$$

where $y(x)$ is an arbitrary test function that is continuous and differentiable in the interval $[a,\ b]$. Obtaining the Taylor series expansions of $y(x_n + jh)$ and $y'''(x_n + jh)$ about x_n and collecting the coefficients of h^p lead to

$$L[y(x_n) : h] = c_0y(x_n) + c_1hy'(x_n) + c_2h^2y''(x_n) + \ldots + c_ph^py^{(p)}(x_n) + \ldots \quad (24)$$

where c_j's for $j = 0,\ 1,\ 2,\ \ldots$ are vectors.

IV. ORDERS AND ERROR CONSTANTS

From (24), if it is obtained that:

$$c_0 = c_1 = c_2 = \cdots = c_p = 0\ :\ c_{p+1} \neq 0$$

then the hybrid block method (9) – (22) is said to be of order p and the error constant is c_{p+1}. Using (24) and following [7], the orders of the method are

$$(7,\ 8,\ 7,\ 7,\ 8,\ 7,\ 7,\ 7,\ 7,\ 7,\ 7,\ 7)^T$$

and the error constants are

$$\left(\frac{1}{368640}, \frac{1}{15482880}, -\frac{59}{258048}, -\frac{1}{860160}, -\frac{113}{464486400}, \frac{1}{860160}, -\frac{13}{430080}, -\frac{59}{258048}, \frac{83}{1290240}, -\frac{71}{1290240}, \frac{83}{1290240}, -\frac{59}{258048}\right)^T.$$

V. ZERO STABILITY OF THE BLOCK METHOD

The main block method given by (9) and (10) is used to obtain the first characteristics polynomial.
Since $y_{n+\frac{1}{2}} = y_{n+\frac{3}{2}} = 0$, then the first characteristics polynomial of the method is given by

$$\rho(r) = |Ar - B|$$

where

$$A = \begin{pmatrix} -1 & -\frac{1}{3} \\ 2 & -2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -\frac{1}{3} - \frac{1}{5760}z \\ 0 & 1 + \frac{1}{960}z \end{pmatrix}$$

$$Det(Ar - B) = \begin{vmatrix} -r & -\frac{1}{3}r + \frac{1}{3} + \frac{1}{5760}z \\ 2r & -2r - 1 - \frac{1}{960}z \end{vmatrix}$$

$$= \frac{1}{3}r + \frac{8}{3}r^2 + \frac{r}{1440}z$$

Setting $Det(Ar - B) = 0$ with $z = 0$, gives

$$\frac{1}{3}r + \frac{8}{3}r^2 = 0.$$

Then,

$$r_1 = 0, r_2 = -\frac{1}{8}$$

The block method is said to be zero-stable if as $h \to 0$, the roots, $r_j's: j = 1(1)k$ of the characteristics polynomial, $\rho(r) = 0$. That is,

$$\rho(r) = Det\left[\sum A^{(i)} R^{k-1}\right] = 0$$

satisfies $|R| \leq 1$ and for those roots with $|R| \leq 1$, must have multiplicity not greater than 2.

VI. INTERVAL OF ABSOLUTE STABILITY

The Routh-Hurwitz criterion is applied to determine the interval of absolute stability of the method. For the two-step method;

$$y_{n+2} = y_n - 2y_{n+\frac{1}{2}} + 2y_{n+\frac{3}{2}} + h^3\left[\frac{1}{960}f_n + \frac{7}{120}f_{n+\frac{1}{2}} + \frac{21}{160}f_{n+1} + \frac{7}{120}f_{n+\frac{3}{2}} + \frac{1}{960}f_{n+2}\right].$$

The characteristics polynomial is

$$(1-\tfrac{1}{960}h^3)\lambda^2 - \left(2+\tfrac{7}{120}h^3\right)\lambda^{\frac{3}{2}} - \tfrac{21}{160}h^3\lambda + \left(2-\tfrac{7}{120}h^3\right)\lambda^{\frac{1}{2}} - \left(1+\tfrac{1}{960}h^3\right) = 0. \quad (25)$$

Taking the first, third and fifth terms in equation (25), gives

$$(1-\tfrac{1}{960}h^3)\lambda^2 - \tfrac{21}{160}h^3\lambda - \left(1+\tfrac{1}{960}h^3\right) = 0.$$

Setting $\lambda = \frac{1+z}{1-z}$, leads to

$$-\tfrac{2}{15}h^3 + 4h^3 z + \tfrac{31}{240}h^3 z^2 = 0. \quad (26)$$

The Routh-Hurwitz criterion is satisfied if

$$-\tfrac{2}{15}h^3 > 0$$

$$4h^3 z > 0$$

$$\tfrac{31}{240}h^3 z^2 > 0$$

Solving for h, the interval of absolute stability is

$$-\infty < h < 0.$$

VII. CONSISTENCY OF THE METHOD

A linear multistep method is said to be consistent if it has an order of convergence, $p \geq 1$, [7]. The derived hybrid method is consistent since the order is greater than 1.

VIII. CONVERGENCE OF THE METHOD

A hybrid block method is said to be convergent if and only if it is consistent and zero-stable, [7]. Since the proposed method satisfies the two conditions, then the method converges.

IX. NUMERICAL IMPLEMENTATION OF THE SCHEME

In this section, the effectiveness and validity of the derived method is tested and demonstrated by applying it to solve some third order ordinary differential equations. To compare the results obtained by the method with those in literature, the values of h are chosen to be the same. For error calculation, the error formula is given by

$$E_r = |y(x) - y(x_n)|. \quad (27)$$

In (27), $y(x)$ is the exact solution for the problem considered and $y(x_n)$ is the approximate solution obtained using the derived method.

All computations and programmes are carried out with the aid of Mathematica software.

Example 1: Consider the third order ordinary differential equation

$$y''' + y' = 0,\ y(0) = 0,\ y'(0) = -1,\ y''(0) = -2\ :\ h = \tfrac{1}{10}. \quad (28)$$

The exact solution is

$$y(x) = 2(1 - Cosx) + Sinx.$$

Table 1 shows the comparison between the proposed method and One Step Hybrid Block Method, [1] and Block method similar to the block trapezoidal rule, [11].

Example 2: Consider the third order ordinary differential equation

$$y''' = 3Sinx,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -2\ :\ x \in [0,\ 1],\ h = \frac{1}{10}. \quad (29)$$

The exact solution is

$$y(x) = 3Cosx + \frac{1}{2}x^2 - 2.$$

Table 1 shows the comparison between the proposed method, One Step Hybrid Block Method, [1] and an accurate scheme by block method, [12].

Example 3: Consider the third order ordinary differential equation

$$y''' + 4y' = x,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 0 :\ x \in [0,\ 1],\ h = \frac{1}{10}. \quad (30)$$

The exact solution is

$$y(x) = \frac{3}{16}(1 - Cos(2x)) + \frac{1}{8}x^2.$$

Table 3 shows the comparison between the proposed method, One Step Hybrid Block Method, [1] and an accurate scheme by block method, [12].

Example 4: Consider the third order ordinary differential equation

$$y''' = -y,\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 1 :\ x \in [0,\ 1],\ h = 0.1. \quad (31)$$

The exact solution is

$$y(x) = e^{-x}.$$

Table 4 shows the comparison between the proposed method, Differential Transform Method, [13] and Adomian Decomposition Method, [14].

Example 5: Consider the third order ordinary differential equation

$$y''' = e^x,\ y(0) = 3,\ y'(0) = 1,\ y''(0) = 5 :\ x \in [0,\ 1],\ h = 0.1. \quad (32)$$

The exact solution is

$$y(x) = 2 + 2x^2 + e^x.$$

Table 5 shows the comparison between the proposed method, Differential Transform Method, [13] and A seven-step block method, [15].

Tables of Results

TABLE 1

Numerical Results for Example 1: Comparison between the absolute errors in the proposed method and other methods in literature

x	Exact Solution	Results by the new Method	Error	Error in [1]	Error in [11]
0.0	0.0000000000000000	0.0000000000000000	0.0000000	0.000000000000	0.00000000
0.1	-0.1098250858850489	-0.1098250858850489	0.0000000	2.40291100E-10	1.5405E-09
0.2	-0.238536172218694	-0.238536172218694	0.0000000	2.89388600E-09	9.8455E-09
0.3	-0.384847213583321	-0.384847213583321	0.0000000	1.48268090E-08	2.3652E-08
0.4	-0.547296306677023	-0.547296306677023	0.0000000	4.76258570E-08	4.3273E-08
0.5	-0.724260297727691	-0.724260297727691	0.0000000	1.17095759E-07	3.9018E-08
0.6	-0.913971000905724	-0.913971000905724	0.0000000	2.42669956E-07	6.9700E-08
0.7	-1.114532865923590	-1.114532865923590	0.0000000	4.46745120E-07	5.2032E-08
0.8	-1.323941918254150	-1.323941918254150	0.0000000	7.53951040E-07	1.3527E-07
0.9	-1.540105782718330	-1.540105782718330	0.0000000	1.19036782E-06	4.7483E-07
1.0	-1.760864590367610	-1.760864590367610	0.0000000	1.78270401E-06	1.0693E-07

TABLE 2
Numerical Results for Example 2: Comparison between the absolute errors in our method and other methods in literature

x	Exact Solution	Results by the new Method	Error	Error in [1]	Error in [12]
0.0	1.000000000000000	1.000000000000000	0.0000000	0.00000000	0.00000000
0.1	0.990012495834090	0.990012495834090	0.0000000	1.0000E-14	1.6000E-10
0.2	0.960199733524001	0.960199733524001	0.0000000	2.7100E-13	4.7000E-10
0.3	0.911009467377965	0.911009467377965	0.0000000	1.1450E-12	6.2000E-10
0.4	0.843182982011622	0.843182982011622	0.0000000	2.9620E-12	1.9000E-10
0.5	0.757747685677192	0.757747685677192	0.0000000	6.0710E-12	3.2000E-10
0.6	0.656006844739830	0.656006844739830	0.0000000	1.0800E-11	1.3000E-09
0.7	0.539526561870899	0.539526561870899	0.0000000	1.7439E-11	4.8000E-09
0.8	0.410120128067774	0.410120128067774	0.0000000	2.6274E-11	1.1000E-08
0.9	0.269829904849588	0.269829904849588	0.0000000	3.7598E-11	2.0000E-08
1.0	0.120906917656045	0.120906917656045	0.0000000	5.1625E-11	3.5000E-08

TABLE 3
Numerical Results for Example 3: Comparison between the exact solution and the approximate solution by our method for $h = 0.1$

x	Exact Solution	Results by the new Method	Error	Error in [1]	Error in [12]
0.0	1.000000000000000	1.000000000000000	0.0000000	0.00000000	0.00000000
0.1	0.004987516654767	0.004987516654767	0.0000000	2.9700E-08	1.6665E-06
0.2	0.019801063624459	0.019801063624459	0.0000000	1.9880E-07	3.8100E-05
0.3	0.043999572204435	0.043999572204435	0.0000000	6.5080E-07	1.5660E-04
0.4	0.076867491997407	0.076867491997407	0.0000000	1.5480E-06	3.9860E-04
0.5	0.117443317583462	0.117443317583462	0.0000000	3.0620E-06	7.9590E-04
0.6	0.164557921035624	0.164557921035624	0.0000000	5.3625E-06	1.3680E-03
0.7	0.216881160706205	0.216881160706205	0.0000000	8.6068E-06	2.1190E-03
0.8	0.272974910431492	0.272974910431492	0.0000000	1.2926E-05	3.0390E-03
0.9	0.331350392754954	0.331350392754954	0.0000000	1.8418E-05	4.1000E-03
1.0	0.390527531852589	0.390527531852589	0.0000000	2.5129E-05	5.2610E-03

TABLE 4

Numerical Results for Example 4: Comparison between the absolute errors in our method and other methods in literature

x	Exact Solution	Results by the new Method	Error	Error in [13]	Error in [14]
0.0	1.000000000000000	1.000000000000000	0.0000000	0.00000000	0.0000000000
0.1	0.904837418035960	0.904837418035960	0.0000000	0.00000000	2.138401E-12
0.2	0.818730753077982	0.818730753077982	0.0000000	0.00000000	6.055156E-13
0.3	0.740818220681718	0.740818220681718	0.0000000	0.00000000	7.395751E-12
0.4	0.670320046035639	0.670320046035639	0.0000000	0.00000000	2.158163E-12
0.5	0.606530659712633	0.606530659712633	0.0000000	0.00000000	1.484579E-11
0.6	0.548811636094026	0.548811636094026	0.0000000	0.00000000	1.098521E-11
0.7	0.496585303791410	0.496585303791410	0.0000000	0.00000000	3.142886E-11
0.8	0.449328964117222	0.449328964117222	0.0000000	0.00000000	2.309530E-11
0.9	0.406569659740599	0.406569659740599	0.0000000	0.00000000	5.154149E-11
1.0	0.367879441171442	0.367879441171442	0.0000000	0.00000000	8.200535E-11

TABLE 5

Numerical Results for Example 5: Comparison between the absolute errors in our method and other methods in literature

x	Exact Solution	Results by the new Method	Error	Error in [13]	Error in [15]
0.0	3.000000000000000	3.000000000000000	0.0000000	0.00000000	0.0000000000
0.1	3.125170918075650	3.125170918075650	0.0000000	0.00000000	2.531308E-14
0.2	3.301402758160170	3.301402758160170	0.0000000	0.00000000	1.612044E-13
0.3	3.529858807576000	3.529858807576000	0.0000000	0.00000000	4.023448E-13
0.4	3.811824697641270	3.811824697641270	0.0000000	0.00000000	7.536194E-13
0.5	4.148721270700130	4.148721270700130	0.0000000	0.00000000	1.212364E-12
0.6	4.542118800390510	4.542118800390510	0.0000000	0.00000000	1.780798E-12
0.7	4.993752707470480	4.993752707470480	0.0000000	0.00000000	2.456702E-12
0.8	5.505540928492470	5.505540928492470	0.0000000	0.00000000	2.212097E-11
0.9	6.079603111156950	6.079603111156950	0.0000000	0.00000000	5.231993E-11
1.0	6.718281828459050	6.718281828459050	0.0000000	0.00000000	8.860113E-11

X. CONCLUSION

In this paper, a two-step block method with two hybrid points for the numerical solution of third order ordinary differential equations is derived and implemented. The method was derived through interpolation of the assumed power series solution at the point $x = x_n$ and the two off-step points. The third ordered derivative of the assumed solution was collocated at all the step and off-step points in the interval of consideration. The properties of the method including order and error constants, consistency, zero-stability, interval of absolute stability and convergence were discussed. Numerical results as presented in Tables 1-5 show that the method converges to the exact solution in most cases in terms of the absolute error (which in most cases are zeros) obtained. Furthermore, the hybrid block method produces the values for the first and second ordered derivations of the solution at the step and off-step points. Although, the method requires a bit of computational efforts yet with the use of soft-ware, the two-step hybrid block method is implemented with ease. Execution of appropriate programs in future would go a long way in lessening the amount of work involved in the implementation of the method.

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