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A Review on Family of Distributions

Anusha.R[1], Bhatracharyulu N. Ch[2] and Jayasree. G[2]
[1]*TSWRDC(W), Karimnagar, Satavahana University, Karimnagar*
[2]*Department of Statistics, University College of Science, Osmania University, Hyderabad-7*
anusha.rathnapurapu@gmail.com, dwarakbhat@osmania.ac.in and drgjaysree@gmail.com
Corresponding Author: Jayasree. G

ABSTRACT*: It is very difficult to identify which distribution is best fit to the data, out of thousands of probability distributions that are available in the literature. So, it is necessary to group the distributions further according to their nature of curve or common properties they satisfy. In this regard initially Karl Pearson, Johnson, Koopman, Kumaraswamy etc. made attempts to identify the family of distributions. In this paper, we made an attempt to explore the literature available on the families of distributions.*
Keywords*: Pearsonian, Johnson, Kumaraswamy-G, T-X, transmuted, Power series, Exponential and Alpha power transformation.*

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I. INTRODUCTION

Probability distributions are playing a key role to study the properties of the population of a given data. There are many probability distributions and many classifications exist. In general, based on the type of random variables the probability distributions are broadly classified as: Discrete and Continuous Probability Distributions. If the random variable that takes only the integral real values, corresponding probability models are said to be discrete probability distributions. If the random variable takes real values in an interval (i.e. range of real values), the corresponding probability models are said to be continuous Probability models.

It is very difficult to identify which distribution is best fit to the data, out of thousands of Probability distributions. In this aspect, Karl-Pearson opine that it is necessary to group the distributions further according to their nature of curve / common properties. A group of probability distributions that possess some common properties they are exhibiting is called a family of distributions. In this aspect few authors made attempts to group the probability distributions based on the common properties they are exhibiting. It aims to explore the various families of probability distributions that are available in the literature and understanding on their common properties.

Initially, [10] **Karl Pearson (1894,1895)** attempted to propose a family of distributions, identifying the normal distribution (originally described as type V) as well as four further types of distributions (Type-I to Type-IV). "The distributions' support on a bounded interval, a half-line, or the entire real line, as well as whether or not they were symmetric or potentially skewed, determine how the data are classified." Later [11] **Karl Pearson (1901)** introduced the type VI distribution and defined the type V distribution, which was once only the normal distribution but is now the inverse gamma distribution. **Rhind (1909)** created an easy approach for visualising the Pearson system's parameter space, and Pearson later adopted it (1916). [12] **Pearson (1916)** introduced more special conditions and subtypes (VII through XII) in his third article. Two numbers that are sometimes referred to as β_1 and β_2 define the Pearson types. [8] **Normal L. Johnson (1949)** certain curve systems that were created using the translation method. [7] **Noack, A. (1950)** Certain power series can be used to construct a wide category of random variables with discrete probability distributions.

John (1960) The creation of extended tables of percentage points for the Pearson system of distributions was reported in his thesis, "Some contributions to the evaluation of Pearsonian Distribution functions." [9] **Ord (1967)** a system of discrete distributions is developed using difference equation. **Anthony (1971)** found Relatively simple approximating function to express the probability integral in closed form. **Mitra and Romaniuk (1973)** - created new approaches for calculating the Pearsonian Type-I curve's parameters, which are especially flexible in response to variables affecting the pattern of age-specific fertility rates. [4] **Ollero and Ramos (1995)** demonstrated that generalized-binomial distributions can be used to characterise a subclass of discrete Pearson system distributions, which includes the Polya's distribution without replacement and, consequently, the hypergeometric distribution. **Sankaran and Unnikrishnan Nair (1998)** established a connection between the higher order moments of residual life and the failure rate, which defined the Pearson family of distributions. A characterisation theorem of the IFR (DFR) class of distributions in the Pearson family was also provided by them.

Mahanta and Dilip (2007) estimated parameters for Type II of Pearsonian system of distribution curves by using the method of maximum likelihood. **Mohammad Shakil** et.al, **(2010, 2011)** developed a new family of distributions that are a logical extension of the generalised inverse Gaussian distribution and are founded on the generalised Pearson differential equation. A few properties of the new distribution were discovered. There were tables containing percentiles, skewness and kurtosis values, and plots for the probability density function, hazard function, and cumulative distribution function. The statistical application of these findings to the forestry data has been examined as an incentive. It is discovered that compared to gamma, log-normal, and inverse Gaussian distributions, this recently suggested model fits data better. Additionally, they investigated the distributional connections and created several new classes of continuous probability distributions based on the generalised Pearson differential equation.

G.M Cordeiro and M. Castro (2011) expanded on a number of well-known distributions, such as the gamma, Gumbel, inverse Gaussian, Weibull, and normal distributions, by creating a new family of generalised distributions. studied some of the special distributions of the new family, such as the Kw-inverse Gaussian, Kw-gamma, Kw-Gumbel, Kw-normal, and Kw-Weibull distributions. The ordinary moments of any Kw generalised distribution can be expressed as linear functions of probability weighted moments of the parent distribution. We produced the ordinary moments of order statistics by calculating the weighted moments of the baseline distribution as functions of probability.

Saralees Nadarajah (2011) developed an easily understood linear combination of exponentiated-G distributions to express the Kumaraswamy-G density function. They suggest a few new distributions as this family's sub-models. **George and Ramachandran (2011)** A new approach is put forth to estimate the parameters of Johnson's distribution. The MLE-Least Squares technique is the name of this algorithm. **Mohammad** et.al, **(2012)** also used Pearsonian system of frequency curves for the analysis of stock returns. [5] **Mahanta and Dilip (2012)** applied the Pearsonian Type - III curves and its potentials in the analysis of insurance data. **Raykundaliya** et. al, **(2013)** applied Pearsonian type - IV distribution, to improvise the confidence limits of coefficient of variation of the data on the yield of wheat crop.

[2] **Lahcene (2013)** proposed a new extended model of Pearsonian distribution and named it as the extended generalized distribution. This new family is studied for a variety of characteristics. **Raid Al-Aqtash (2014)** The Gumbel-Weibull distribution's applications are highlighted. **Shakil et al (2016)** the expanded Pearson system of differential equations, which can produce these new types of continuous probability distributions, was discussed. In another study, several characterizations of a new type of generalized Pearson distribution were offered using truncated moments. **Morad Alizadeh (2016)** The new density function can be expressed as a linear combination of exponentiated densities with respect to the same baseline distribution.

AL – Kadim and Mohammed (2017) created the cubic transmuted Weibull distribution, or lifespan distribution, and talked about some of its statistical characteristics. **Mahdavi, A. and Kundu, D. (2017)** A novel approach has been proposed to increase the flexibility of a family of distributions by including an extra parameter. One specific example, the one-parameter exponential distribution, has been studied in detail. Various features of the proposed distribution are generated, including different explicit formulations for the moments, quantiles, mode, mean residual lifetime, order statistics, stochastic ordering, and expression of the entropies.

Mustafa Unlu et al (2019) demonstrated that Pearson Type IX probability density functions can be used to generate desired distributions and their accompanying quality-loss functions. **Hongjie Wan et al (2019)** estimated the parameters of an autoregressive model, and a Bayesian model is constructed using the Pearson type VII distribution as the noise model. **Tegos et al (2020)** provided new results for Pearson distributions of Type - III and used them to look at the statistical behaviour of wireless power transfer for the first time in the literature. **Rahman (2020)** provided a summary of the transmuted families of distributions, a list of the transmuted distributions that can be found in the literature, and some finishing comments. **Provost et al (2022)** presented a method for approximating density based on moments, which is based on a generalisation of Pearson's system of frequency curves. **Mohiuddin (2022)** a thorough analysis of several distribution families is provided. Twenty-six related distributions are investigated and a total of roughly eighteen approaches for creating new families of distributions are discussed. **Aneeqa Khadim et al (2022)** A brief summary of the family of distributions produced by the {T-X} transformed-transformer is given. Included are comprehensive reviews, recommendations for specific enlarged versions of these distributions, and a list of pertinent research publications on the {T-X} family of distributions.

II. SOME POPULAR FAMILIES OF DISTRIBUTIONS:

1. PEARSONIAN FAMILY: The two primary quantities that define the Pearson distributions are sometimes referred to as Pearsonian constants β_1 and β_2, where $\beta_1 = \mu_3^2 / \mu_2^3$; $\beta_2 = \mu_4 / \mu_2^2$ Where μ_2, μ_3 and μ_4 are the second, third and fourth central moments of a probability distribution. These parameters are describing the nature of the curve of the distribution and the properties of centrality of the data, variance in the data, symmetricity / skewness and peaked Ness / kurtosis of the curve. Pearson initially classified the distributions into seven types of distributions named as Types I to Type-VII based on the values for the parameters. Later, identified 12 types which are variants of three basic distributions (Beta, Gamma and Normal). Pearson defined initially seven types of distributions based on the solutions to the below differential equation

$$f(x)\,dx + \frac{[b_0 + b_1 x + b_2 (x)^2\,]}{a + x}\,df(x) = 0$$

$$\boldsymbol{a} = \boldsymbol{b_1} = \sqrt{\mu_2}\sqrt{\beta_1}\,\frac{\beta_2+3}{10\beta_2-12\beta_1-18}\ ;\ \boldsymbol{b_0} = \frac{4\beta_2-3\beta_1}{10\beta_2-12\beta_1-18}\mu_2,\ \text{and}\ \boldsymbol{b_2} = \frac{2\beta_2-3\beta_1-6}{10\beta_2-12\beta_1-18}.$$

where $k = \frac{1}{4}(b_1)^2(b_0 b_2)^{-1}$.

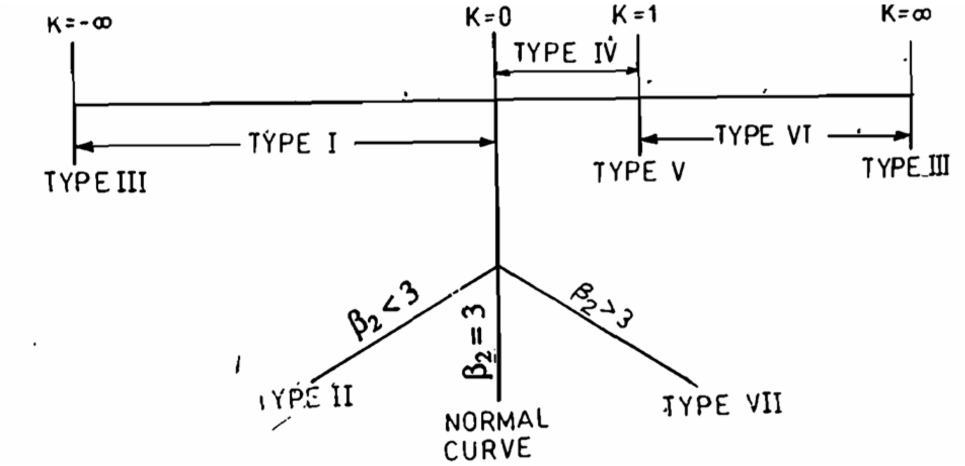

Figure 1. Seven types of Pearsonian distributions for different values of k

Types	Density function	Range of r.v.	Parameters
Type – I	$f(y) = y_0\left(1+\frac{x}{a_1}\right)^{m_1}\left(1-\frac{x}{a_2}\right)^{m_2}$	$-a_1 < x < a_2$	$m_1, m_2 > 0$
Type – II	$f(y) = y_0\left(1-\frac{x^2}{a^2}\right)^{m}$	$-a < x < a$	$m > -1$
Type – III	$f(y) = y_0\left(1+\frac{x}{a}\right)^{\mu a} e^{-\mu x}$	$-a < x < \infty$	$\mu, a > -1$
Type – IV	$f(y) = y_0\left(1+\frac{x^2}{a^2}\right)^{-m} e^{-\mu\tan^{-1}(x/a)}$	$-\infty < x < \infty$	$a, \mu, m > 0$
Type – V	$f(y) = y_0 x^{-p} e^{-(\alpha/x)}$	$0 < x < \infty$	$\alpha > 0,\ p > 1$
Type – VI	$f(y) = y_0 x^{-p}(x-a)^{q}$	$a < x < \infty$	$q > -1,\ p < 1$
Type – VII	$f(y) = y_0\left(1+\frac{x^2}{a^2}\right)^{-m}$	$-\infty < x < \infty$	$m > \frac{1}{2}$
Type-VIII	$f(y) = y_0\left(1+\frac{x}{a}\right)^{-m}$	$-a < x \le 0$	$m > 1$
Type – IX	$f(y) = y_0\left(1+\frac{x}{a}\right)^{m}$	$-a < x \le 0$	$m > -1$
Type – X	$f(y) = y_0 e^{-(x/\sigma)}$	$0 \le x < \infty$	$\sigma > 0$
Type – XI	$f(y) = y_0 x^{-m}$	$b \le x < \infty$	$m > 1$
Type – XII	$f(y) = y_0\left(\frac{g+x}{g-x}\right)^{m}$	$-g \le x \le g$	$\lvert m\rvert < 1$

Table 1. Pearsonian Family of Distribution density functions

2. JOHNSON'S FAMILY: Johnson (1949), introduced a system of distributions, based on the transformation of the standard normal variable. Johnson proposed a system of distributions that are generated using normalization transformation. Let $f(.)$ is the transformation function and γ, δ, λ, and ξ are parameters (δ and λ are positive), then the general form of Z is

$$Z = \gamma + \delta f\left(\frac{x-\xi}{\lambda}\right)$$

The distributions possessing this transformation (or translation) are said to be Johnson system of distributions. The distribution can be bounded on the lower end, upper end or both ends. This family is further divided as S_L (lognormal system), the S_L system defined with the transformation

$$Z = \gamma + \delta ln\left(\frac{x-\xi}{\lambda}\right), x > \xi;$$

The S_U system defined based on the transformation

$$Z = \gamma + \delta \sinh^{-1}\left(\frac{x-\xi}{\lambda}\right), -\infty < x < \infty.$$

The S_B system of distributions defined by

$$Z = \gamma + \delta ln\left(\frac{x-\xi}{\xi+\lambda-x}\right), \xi < x < \xi + \lambda$$

3. EXPONENTIAL FAMILY: The probability distribution expressed in the form

$$f(x, \theta) = \exp\{A(x) + B(\theta) + C(x). D(\theta)\}$$

where θ is the parameter(s), then the distribution belongs to exponential family of distributions. The Exponential family is an application oriented and is widely used in unified family of distributions on finite dimensional Euclidean spaces parameterized by a finite dimensional parameter vector as

$$f(X,\theta) = \exp\{A(X) + B(\theta) + \sum_{i=1}^{k} C_i(X)D_i(\theta)\}.$$

Where θ is the vector of k- parameters with the exists of k-real valued functions $D_1, D_2, \dots D_k$, and B defined on Θ and Borel – measurable functions $C_1, C_2, \dots C_k$ and A on R^n.

Distribution	A(x)	B(θ)	C(x)	D(θ)
Binomial	$\log\binom{n}{x}$	Nlogq	X	$\log\left(\frac{p}{q}\right)$
Poisson	$-\log(x!)$	$-\lambda$	X	$\log\lambda$
Negative Binomial	$\log\binom{x+r-1}{r-1}$	Rlogp	X	logq
Normal (When μ is known)	0	$-\log(\sigma\sqrt{2\pi})$	$(x-\mu)^2$	$-1/2\sigma^2$
Normal (when σ^2 is known)	-μ/2	$-((x/2\sigma^2)+(1/2)(\log(2\pi\sigma^2)$	X	M
Normal (μ, σ^2)	0	$-((x-\mu)2/2\sigma^2)+(1/2)(\log(2\pi\sigma^2))$	x, x2	$-1/2\sigma^2$, μ^2/σ^2

Table 2. Some Exponential Family of Distributions

4. POWER SERIES FAMILY: Noack (1950) introduced a family of Power series distributions. A probability distribution $f(x, \theta)$ is said to be a 'Power Series family of distributions' if it can be expressed in the form

$$P[X = x] = \frac{a^x \theta^x}{\sum_x^\infty a^x \theta^x}, \quad x = 0,1,2,\dots \text{ and } \theta > 0.$$

Some of the Power Series distributions are: Binomial, Poisson, Geometric, Negative Binomial, Logarithmic Series, etc.

Distribution	Probability function	a_x	θ^x	$f(\theta)$
Binomial	$\binom{n}{x}\left(\frac{p}{1-p}\right)^x [(1-p)^{-n}]^{-1}$	$\binom{n}{x}$	$\left(\frac{p}{1-p}\right)^x$	$(1-p)^{-n}$
Poisson	$\left(\frac{1}{x!}\right)(\lambda^x)(e^\lambda)^{-1}$	$\frac{1}{x!}$	λ^x	e^λ
Geometric	$(1)(1-p)^x(p^{-1})^{-1}$	1	$(1-p)^x$	p^{-1}
Negative Binomial	$(-1)^x\binom{-r}{x}(1-p)^x(p^{-r})^{-1}$	$(-1)^x\binom{-r}{x}$	$(1-p)^x$	p^{-r}
Logarithmic	$\left(\frac{1}{x}\right)(p^x)[-\log(1-p)]^{-1}$	$\frac{1}{x}$	p^x	$-\log(1-p)$

Table 3. Some Power Series Family of Distributions

5. T-X Family: Let f (t) be the probability density function and F(t) be its cumulative distribution function (cdf) of a r.v T defined on the range $-\infty < a < t < b < \infty$. Let 'c' (a < c < b) be a function of F(x) of some baseline continuous r.v X, with distribution function: $G(x) = \int_a^c f(t)dt$. Distributions possess such property are said to be T-X family of distributions. Some of the existing continuous and discrete distributions are found to belong to the first generated distributions.
Note: In case of t is discrete: $g(x) = G(x) - G(x-1)$ where $G(x) = \sum_a^c p(t)$ where $p(t)$ is Probability mass function.

	Weibull – Exponential	**Weibull – Rayleigh**	**Gumbel – Weibull**
$\boldsymbol{f(t)}$	$\frac{n}{\theta}\left(\frac{t}{\theta}\right)^{n-1} e^{-\left(\frac{t}{\theta}\right)^n}$	$\frac{n}{\theta}\left(\frac{t}{\theta}\right)^{n-1} e^{-\left(\frac{t}{\theta}\right)^n}$	$\left(\frac{1}{\sigma}\right) e^{-\frac{(t-v)}{\sigma}} exp\left[-e^{-\frac{(t-v)}{\sigma}}\right]$
$\boldsymbol{c}$	$-\log(1-F(x))$	$\frac{F(x)}{1-F(x)}$	$log\left[\frac{F(x)}{1-F(x)}\right]$
$\boldsymbol{G(x)}$	$1 - exp\left[-\alpha\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{\beta}\right]$	$1 - e^{\left[-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}\right]}$	$\exp\left\{e^{-\frac{v}{\sigma}}\left(e^{\left(\frac{x}{\lambda}\right)^a}-1\right)^{-\frac{1}{\sigma}}\right\}$
$\boldsymbol{g(x)}$	$\alpha\beta\lambda e^{-\lambda x}\left[\frac{\left(1-e^{-\lambda x}\right)^{\beta-1}}{(e^{-\lambda x})^{\beta+1}}\right] exp\left[-\alpha\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{\beta}\right]$	$\alpha\beta\theta x e^{\frac{\theta}{2}x^2}\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta-1} e^{\left[-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}\right]}$	$\frac{ae^{\frac{v}{\sigma}}}{\lambda\sigma}\left(\frac{x}{\lambda}\right)^{a-1} e^{\left(\frac{x}{\lambda}\right)^a}\left(e^{\left(\frac{x}{\lambda}\right)^a}-1\right)^{-1-\frac{1}{\sigma}} \exp\left\{e^{-\frac{v}{\sigma}}\left(e^{\left(\frac{x}{\lambda}\right)^a}-1\right)^{-\frac{1}{\sigma}}\right\}$

Table 4. T-X family of distributions

6 TRANSMUTED FAMILY: [14] Shaw and Buckley (2009) pioneered another prominent family of distributions by including a parameter λ . Let $F(x;\theta)$ be the cdf with a parameter θ, then the transmuted family of distributions with distribution function $G(x;\lambda,\theta)$ is defined as:

$$G(x;\lambda,\theta) = (1+\lambda)F(x;\theta) - \lambda F^2((x;\theta); \quad \theta > 0; |\lambda| \leq 1, x \in \mathbb{R}$$

Note: If $\lambda = 0$, we obtain the baseline distribution, i.e., $F(x;\theta) = G(x;\theta)$.

Distribution	$\boldsymbol{F(x)}$	$\boldsymbol{G(x)}$
Transmuted Burr	$F(x) = 1-(1+x^c)^{-k}$	$G(x) = 1-(1+x^c)^{-k} + \lambda\frac{[1-(1+x^c)^{-k}][1+x^c]^{-k}}{2-(1+x^c)^{-k}}$
Transmuted Gompertz	$F(x) = 1 - exp\left[-\frac{a}{b}(e^{bx}-1)\right]$	$G(x) = \left[1 - exp\left[-\frac{a}{b}(e^{bx}-1)\right]\right]\left[1 + \lambda\frac{exp\left[-\frac{a}{b}(e^{bx}-1)\right]}{2-exp\left[-\frac{a}{b}(e^{bx}-1)\right]}\right]$
Transmuted Weibull	$F(x) = 1-e^{-\alpha x^{\beta}}$	$G(x) = 1-e^{-\alpha x^{\beta}} + \lambda\frac{\left[1-e^{-\alpha x^{\beta}}\right]}{2-e^{-\alpha x^{\beta}}}e^{-\alpha x^{\beta}}$
Transmuted gamma	$F(x) = \frac{\gamma(a,b,x)}{\Gamma(a)}$ $\gamma(a,b,x) = b^a\int_0^x t^{a-1}e^{-bt}dt$	$G(x) = \frac{\gamma(a,b,x)}{\Gamma(a)}\left[1+\lambda\frac{1-\frac{\gamma(a,b,x)}{\Gamma(a)}}{1+\frac{\gamma(a,b,x)}{\Gamma(a)}}\right]$

Table 5. Transmuted family of distributions

7. GENERALIZED TRANSMUTED FAMILY: Let f (t) be the pdf of a r.v $T \in [a,b]$ for $-\infty < a < b < \infty$ and let $c = [G(x)]^{\alpha}$ be a function of the cdf of a r.v X such that (i) a < c < b; (ii) c is differentiable; and (iii) monotonically non- decreasing, then the Generalized transmuted family of distribution is defined as

$$\mathrm{F}(x) = 1 - \{1 - \lambda[\mathrm{G}(x)]^{\alpha}\}\{1 - [\mathrm{G}(x)]^{\alpha}\}, \quad \alpha > 0, \ |\lambda| \leq 1$$

$$f(x) = \alpha g(x)\, [G(x)]^{\alpha-1} \{1 + \lambda - 2\lambda[G(x)]^{\alpha}\}, \ \alpha > 0, \ |\lambda| \leq 1$$

where $g(x)$ and $G(x)$ are the baseline pdf and cdf.

Distribution	$\boldsymbol{g(x)}$	$\boldsymbol{G(x)}$	$\boldsymbol{f(x)}$
Generalized transmuted Normal	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$	$\alpha\phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha-1} \left\{1 + \lambda - 2\lambda\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}\right\}$
Generalized transmuted Exponential	$\theta e^{-\theta x}$	$1 - \exp(-\theta x)$	$\alpha\theta \exp(-\theta x)\,(1 - \exp(-\theta x))^{\alpha-1} [1 + \lambda - 2\lambda(1 - \exp(-\theta x))^{\alpha}]$
Generalized transmuted Weibull	$\frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1} \exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)$	$1 - exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)$	$\alpha\frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1} \exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\left(1 - exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right)^{\alpha-1}\left[1 + \lambda - 2\lambda\left(1 - exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right)^{\alpha}\right]$

Table 6. Generalized Transmuted Family

8. CUBIC TRANSMUTED FAMILY: [1] AL- Kadim and Mohammed (2017) proposed cubic transmuted family of distributions, by substituting the value of k=2 in a general transmuted family of distributions: $\mathrm{G}(x) = \mathrm{F}(x) + (1 - \mathrm{F}(x)) \sum_{i=1}^{k} \lambda_i\, F(x)^i$; $\theta > 0$, x ∈ ℝ, with λ_i ∈ [-1,1] for i = 1, 2,..., k and $-\mathrm{k} \leq \sum_{i=1}^{k} \lambda_i \leq 1$. Then the by setting k=2, a cubic transmuted family of distributions is defined with distribution function G(x) as

$$G(x) = (1 + \lambda_1)F(x) + (\lambda_2 - \lambda_1)F^2(x) - \lambda_2 F^3(x);\ \lambda_1, \lambda_2\ \in [-1,1] \text{and } -2 \leq \lambda_1 + \lambda_2 \leq 1$$

Note: The general transmuted family reduces to the base distribution for $\lambda_i = 0$ for all 'i'.

Cubic Transmuted Distribution	$\boldsymbol{F(x)}$	$\boldsymbol{G(x)}$
Normal	$\frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$	$(1 + \lambda_1)\Phi(x) + (\lambda_2 - \lambda_1)\Phi^2(x) - \lambda_2\Phi^3(x)$
Log-logistic	$\frac{x^{\beta}}{\alpha^{\beta} + x^{\beta}}$	$\frac{x^{\beta}\left[\lambda_1\alpha^{\beta}(\alpha^{\beta} + x^{\beta}) + \lambda_2\alpha^{\beta}x^{\beta} + (\alpha^{\beta} + x^{\beta})^2\right]}{(\alpha^{\beta} + x^{\beta})^3}$
Pareto	$1 - \left(\frac{k}{x}\right)^{\theta}$	$\left[\left(\frac{k}{x}\right)^{\theta} - 1\right]\left[-\lambda_1\left(\frac{k}{x}\right)^{\theta} + \lambda_2\left\{\left(\frac{k}{x}\right)^{\theta} - 1\right\}\left(\frac{k}{x}\right)^{\theta} - 1\right]$
Rayleigh	$1 - e^{-\frac{x^2}{2\sigma^2}}$	$\left(e^{-\frac{3x^2}{2\sigma^2}}\right)\left(e^{\frac{x^2}{2\sigma^2}} - 1\right)\left[\lambda_1 e^{\frac{x^2}{2\sigma^2}} + \lambda_2\left(e^{\frac{x^2}{2\sigma^2}} - 1\right) + e^{\frac{x^2}{2\sigma^2}}\right]$

Table 7. Cubic Transmuted Distribution

9. KUMARASWAMY – G FAMILY: [13] Kumaraswamy (1980) proposed a two-parameter distribution on (0,1), called Kumaraswamy distribution, whose density function is

$$g(x;\alpha,\beta) = \alpha.\beta.x^{\alpha-1}.(1 - x^{\alpha})^{\beta-1} \text{ where } \alpha > 0,\ \beta > 0 \text{ and } 0 < x < 1.$$

with a distribution function $G(x;\alpha,\beta) = 1 - (1 - x^{\alpha})^{\beta}$.

For any baseline distribution function $G(x)$, [3] Cordeiro and Castro proposed the Kumaraswamy-G family of distributions with additional two shape parameters a, b > 0 as

$$f(x) = abg(x)G^{a-1}(x)\{1 - G^{a}(x)\}^{b-1} \quad \text{and} \quad F(x) = 1 - (1 - \mathrm{G}^{a}(x))^{b}$$

Distribution	$G(x)$	$f(x)$
Kw – Weibull	$1-exp\{-(\lambda x)^c\}$	$abc\lambda^c x^{c-1}exp\{-(\lambda x)^c\}[1-exp\{-(\lambda x)^c\}]^{a-1}\{1-[1-exp\{-(\lambda x)^c\}]^a\}^{b-1}$
Kw- Exponential	$1-exp\{-\lambda x\}$	$ab\lambda exp\{-\lambda x\}[1-exp\{-\lambda x\}]^{a-1}\{1-[1-exp\{-\lambda x\}]^a\}^{b-1}$
Kw – Gumbel	$1-exp\left\{-exp\left(-\frac{(x-\mu)}{\sigma}\right)\right\}$	$\frac{ab}{\sigma}exp\left\{\frac{x-\mu}{\sigma}-exp\left(\frac{x-\mu}{\sigma}\right)\right\}\left[1-exp\left\{-exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{a-1}\left\{1-\left[1-exp\left\{-exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^a\right\}^{b-1}$

Table 8. Kumaraswamy – G Family

10 ALPHA POWER TRANSFORMATION: [6] Mahdavi and Kundu (2017) proposed Alpha Power Transform class of distributions to the baseline distribution with an additional parameter θ. The distribution function of this transformation class defined based on base line distribution function G(x) as

$$F(x)=\begin{cases}\frac{\theta^{G(x)}-1}{\theta-1} & if\ \theta>0,\ \theta\neq 1\\ G(x) & if\ \theta=1\end{cases}$$

and the corresponding density function takes the form

$$f(x)=\begin{cases}\frac{\log(\theta)g(x)\theta^{G(x)}}{\theta-1} & if\ \theta>0,\ \theta\neq 1\\ g(x) & if\ \theta=1\end{cases}$$

Distribution	$f(x)$	$F(x)$
Alpha Power Exponential Weibull	$\begin{cases}\frac{log\alpha}{(\alpha-1)\exp(\theta x^\beta)}\lambda\theta\beta x^{\beta-1}\left(1-e^{-\lambda x^\beta}\right)^{\theta-1}\alpha^{\left(1-e^{-\lambda x^\beta}\right)^\theta}; & x>0,\beta,\lambda,\theta>0,\alpha\neq 1\\ \lambda\theta\beta x^{\beta-1}\left(1-e^{-\lambda x^\beta}\right)^{\theta-1}e^{-\theta x^\beta}; & x>0,\ \alpha=1,\beta,\lambda,\theta>0\end{cases}$	$\begin{cases}\frac{\alpha^{\left(1-e^{-\lambda x^\beta}\right)^\theta}-1}{\alpha-1}; & x>0,\beta,\lambda,\theta>0,\alpha\neq 1\\ \left(1-e^{-\lambda x^\beta}\right)^\theta; & x>0,\ \alpha=1,\beta,\lambda,\theta>0\end{cases}$
Alpha Power transformation Lindley	$\begin{cases}\frac{log\alpha}{(\alpha-1)}\left(\frac{\theta^2}{\theta+1}\right)(1+x)\exp(-\theta x)\,\alpha^{1-(1+\theta+\frac{\theta x}{\theta+1}}; & x,\alpha,\theta>0,\alpha\neq 1\\ \left(\frac{\theta^2}{\theta+1}\right)(1+x)\exp(-\theta x); & x,\theta,\alpha>0,\alpha=1\end{cases}$	$\begin{cases}\frac{\alpha^{1-\left(\frac{1+\theta+\theta x}{\theta+1}\right)}-1}{\alpha-1}; & x,\alpha,\theta>0,\alpha\neq 1\\ 1-\left(\frac{1+\theta+\theta x}{\theta+1}\right); & x,\theta,\alpha>0,\alpha=1\end{cases}$
Alpha Power transformed Pareto	$\begin{cases}\frac{log\alpha}{(\alpha-1)}\frac{\beta}{x^{\beta+1}}\alpha^{\left(1-\frac{k}{x}\right)^\beta}; & \beta,\alpha,k>0,x\geq k,\alpha\neq 1\\ \frac{\beta}{x^{\beta+1}}\alpha^{\left(1-\frac{k}{x}\right)^\beta}; & \beta,\alpha,k>0,x\geq k,\alpha=1\end{cases}$	$\begin{cases}\frac{\alpha^{\left(1-\frac{k}{x}\right)^\beta}-1}{\alpha-1}; & \beta,\alpha,k>0,x\geq k,\alpha\neq 1\\ \left(1-\frac{k}{x}\right)^\beta; & \beta,\alpha,k>0,x\geq k,\alpha=1\end{cases}$

Table 9. Alpha Power Transformation

REFERENCES:

[1]. AL – Kadim and Mohammed (2017): The Cubic Transmuted Weibull Distribution, Journal of Babylon University/Pure and Applied Sciences/No. (3)/Vol. (25), pp 862-876.
[2]. Bachioua Lahcene (2013): “On Pearson families of distributions and its applications” Vol. 6(5), pp. 108-117.
[3]. G.M. Cordeiro and M. de Castro, A new family of generalized distributions, J. Statist. Comput. Simul. 81 (2011), pp. 883–898.
[4]. Jorge Ollero and Hector M. Ramos (1995): Description of a sub family of the discrete Pearson system as Generalized binomial distribution, Ital. Statist. Soc.: pp 235-249.
[5]. Lipi B. Mahanta and Dilip C. Nath (2012): “The Pearsonian Type III Curve and its Potentials in Projecting Insurance Data”, Vol. 2. pp. 499-503.
[6]. Mahdavi, A. and Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. Communications in Statistics - Theory and Methods, 46:6543 - 6557.
[7]. Noack, A. (1950): “A class of random variable with discrete distribution” Annals of Institute of Statistics &Mathematics, vol 21(1), pp 127-132.
[8]. Norman. L. Johnson (1949): Systems of Frequency Curves Generated by Methods of Translation, Biometrika, Vol. 36, No. 1/2, pp. 149-176.
[9]. Ord J.K. (1967): On a system of discrete distributions, Biometrika 54: pp. 649-656
[10]. Pearson K (1895): "Contributions to the mathematical theory of evolution, II: Skew variation in homogeneous material.” Philos. Trans. Royal Soc. London. A, Vol. 186, pp. 343-414.
[11]. Pearson K (1901): "Mathematical contributions to the theory of evolution, X: supplement to a memoir on skew variation”, Philosophical Transactions of the Royal Society of London. A, Vol. 68, pp. 442-450.
[12]. Pearson K (1916): "Mathematical contributions to the theory of evolution, XIX: Second supplement to a memoir on skew variation". Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character, Vol. 216, pp. 429-457.
[13]. P. Kumaraswamy (1980): Generalized probability density-function for double-bounded random-processes, J. Hydrol. 46, pp. 79–88.
[14]. Shaw W.T. and Buckley I. R. (2009): The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, arXiv:0901.0434, pp 1-8.