International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 12 Issue 2 || Mar.-Apr. 2024 || PP-48-51

Support points of some subclasses of univalent analytic functions with negative coefficients

[1]N.C. Sobha Rani, [2]S. Lalitha
[1]*Department of Mathematics, Kamala Institute of Technology and Science, Karimnagar*
[2]*Department of Mathematics, Geethanjali College of Engineering and Technology, Hyderabad*
Corresponding Author: S. Lalitha

ABSTRACT: *The main aim of the present paper is to find support points of some subclasses of univalent analytic functions with negative coefficients by fixing second ant third Taylor coefficients.*
KEYWORDS: *Support points, Univalent functions .*

Date of Submission: 25-04-2024 Date of acceptance: 02-05-2024

I. INTRODUCTION

Let S denote the class of normalized univalent analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. where $\mathbb{C}$ is the set of complex numbers. Silverman [2] introduced the concept of univalent analytic functions in the open unit disc U with negative Taylor coefficients from the second and denoted the class by T. The class T is a subclass of S, consisting of functions which are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, z \in U, \; a_n \geq 0 \text{ for } n \geq 2. \quad (1)$$

that are analytic and univalent in U. The subclasses of T that are starlike of order α, convex of order α are denoted by $T^*(\alpha)$ and $C(\alpha)$ respectively for $0 \leq \alpha < 1$. Silverman[2] obtained necessary and sufficient condition for the function

$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $z \in U$, $a_n \geq 0$ to be in $T^*(\alpha)$, as

$$\sum_{n=2}^{\infty} (n - \alpha) a_n \leq 1 - \alpha \quad (2)$$

and the necessary and sufficient condition for $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ to be in $C(\alpha)$ as

$$\sum_{n=2}^{\infty} n(n - \alpha) a_n \leq 1 - \alpha. \quad (3)$$

Silverman [3] defined a subclass of T which contains functions having univalent derivatives and denoted this class by T_1. He derived a necessary and sufficient condition for $f \in T$ of the form (1) to be in T_1 as

$$\sum_{n=3}^{\infty} (n-1)n \, a_n \leq 2a_2 \quad (4)$$

for $a_2 > 0$.

By imposing generalized form of condition (4), Srinivas [5] introduced subclasses $T(b, B_n)$, $C(b, B_n)$, $T^*(b, B_n)$ of T by fixing second Taylor coefficient a_2 and obtained extreme points and support points. Applying the generalized form of condition (4), Lalitha and Srinivas[6] introduced the class $T(b, c, B_n)$ by fixing second and third Taylor coefficients a_2 and a_3.

$T(b, c, B_n) =$
$\{f(z) \in T: \; f(z) = z - a_2 z^2 - a_3 z^3 - \sum_{n=4}^{\infty} a_n z^n$ and $\sum_{n=3}^{\infty} B_n a_{n+1} \leq 2b - cB_2 \}$
where $0 < a_2 = b \leq \frac{1}{4}$, $0 < a_3 = c \leq \frac{1}{12}$ and $B_n \geq n(n+1)$ for $n \geq 2$.

They [6] also introduced starlike and convex subclasses $S_\alpha^*(b, c, B_n)$ and $C_\alpha(b, c, B_n)$ of T as

$S_\alpha^*(b, c, B_n) =$
$\left\{ f(z) \in T(b, c, B_n): \; \sum_{n=3}^{\infty} \frac{(n+1-\alpha)}{n(n+1)} B_n a_{n+1} \leq (1-\alpha) - (2-\alpha)b - (3-\alpha)c \right\}$

$C_\alpha(b,c,B_n) =$
$\left\{f(z) \in T(b,c,B_n) \colon \sum_{n=3}^{\infty} \frac{(n+1-\alpha)}{n} B_n a_{n+1} \leq (1-\alpha) - (2-\alpha)2b - (3-\alpha)3c\right\}$
for $0 \leq \alpha < 1$.

We have $S^*(b,c,B_n) = \ S_0^*(b,c,B_n)$ and $C(b,c,B_n) = C_0(b,c,B_n)$.

In [6], a characterization for a function to be in $T(b,c,B_n)$ was obtained as follows:

Theorem A: $f \in T(b,c,B_n)$ if and only if $f(z)$ can be expressed as
$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z),\ z \in U$ where $\lambda_n \geq 0$ for $n \geq 2$ and $\sum_{n=2}^{\infty} \lambda_n = 1$,
$f_2(z) = z - bz^2 - cz^3,\ f_n(z) = z - bz^2 - cz^3 - \frac{2b-6c}{B_n} z^{n+1},\ n \geq 3$
and $B_n \geq n(n+1)$ for $n \geq 2$, $0 < b \leq \frac{1}{4}$, $0 <\leq \frac{1}{12}$ and $z \in U$.

The Extreme points of $T(b,c,B_n)$ were obtained in [6] as
$f_2(z) = z - bz^2 - cz^3,\ f_n(z) = z - bz^2 - cz^3 - \frac{2b-6c}{B_n} z^{n+1},\ n \geq 3.$

Similarly Extreme points of $C(b,c,B_n)$ were obtained in [6] as
$f_2(z) = z - bz^2 - cz^3,\ f_n(z) = z - bz^2 - cz^3 - \frac{n(1-4b-9c)}{(n+1)B_n} z^{n+1},\ n \geq 3.$

Further in [6], the Extreme points of $S^*(b,c,B_n)$ were obtained as
$f_2(z) = z - bz^2 - cz^3,\ f_n(z) = z - bz^2 - cz^3 - \frac{n(1-2b-3c)}{B_n} z^{n+1},\ n \geq 3.$

Results analogous to Theorem A for the classes $C(b,c,B_n)$ and $S^*(b,c,B_n)$ were found in [6].

Support Points: A function $f(z)$ in the class $T(b,c,B_n)$ is said to be a support point of $T(b,c,B_n)$ if there exists a continuous linear functional J on

$$A = \{f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0,\ f(z) \text{ is analytic in } U\},$$

such that

$Re\{J(f)\} \geq Re\{J(g)\}$ for all $g \in T(b,c,B_n)$ and $Re(J)$ is non-constant on $T(b,c,B_n)$.

We denote by $Supp\{T(b,c,B_n)\}$ the set of support points of $T(b,c,B_n)$ and the set of extreme points of $T(b,c,B_n)$ is denoted by $ext\{T(b,c,B_n)\}$.

In a similar fashion, we can define $Supp\{C(b,c,B_n)\}$ and $Supp\{S^*(b,c,B_n)\}$.

Now we find the support points using the following Lemma proved by Brickman et.al.[1] .

Lemma: Let $\{b_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} sup|b_n|^{\frac{1}{n}} < 1$ and set $J(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A$. Then J is a continuous linear functional on A.

Conversely, any continuous linear functional on f on A is given by such a sequence $\{b_n\}$.

II. MAIN RESULTS

Now, we proceed to find the support points of $T(b,c,B_n)$.

Theorem 1: $Supp\{T(b,c,B_n)\} =$
$\left\{f \in T(b,c,B_n) \colon f(z) = z - bz^2 - cz^3 - (2b-6c)\sum_{n=3}^{\infty} \frac{\lambda_n}{B_n} z^{n+1}\right.$ where $\lambda_n \geq 0,\ \sum_{n=3}^{\infty} \lambda_n \leq 1$ and $\lambda_j =$
0 for some $\left.j\right\}$, where $0 < b \leq \frac{1}{4},\ 0 < c \leq \frac{1}{12}$ and $B_n \geq n(n+1)$ for $n \geq 2$. (5)

Proof:
Let the function $f(z)$ be in the class $T(b,c,B_n)$
and let $f_0(z) = z - bz^2 - cz^3 - (2b-6c)\sum_{n=3}^{\infty} \frac{\lambda_n}{B_n} z^{n+1}$
where $\lambda_n \geq 0$, $\sum_{n=3}^{\infty} \lambda_n \leq 1$ and $\lambda_j = 0$ for some $j \geq 3$.

If $b_1 = b_2 = b_3 = b_{j+1} = 1$ and $b_n = 0$ for $n \geq 4$ and $n \neq j + 1$ then

$$\lim_{n\to\infty} \sup|b_n|^{1/n} < 1.$$

Then, with the help of Lemma, we can define the continuous linear functional f on A given by the sequence $\{b_n\}$.

It follows from the above that

$$J(f_0) = 1 - b - c \text{ and } J(f) = 1 - b - c - \mu_j \frac{(2b-6c)}{B_j}$$

where $f(z) = z - bz^2 - cz^3 - \sum \frac{(2b-6C)}{B_n} \mu_n z^{n+1} \in T(b, c, B_n)$

and there exist μ_n, such that $\mu_n \geq 0$ and $\sum_{n=2}^{\infty} \mu_n \leq 1$.

Thus we have

$$Re\{J(f)\} \leq Re\{J(f_0)\} \text{ for all } f \in T(b, c, B_n).$$

Now

$$f_j(z) = z - bz^2 - cz^3 - \frac{(2b-6c)z^{j+1}}{B_j} \in T(b, c, B_n)$$

and $J(f_j) = 1 - b - c - \frac{2b-6c}{B_j} \leq J(f_0)$.

This shows that $f_0 \in Supp\{T(b, c, B_n)\}$

Conversely, let $f_0 \in Supp\{T(b, c, B_n)\}$.

Then there exists a continuous linear fractional J on A such that $Re\{J\}$ is non-constant on $T(b, c, B_n)$ and

$$Re\{J(f_0)\} = \max_{T(b,c,B_n)} Re\{J(g)\}.$$

Now,

Define the class $G =$

$\{f \in T(b, c, B_n): Re\{J(f)\} = \max_{T(b,c,B_n)} Re\{J(G)\} = Re\{J(f_0)\}\}$.

The class G is closed , convex and locally uniformly bounded.

Thus G is compact.

Since $f_0 \in G$, by Krein-Milman theorem

$Ext\{G\} \equiv \{f \in G: f$ is an extreme point of $G\}$ is nonempty.

Claim: $Ext\{G\} \subseteq Ext\,\{T(b, c, B_n\}$.

Proof of claim: Let $h \in Ext\{G\}$.

If $h \notin Ext\, T(b, c, B_n)$, $\exists\, h_1, h_2 \in T(b, c, B_n)$ and $0 < \lambda < 1$

such that $h = \lambda h_1 + (1 - \lambda)h_2$.

Since $h \in G$, this implies h_1, $h_2 \in G$.

But h is an extreme point of G.

Therefore $h \in Ext\, T(b, c, B_n)$.

Since $Re\{J\}$ is non constant on $T(b, c, B_n)$,

We obtain that $Ext\{G\} \subset Ext\, T(b, c, B_n)$ properly.

Thus there exists a $j \in N$ such that the extreme point of $T(b, c, B_n)$, $f_j \notin Ext\,\{G\}$.

Let k be the smallest of such j's.

Suppose $k > 1$.

In this context, $f_0 \in G$ influences that

$f_0(z) = z - bz^2 - cz^3 - (2b - 6c) \sum_{n=3}^{\infty} \frac{\lambda_n}{B_n} z^{n+1}, z \in U$

where $0 \leq \lambda_n$ and $\sum_{n=3}^{\infty} \lambda_n \leq 1$ by applying Theorem A.

Thus $f_0 \in RHS\ of$ (5)

Now consider $k = 1$.

Here $Ext\{G\} \subset \{f_j \in Ext\{T(b, c, B_n): j \geq 2\}$

Since B_n tends to ∞ as n goes to infinity and G is closed, there are only a finite number of

f_n's $\in Ext\{G\}$.

Now by applying Theorem A and $f_0 \in G$, we get

$f_0 \in RHS\ of$ (5)

This completes the proof of the theorem.

Next, we find the support points of the class $C(b,c,B_n)$.

Theorem 2: $Supp\{C(b,c,B_n)\} = \left\{f \in C(b,c,B_n): f(z) = z - bz^2 - cz^3 - (1-4b-9c)\sum_{n=3}^{\infty}\frac{\lambda_n}{(n+1)B_n}z^{n+1},\right.$ $z \in U$ where $\lambda_n \geq 0$, $\sum_{n=3}^{\infty}\lambda_n \leq 1$ and $\left.\lambda_j = 0 \text{ for some } j\right\}$, where $0 < b \leq \frac{1}{4}$ and $0 < c \leq \frac{1}{12}$, $B_n \geq \frac{(n+1)^2(2b-6c)}{1-4b-9c}$.

Proof: Similar to that of Theorem1.

Finally, we find the support points of the class $S^*(b,c,B_n)$

Theorem 3: $Supp\{S^*(b,c,B_n\} =$ $\left\{f \in S^*(b,c,B_n) : f(z) = z - bz^2 - cz^3 - (1-2b-6c)\sum_{n=3}^{\infty}\frac{\lambda_n}{B_n}z^{n+1}, z \in U \text{ where } \lambda_n \geq 0,\ \sum_{n=3}^{\infty}\lambda_n \leq 1 \text{ and } \lambda_j = 0 \text{ for some } j\right\}$

where $0 < b \leq \frac{1}{4}$ and $0 < c \leq \frac{1}{12}$, $B_n \geq \frac{(n+1)(2b-6c)}{(1-2b-3c)}$

Proof: Similar to that of Theorem 1.

REFERENCES

[1]. Brickman, L. MacGregor, T.H. and Wilken, D.R.- Convex hulls of some classical families of univalent functions, Trans. Amer. Math Soc. 156(1971), pp. 91-107.

[2]. Herb Silverman- Univalent functions with negative coefficients, Proceedings of the American Mathematical Society, Volume 1, Number1, August 1975, pp. 109-116.

[3]. Silverman. H.-Univalent functions having univalent derivatives, Rocky Mountain J. Maths. 16, (1986), pp. 55-61.

[4]. Patel, Jagannath.- A study of univalence of Gelfond-Leontev derivatives of analytic functions and related function classes, Ph.D. Thesis, Jan-1987, IIT, Kanpur.

[5]. V. Srinivas - On certain classes of analytic functions with fixed second coefficient, J. Ramanujan Math Society-6 Nos 1 & 2[1991], pp. 151-166.

[6]. S. Lalitha, V. Srinivas - International Journal of Creative Research Thoughts, Volume 6, Issue 2, April 2018 | ISSN: 2320-2882, pp. 89-93.