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NEVER PRIME!

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Abstract
We find pairs, (a, b) of opposite parity, where a and b are natural numbers greater than 1 such that $f(n) = a^n + b$ is composite for all $n \geq 2$.

Let $f(n) = a^n + b$, where a, b, and n are natural numbers greater than 1. Note that if a and b have the same parity, then a^n and b have the same parity, in which case $f(n)$ is even and is, therefore, composite. Can we find pairs, (a, b) of opposite parity such that $f(n)$ is composite for all $n \geq 2$? We will show, as an illuminating example, that $f(n) = 14^n + 11$ is composite for all $n \geq 2$. We shall take the moduli of both sides of $f(n) = 14^n + 11$, mod 15, obtaining

$$f(n) = (-1)^n + 11 \pmod{15} \qquad (*)$$

Case 1: n even. (*) becomes $f(n) = 1 + 11 = 12 \pmod{15}$, implying that $f(n) = 12 + 15k$, for some integer, k, so $3 \mid f(n)$.

Case 2: n odd. (*) becomes $f(n) = -1 + 11 = 10 \pmod{15}$, implying that $f(n) = 10 + 15k$, for some integer, k, so $5 \mid f(n)$. Done.

Remark: $f(n) = 14^n + (11 + 30k)$ is composite for all $n \geq 2$, and for all integers, k. Note that $11 + 30k$ will assume positive and negative values.
We have the following theorems:

Theorem 1: Let m be a given positive *odd* integer > 1, and let $f(n) = (2m + 1)^n + (m - 1)$, that is, $a = 2m + 1$, which is odd, and $b = m - 1$, which is even. Then $f(n)$ is composite for all $n \geq 2$.

Proof: By the Binomial Theorem, we have

$$f(n) = (2m)^n + \binom{n}{1}(2m)^{n-1} + \binom{n}{2}(2m)^{n-2} + \binom{n}{3}(2m)^{n-3} + \cdots + \binom{n}{n-1}(2m) + 1 + (m-1) =$$

$$(2m)^n + \binom{n}{1}(2m)^{n-1} + \binom{n}{2}(2m)^{n-2} + \binom{n}{3}(2m)^{n-3} + \cdots + \binom{n}{n-1}(2m) + m$$

Since every term in this last expression contains a factor, m, we see that $3 \mid f(n)$. ■
We can generalize the Theorem by changing $(m - 1)$ to $(km - 1)$ for any odd natural number, k.

Theorem 2: Let $f(x) = x^2 + x + 2$, where x is a natural number. Even though $f(x)$ can't be factored algebraically, it never assumes a prime value.

Proof: Since x and x^2 have the same parity (both even or both odd), their sum, $x^2 + x$, is even. Then $f(x) = x^2 + x + 2$ is always even. As the only even prime is 2 and since $f(x) > 2$, we are done. ■

Remark: The repunit, R_m, consists of m '1's. Let $f(n, k) = 10^n + R_{3k+2}$, where $n \geq 2$ and $k \geq 0$. Then $f(n, k)$ never assumes prime values. This follows from the fact that $f(n, k)$ has exactly 3 '1's and any number of '0's, so its digit sum is 3.

References

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